Long-Wavelength Oscillations in an Inhomogeneous One-Component Plasma

Joel L. Lebowitz
Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

and

Ph. A. Martin
Institut de Physique Théorique, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
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The perfect screening of charge fluctuations in an equilibrium plasma is extended to the time-displaced structure function of a general inhomogeneous one-component plasma. We find that the long-wavelength modes oscillate undamped with a single frequency $\overline{\omega}$, $\overline{\omega}^2$ being an angular average of squares of plasma frequencies $\omega_p^2 = 4\pi e^2\rho/m$ in uniform systems with density ρ. Our results are derived rigorously from the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy under some reasonable assumptions on the spatial decay of correlations and contain as special cases previously obtained results of this kind.

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It is known that the static structure function $S(q_1|q_2)$ describing charge fluctuations in an equilibrium classical d-dimensional plasma obeys a perfect screening condition which can be written in the following form[1] where q_1 and q_2 are vectors in d-dimensional space (our results hold for all $d \geqslant 2$, but for simplicity we formulate them explicitly only for $d=3$):

$$\int dq \left[\int dq_1 \, |q_1|^{-1} S(q_1|q)\right] = k_B T. \tag{1}$$

In the homogeneous case, Eq. (1) reduces to the usual Stillinger-Lovett second-moment condition $\tilde{S}(k) \simeq (k_B T/4\pi)|k|^2$ as $k \to 0$ with $\tilde{S}(k) = \int dq \exp(ik\cdot q) S(q|0)$. In Ref. 1, Eq. (1) was established under the assumption that the direct correlation function behaves as the potential at large distances; it can also be shown to be an exact consequence of the Born-Green-Yvon equations.[2]

In this Letter, we derive a dynamical generalization of Eq. (1) for a large class of nonuniform one-component plasmas (OCP). Consider an OCP with particles of charge e and mass m, having a background density $\rho_b(q)$ which is asymptotically constant in (almost) all directions, i.e., $\lim_{r\to\infty} \rho_b(r,\Omega) = \rho_b(\Omega)$ exists (for almost every Ω), with $q=(r,\Omega)$, $r=|q|$, and Ω the angles of q. Let $N(q,t) = \sum_j \delta[q - q_j(t)]$ be the particle number density, $\rho(q) = \langle N(q,0)\rangle_{\rm eq}$ the corresponding equilibrium average density, and

$$S(q_1,t|q_2) = e^2[\langle N(q_1,t)N(q_2,0)\rangle_{\rm eq} - \rho(q_1)\rho(q_2)]$$

the time-displaced charge-charge fluctuations. Then the proper generalization of Eq. (1) is

$$\int dq \left[\int dq_1 \, |q_1|^{-1} S(q_1,t|q)\right] = k_B T \cos(\overline{\omega} t), \tag{2}$$

with

$$\overline{\omega}^2 = (4\pi e^2/m)\left[\lim_{r\to\infty} \int (d\Omega/4\pi)\rho_b(r,\Omega)\right]. \tag{3}$$

As sketched below, Eqs. (2) and (3) are derived from the time-dependent Boguliubov-Born-Green-Kirkwood-Yvon hierarchy under some reasonable spatial-clustering assumptions. Let us first note that Eqs. (2) and (3) include various known cases, obtained from linear response and macroscopic electrodynamic considerations:

(i) For a uniform OCP with background density ρ_b, $\overline{\omega} = \omega_p = (4\pi e^2\rho_b/m)^{1/2}$ is the usual plasmon frequency, and Eq. (2) reduces to

$$\tilde{S}(k,t) \simeq [(k_B T/4\pi)\cos(\omega_p t)]|k|^2, \quad k \to 0,$$

the known long-wavelength behavior.[3,4]

(ii) For a semi-infinite OCP bounded by an impenetrable plane wall,

$$\rho_b(q) = \begin{cases} 0, & x<0, \\ \rho_b, & x>0, \end{cases}$$

$q=(x,y)$ (y the component of vector q parallel to the wall), we have $\overline{\omega} = \omega_p/\sqrt{2}$ and Eq. (2) is equivalent to a sum rule obtained by Jancovici.[5] Indeed, with introduction of the partial Fourier transform in the y direction,

$$\tilde{S}(x_1,k,t|x_2)=\int dy_1\, e^{ik\cdot y_1}S(x_1,y_1,t|x_2,0),$$

and the fact that

$$\int dy\, e^{ik\cdot y}(x^2+|y|^2)^{-1/2}=2\pi|k|^{-1}e^{-|k||x|},$$

our Eq. (2) becomes

$$\int_0^\infty dx_1\int_0^\infty dx_2\, e^{-|k||x_1|}\tilde{S}(x_1,k,t|x_2)\simeq[(k_BT/2\pi)\cos(\omega_p t/\sqrt{2})]|k|,\quad k\to 0, \tag{4}$$

which is the same as the classical limit of Eq. (3.11) of Ref. 5.

(iii) For a two-density OCP,

$$\rho_b(q)=\begin{cases}\rho_+, & x>0,\\ \rho_-, & x<0,\end{cases}$$

Eq. (3) gives $\bar{\omega}^2=\frac{1}{2}(\omega_+^2+\omega_-^2)$, and in a form analogous to (4), Eq. (2) is the result of Sec. 5 of Ref. 5.

We now formally give the main argument of the proof of Eq. (2). Details and additional results will be given by Jancovici, Lebowitz, and Martin.[6] The BBGKY hierarchy yields the following expression for the second time derivative of spatial averages over the time-dependent structure function:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int dq_1 f(q_1)S(q_1,t|q)\\ &=m^{-1}\int dq_1\,\nabla_1 f(q_1)\cdot[\rho(q_1)\int dq_2\,F(q_1-q_2)S(q_2,t|q)]\\ &\quad+m^{-1}\int dp_1\int dq_1(p_1\cdot\nabla_1)f(q_1)(p_1\cdot\nabla_1)S(p_1,q_1,t|q)\\ &\quad+m^{-1}\int dq_1\,\nabla_1 f(q_1)\cdot[eE(q_1)S(q_1,t|q)+e^2\int dq_2\,F(q_1-q_2)\rho_T(q_1,q_2,t|q)],\end{aligned} \tag{5}$$

where $F(q_1-q_2)=-\nabla_1 e^2/|q_1-q_2|$ is the Coulomb force and $f(q_1)$ is an arbitrary space-dependent function. $S(p_1,q_1,t|q)$ is a generalized momentum- (p_1-) dependent structure function. $E(q_1)$ is the electric field due to the total charge density and $\rho_T(q_1,q_2,t|q)$ is a fully truncated three-point function. Choosing $f(q_1)=1$ in Eq. (5) gives

$$\frac{\partial^2}{\partial t^2}\int dq_1\,S(q_1,t|q)=0.$$

This implies the validity of the electroneutrality sum rule for all times,

$$\int dq_1\,S(q_1,t|q)=0, \tag{6}$$

as a consequence of the fact that the same rule holds in equilibrium at $t=0$.[7] If we now choose $f(q_1)=|q_1|^{-1}$ and integrate over q, the first term on the right-hand side of (5) can be written, after an integration by parts and an exchange of integrals, as

$$(e^2/m)\int dq_1\,\nabla_1\cdot[\rho(q_1)\nabla_1|q_1|^{-1}]\{\int dq[\int dq_2\,|q_1-q_2|^{-1}S(q_2,t|q)]\}. \tag{7}$$

If we set

$$\alpha(q_1,t)=\int dq[\int dq_2\,|q_1-q_2|^{-1}S(q_2,t|q)],$$

then the Poisson equation gives

$$\nabla_1^2\alpha(q_1,t)=-4\pi\int dq\,S(q_2,t|q)=0, \tag{8}$$

where we have used $S(q_1,t|q)=S(q,-t|q_1)$ (stationarity of the equilibrium state) and Eq. (6). Thus $\alpha(q_1,t)$ is harmonic everywhere. Hence, since it is uniformly bounded, $\alpha(q_1,t)$ is constant with respect to q_1. Therefore, with the definition

$$\bar{\omega}^2 = -\frac{e^2}{m}\int dq_1 \nabla_1 \cdot \left[\rho(q_1)\nabla_1 \frac{1}{|q_1|}\right] = -\frac{e^2}{m}\lim_{r\to\infty}\int_{|q_1|=r}\rho(q_1)\nabla_1\frac{1}{|q_1|}\cdot d\sigma_1$$

$$= \frac{e^2}{m}\lim_{r\to\infty}\int \rho(r,\Omega)\,d\Omega = \frac{e^2}{m}\lim_{r\to\infty}\int \rho_b(r,\Omega)\,d\Omega \tag{9}$$

$[\rho(r,\Omega)-\rho_b(r,\Omega)\rightarrow 0$, $r\rightarrow\infty$, because of neutrality], the expression (7) is simply $-\bar{\omega}^2\alpha(0,t)$. Moreover, it can be shown that with $f(q_1)=|q_1|^{-1}$, the second and third terms on the right-hand side of (5) vanish when integrated on q. One proceeds by comparing these terms with the corresponding ones for uniform sytems: The argument involves the charge sum rules, as well as a time-dependent generalization of the dipole sum rule of Ref. 7. With all this, Eq. (5) reduces to the simple second-order differential equation

$$\frac{\partial^2}{\partial t^2}\alpha(0,t) = -\bar{\omega}^2\alpha(0,t). \tag{10}$$

Supplemented with the initial condition (1), and $\partial\alpha(0,t)/\partial t|_{t=0}=0$ [$\alpha(0,t)$ is even in time], the solution of Eq. (10) is our result, Eqs. (2) and (3).

A complete proof requires a specification of the cluster properties needed for the convergence and permutation of integrals, and the vanishing of surface terms at infinity. The main assumption is the following: At any fixed time the correlations of the inhomogeneous OCP with all arguments going to infinity in a fixed direction Ω converge sufficiently fast to the correlations of a uniform OCP with density $\rho_b(\Omega)$. If the system is bounded by hard walls (as the semi-infinite OCP) and the particles undergo elastic collisions at the walls, Eq. (5) can still be used but one has to take care of boundary contributions; the formulas (2) and (3) remain true, however.

Finally, it should be stressed that, whereas the static condition (1) holds generally for multicomponent systems, Eq. (2) is valid only for the OCP. Multicomponent systems show a much more complex dynamical behavior with dissipation occurring also in the long-wavelength limit. The reason for this difference is that in the one-component plasma momentum and current go together so that interparticle collisions (which always conserve momentum) also conserve current. This is particularly transparent in the homogeneous case where total momentum and total current are conserved quantities. The interesting thing about our result is that we still get a well-defined, nondissipative, oscillation even in the homogeneous case.

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