New systematic expansion of the electric field distribution in plasmas

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We derive a new systematic expansion of the electric field distribution at a test charge immersed in an infinite two- or three-dimensional one-component plasma. The lowest-order truncation of this expansion leads to a mean-field theory very similar to the adjustable-parameter exponential approximation (APEX). The next-order corrections to this mean-field theory are explicitly computed in terms of the distribution functions of the plasma particles. All these approximations are compared to the Monte Carlo results for a two-dimensional system at $\Gamma=2$ and various test charges. The systematic approximations appear to be useful. Even the zeroth-order approximation is quite accurate for large test charges or strongly coupled systems and the next order improves on it. Still, APEX is found to be most reliable (as it is also in three dimensions) and remains accurate in the practical interesting limit where the test charge vanishes, i.e., at a neutral atom.

I. INTRODUCTION

The spectral line shapes of a radiator (atom or ion) immersed in a plasma provide useful information about the system.[1] The line shapes are modified (broadened and shifted) by the fluctuating electric field created by the charges of the plasma. Under some assumptions,[2,3] the determination of this can be deduced from the computation of the electric field distribution at the radiator.

Various approximate theories[2–10] have been proposed for computing the electric field distribution. These approximations generally consider the correlations between the plasma particles as small perturbations and thus fail in the strong-coupling regime. Recently, Iglesias *et al.*[11] have proposed the adjustable parameter exponential approximation (APEX) which appears to be very accurate for strongly coupled plasmas. APEX involves inspired guesses and/or *ad hoc* assumptions. The main motivation of the present work is to derive an approximation very similar to APEX in a systematic way.

In this paper we derive a new systematic expansion of the electric field distribution whose lowest-order truncation leads to a mean-field theory very similar to APEX. Both approximations exactly give the second moment of the field distribution and replace the Coulomb field created by one plasma particle by a suitable screened mean field. The corrections to the mean-field approximation are related to the variations of the field fluctuations when one plasma particle is kept fixed. The essential novelty of the present expansion is that part of the correlation effects is treated "exactly" and included in the lowest-order term. In particular, the mean field appearing in this term is screened; this property is in agreement with the intuitive ideas about screening effects in plasmas.

For simplicity in presentation, the present expansion is explicitly written only for the case of a test charge immersed in an infinite two- or three-dimensional one-component plasma. Calculations are further restricted to the two-dimensional system at a special value of the temperature for which many exact results are available. That provides an unambiguous test of the considered approximations.

The paper is arranged as follows. In Sec. II we define the models and describe the coupling-parameter integration method,[10] which expresses the Fourier transform of the field distribution in terms of a formal distribution function. We also recall the general form of APEX and extend it to the practical interesting case of a neutral point (i.e., when the radiator is an atom). Starting from the formal coupling-parameter expression, we derive the required expansion in Sec. III. A drawback of the latter in two dimensions is briefly discussed. The truncation of this expansion provides a scheme of successive approximations. The two approximations corresponding to the two first terms of this expansion are explicitly computed. In Sec. IV both approximations and APEX are applied to the specific model described above and compared to the Monte Carlo computations. Different magnitudes of the test charge are considered, as well as the neutral-point case. The results are interpreted on the basis of very general arguments. Our results are summarized in Sec. V.

II. COUPLING-PARAMETER INTEGRATION METHOD AND APEX

A. Formulation of the problem

We consider a system of N moving charges e and one fixed test charge γe ($\gamma > 0$) located at $\vec{r}_0$, embedded in a uniform neutralizing rigid background. The system is contained in a box with volume Λ, at temperature T ($\beta = 1/k_B T$). The total interaction potential V is

$$V = \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{N} v(r_{ij}) + \gamma \sum_{j=1}^{N} v(r_{0j}) + V_B , \tag{2.1}$$

where V_B includes all the interactions involving the background; $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$, and $v(r)$ is the Coulomb interaction potential between two charges e at a distance r,

$$v(r) = -e^2 \ln \left[\frac{r}{L} \right] \tag{2.2}$$

in two dimensions (2D) and

$$v(r) = \frac{e^2}{r} \tag{2.3}$$

in three dimensions (3D) [in (2.2) L is an irrelevant scale length]. The density of plasma particles is $\rho = N/\Lambda$, $\rho = 1/\pi a^2$ in 2D and $\rho = 3/4\pi a^3$ in 3D where a is the mean interparticle distance. The coupling constant Γ characterizes the strength of electrostatic interactions, $\Gamma = \beta e^2$ in 2D and $\Gamma = \beta e^2/a$ in 3D. The system is assumed to be overall neutral.

We compute the electric field distribution $W(\vec{E})$ at the test charge γe in the thermodynamic limit (TL), i.e., when N and Λ go to infinity, with ρ being kept fixed. For notational convenience we will not explicitly specify in the equations that the TL must be taken. Furthermore, we assume that all the quantities appearing in the computations are well defined in this limit.[12] $W(\vec{E})$ is defined by

$$W(\vec{E}) = \frac{\int d\vec{r}_1 \cdots d\vec{r}_N \exp(-\beta V) \delta(\vec{E}_0 - \vec{E})}{\int d\vec{r}_1 \cdots d\vec{r}_N \exp(-\beta V)} , \tag{2.4}$$

where $\vec{E}_0$ is the total electric field created by the charges e and the background at $\vec{r}_0$,

$$\vec{E}_0 = -\frac{1}{\gamma e} \vec{\nabla}_0 V . \tag{2.5}$$

Instead of working directly on $W(\vec{E})$, it is more convenient to introduce the Fourier transform of $W(\vec{E})$

$$A(\vec{k}) = \int d\vec{E} \exp(i\vec{k}\cdot\vec{E}) W(\vec{E}) , \tag{2.6}$$

which can be rewritten using (2.4) as

$$A(\vec{k}) = \frac{\int d\vec{r}_1 \cdots d\vec{r}_N \exp(-\beta V) \exp(i\vec{k}\cdot\vec{E}_0)}{\int d\vec{r}_1 \cdots d\vec{r}_N \exp(-\beta V)} . \tag{2.7}$$

In the TL, $W(\vec{E})$ and $A(\vec{k})$ become isotropic functions. The distribution of the modulus of the electric field is then

$$P(E) = 2\pi(d-1) E^{d-1} W(E) , \tag{2.8}$$

where d is the dimensionality.

B. Coupling-parameter integration method

Using a coupling-parameter integration technique, one of us[10] has shown that $A(k)$ can be formally expressed as

$$A(k) = \exp\left[i\hat{k} \cdot \int_0^k d\lambda \int d\vec{r}_1 \vec{e}_0(\vec{r}_1) [\rho_\lambda(\vec{r}_1) - \rho] \right] , \tag{2.9}$$

where $\vec{e}_0(\vec{r}_1)$ is the electric field created at $\vec{r}_0$ by a charge e located at $\vec{r}_1$, and $\rho_\lambda(\vec{r}_1)$ is the one-body density of the plasma particles (with charge e) when the extra coupling

$$-\frac{i\lambda}{\beta} \hat{k} \cdot \vec{E}_0 \tag{2.10}$$

between the test charge and the plasma particles is added to the interaction potential V. That is,

$$\rho_\lambda(\vec{r}_2) = N \frac{\int d\vec{r}_2 \cdots d\vec{r}_N \exp[-\beta V(\lambda)]}{\int d\vec{r}_1 \cdots d\vec{r}_N \exp[-\beta V(\lambda)]} \tag{2.11}$$

with

$$V(\lambda) = V - \frac{i\lambda}{\beta} \hat{k} \cdot \vec{E}_0 . \tag{2.12}$$

The crucial problem now lies in the computation of $\rho_\lambda(\vec{r}_1)$. Various approximations have been described elsewhere.[10,11] Here we only recall the most accurate one, namely, APEX.

C. APEX

In APEX one makes the *ad hoc* assumption[11]

$$\rho_\lambda(\vec{r}_1) = \rho_0(\vec{r}_1) \exp[i\lambda \hat{k} \cdot \vec{e}_0^{\,*}(\vec{r}_1)] , \tag{2.13}$$

where $\rho_0(\vec{r}_1)$ is the one-body density of the plasma particles at $\vec{r}_1$ for the system with interaction potential $V(0)$, and $\vec{e}_0^{\,*}(\vec{r}_1)$ is a modified Debye-Hückel screened field

$$\vec{e}_0^{\,*}(\vec{r}_1) = -e\alpha K_1(\alpha r_{01}) \hat{r}_{01} \tag{2.14}$$

in 2D and

$$\vec{e}_0^{\,*}(\vec{r}_1) = \frac{-e}{r_{01}^2} (1 + \alpha r_{01}) \exp(-\alpha r_{01}) \hat{r}_{01} \tag{2.15}$$

in 3D (K_1 is the Bessel function defined in Ref. 13). The parameter α is determined from the exact sum rule[11]

$$\langle \vec{E}_0^2 \rangle = \frac{2(d-1)\pi\rho}{\gamma\beta} . \tag{2.16}$$

We can think of this approximation[11] as one in which the plasma particles are replaced by independent quasiparticles with density $\rho_0(\vec{r}_1) e_0(\vec{r}_1) / e_0^*(\vec{r}_1)$ and screened fields (2.14) in 2D and (2.15) in 3D. The distribution of the Coulomb field due to the particles is then identical to the Holtzmark distribution[2] of the screened field created by the quasiparticles.

A priori, APEX is defined only for γ strictly positive, $\langle \vec{E}_0^2 \rangle$ becoming infinite when $\gamma = 0$. However, we can take the limit $\gamma \to 0$ and apply this approximation to the interesting case $\gamma = 0$ by proceeding as follows. Equation (2.16) is equivalent to

$$A(k) - 1 \sim \frac{-(d-1)\pi\rho}{d\gamma\beta} k^2 \tag{2.17}$$

when $k \to 0$. Thus α is the solution of

$$\int d\vec{r}_1 [\hat{k}\cdot\vec{e}_0(\vec{r}_1)][\hat{k}\cdot\vec{e}_0^{\,*}(\vec{r}_1)]\rho_0(\vec{r}_1) = \frac{2(d-1)\pi\rho}{d\gamma\beta} . \tag{2.18}$$

By expanding the left-hand side (lhs) of (2.18) in Laurent's series with respect to γ, we show in Appendix A that α goes to a finite value when $\gamma \to 0$, namely,

$$\frac{2}{a}\exp\left[-\frac{2\beta U_{\rm exc}(\Gamma)}{\Gamma} - C\right] \tag{2.19}$$

in 2D and

$$-\frac{2}{a}\frac{\beta U_{\rm exc}(\Gamma)}{\Gamma} \tag{2.20}$$

in 3D. $U_{\rm exc}(\Gamma)$ is the excess internal energy per particle of the one-component plasma (OCP) (without any test charge) and C is Euler's constant ($C \simeq 0.577\,22$). Thus APEX gives a finite expression for $A(k)$ when $\gamma \to 0$,

$$A_{\rm APEX}(k) = \exp\left[i\rho\hat{k}\cdot \int_0^k d\lambda \int d\vec{r}_1 \vec{e}_0(\vec{r}_1)\{\exp[i\lambda\hat{k}\cdot\vec{e}_0^{\,*}(\vec{r}_1)] - 1\}\right] , \tag{2.21}$$

where we have replaced $\rho_0(\vec{r}_1)$ by its limit ρ when $\gamma \to 0$.

III. NEW SYSTEMATIC EXPANSION OF THE FIELD DISTRIBUTION

A. Systematic expansion

Equation (2.11) can be rewritten as

$$\rho_\lambda(\vec{r}_1) = \rho_0(\vec{r}_1)\frac{\langle \exp(i\lambda\hat{k}\cdot\vec{E}_0)\rangle_1}{\langle \exp(i\lambda\hat{k}\cdot\vec{E}_0)\rangle} , \tag{3.1}$$

where the statistical averages $\langle f \rangle$ and $\langle f \rangle_1$ of any function f of $\vec{r}_0, \vec{r}_1, \ldots, \vec{r}_N$ are

$$\langle f \rangle = \frac{1}{Q_N(\vec{r}_0)} \int d\vec{r}_1 \cdots d\vec{r}_N \exp(-\beta V) f \tag{3.2}$$

with

$$Q_N(\vec{r}_0) = \int d\vec{r}_1 \cdots d\vec{r}_N \exp(-\beta V) , \tag{3.3}$$

and

$$\langle f \rangle_1 = \frac{1}{Q_N(\vec{r}_0, \vec{r}_1)} \int d\vec{r}_2 \cdots d\vec{r}_N \exp(-\beta V) f \tag{3.4}$$

with

$$Q_N(\vec{r}_0, \vec{r}_1) = \int d\vec{r}_2 \cdots d\vec{r}_N \exp(-\beta V) . \tag{3.5}$$

In $\langle f \rangle$, the average is performed upon all the plasma particles, whereas in $\langle f \rangle_1$, particle 1 is fixed. We stress that these averages are done in the external potential of the charge γe.

The statistical averages appearing in (3.1) can be replaced by their cumulant expansions

$$\langle \exp(i\lambda\hat{k}\cdot\vec{E}_0)\rangle = \exp\{-\tfrac{1}{2}\lambda^2[\langle(\hat{k}\cdot\vec{E}_0)^2\rangle - \langle\hat{k}\cdot\vec{E}_0\rangle^2] + \cdots\} \tag{3.6}$$

and

$$\langle \exp(i\lambda\hat{k}\cdot\vec{E}_0)\rangle_1 = \exp\{i\lambda\hat{k}\cdot\langle\vec{E}_0\rangle_1 - \tfrac{1}{2}\lambda^2[\langle(\hat{k}\cdot\vec{E}_0)^2\rangle_1 - \langle\hat{k}\cdot\vec{E}_0\rangle_1^2] + \cdots\} \tag{3.7}$$

(for symmetry reasons $\langle\vec{E}_0\rangle = 0$). Assuming the *differences* between the fluctuations $\langle\ \rangle_1$ and $\langle\ \rangle$ to be small, we then expand the exponentials in (3.1) involving such differences in a power series with respect to λ. That gives

$$\rho_\lambda(\vec{r}_1) = \rho_0(\vec{r}_1)\exp(i\lambda\hat{k}\cdot\langle\vec{E}_0\rangle_1) \times\{1 - \tfrac{1}{2}\lambda^2[\langle(\hat{k}\cdot\vec{E}_0)^2\rangle_1 - \langle\hat{k}\cdot\vec{E}_0\rangle_1^2 + \langle\hat{k}\cdot\vec{E}_0\rangle^2 - \langle(\hat{k}\cdot\vec{E}_0)^2\rangle] + \cdots\} . \tag{3.8}$$

Inserting (3.8) in (2.9) we obtain the following expansion of $A(k)$:

$$A(k) = \exp[F_0(k) + F_2(k) + \cdots] \tag{3.9}$$

with

$$F_0(k) = i\hat{k}\cdot \int_0^k d\lambda \int d\vec{r}_1 \vec{e}_0(\vec{r}_1)[\rho_0(\vec{r}_1)\exp(i\lambda\hat{k}\cdot\langle\vec{E}_0\rangle_1) - \rho] , \tag{3.10}$$

$$F_2(k) = -\frac{i}{2}\hat{k}\cdot \int_0^k d\lambda\,\lambda^2 \int d\vec{r}_1 \vec{e}_0(\vec{r}_1)\rho_0(\vec{r}_1)\exp(i\lambda\hat{k}\cdot\langle\vec{E}_0\rangle_1)[\langle(\hat{k}\cdot\vec{E}_0)^2\rangle_1 - \langle\hat{k}\cdot\vec{E}_0\rangle_1^2 + \langle\hat{k}\cdot\vec{E}_0\rangle^2 - \langle(\hat{k}\cdot\vec{E}_0)^2\rangle] , \tag{3.11}$$

etc.

In 3D all the functions $F_n(k)$ are well defined. In 2D only the ones with $n < 2 + \gamma\Gamma$ are finite, the other ones being divergent as are the exact moments $\langle E_0^n \rangle$ for $n \geq 2 + \gamma\Gamma$. This is due to the fact that when a plasma particle j approaches the test charge, the Boltzmann factor vanishes in 2D as $\text{const}\times r_{0j}^{\gamma\Gamma}$ instead of $\text{const}\times\exp(-\gamma\beta e^2/r_{0j})$ in 3D. Thus, the expansion (3.9) is really "systematic" only in 3D, in the sense that it provides a formal series representation of $\ln A(k)$ (the problem of the convergence of these series is far beyond the scope of the present paper). In 2D there is a singular part in $\ln A(k)$ which cannot be expressed in terms of the functions $F_n(k)$; this singular part is not controlled by the expansion (3.9).

We turn now to the approximations based on the truncation of (3.9). The general conditions under which these approximations are accurate will be discussed at the end of Sec. IV.

B. Mean-force-field approximation and its corrections

The simplest approximation based on the expansion (3.9) is to keep only $F_0(k)$. This lowest-order approximation has a form and a physical interpretation very similar to APEX. However, the field induced by the quasiparticles is now $\langle\vec{E}_0\rangle_1$ instead of $\vec{\epsilon}_0^{\,*}$ in APEX. $\langle\vec{E}_0\rangle_1$ can be expressed in terms of the one-body density of the plasma particles as follows. Using (2.5) in the definition of $\langle\vec{E}_0\rangle_1$, we have

$$\langle\vec{E}_0\rangle_1=\frac{1}{\gamma\beta e Q_N(\vec{r}_0,\vec{r}_1)}\vec{\nabla}_0 Q_N(\vec{r}_0,\vec{r}_1)\ . \tag{3.12}$$

Writing

$$Q_N(\vec{r}_0,\vec{r}_1)=\frac{1}{N}Q_N(\vec{r}_0)\rho_0(\vec{r}_1)\ , \tag{3.13}$$

and noting that $Q_N(\vec{r}_0)$ becomes independent of $\vec{r}_0$ in the TL, we obtain

$$\langle\vec{E}_0\rangle_1=\frac{1}{\gamma\beta e}\vec{\nabla}_0\ln[\rho_0(\vec{r}_1)/\rho]\ . \tag{3.14}$$

Introducing the familiar mean-force (MF) potential

$$V_{\rm MF}(r_{01})=-\frac{1}{\beta}\ln[\rho_0(\vec{r}_1)/\rho]\ , \tag{3.15}$$

we see that $\langle\vec{E}_0\rangle_1$ is nothing but the mean-force field

$$\vec{E}_{\rm MF}(\vec{r}_{01})=-\frac{1}{\gamma e}\vec{\nabla}_0 V_{\rm MF}(r_{01})\ . \tag{3.16}$$

Let us call this approximation the mean-force-field approximation (MFFA). We point out that MFFA, like APEX, gives correctly the second moment (2.16) of the field distribution. Indeed, as shown by Eqs. (3.10) and (3.11), the k^2 term of the small-k expansion of $A(k)$ only comes from $F_0(k)$. By an explicit calculation using (3.14), it can be easily checked that $F_0(k)$ behaves as (2.17) when $k\to 0$.

Keeping $F_0(k)$ and $F_2(k)$ in (3.9), we get a new approximation which we call the improved-mean-force-field approximation (IMFF). The fluctuation terms appearing in $F_2(k)$ can be expressed in terms of the distribution functions of the plasma particles, in the same way as we have rewritten $\langle\vec{E}_0\rangle_1$ as (3.14). After some algebra we find

$$\begin{aligned}\langle(\hat{k}\cdot\vec{E}_0)^2\rangle_1-\langle\hat{k}\cdot\vec{E}_0\rangle_1^2+\langle\hat{k}\cdot\vec{E}_0\rangle^2-\langle(\hat{k}\cdot\vec{E}_0)^2\rangle\\ =-\frac{1}{\gamma^2\beta e^2}\frac{\partial^2 V_{\rm MF}}{\partial x_{01}^2}(r_{01})+\frac{1}{\gamma\beta e^2}\frac{\partial^2 v}{\partial x_{01}^2}(r_{01})\\ +\frac{1}{\gamma\beta e^2\rho_0(\vec{r}_1)}\int d\vec{r}_2\,\rho_0^{(2,T)}(\vec{r}_1,\vec{r}_2)\frac{\partial^2 v}{\partial x_{02}^2}(r_{02})\ ,\end{aligned} \tag{3.17}$$

where x is the component of $\vec{r}$ along $\hat{k}$ and $\rho_0^{(2,T)}(\vec{r}_1,\vec{r}_2)$ is the truncated two-body distribution function of the plasma particles

$$\rho_0^{(2,T)}(\vec{r}_1,\vec{r}_2)=\rho_0^{(2)}(\vec{r}_1,\vec{r}_2)-\rho_0(\vec{r}_1)\rho_0(\vec{r}_2)\ . \tag{3.18}$$

Unlike the MFFA, which involves only correlations between the test charge and one plasma particle, IMFF takes into account three-body correlations involving the test charge and two plasma particles. Of course IMFF gives exactly the second moment of the field distribution like MFFA.

IV. OCP IN 2D AT $\Gamma=2$

We apply the MFFA, IMFF, and APEX to the OCP in 2D at $\Gamma=2$ with a test charge which is an integer multiple of the charges of the plasma. In this case all the distribution functions of the plasma particles can be computed analytically. The corresponding approximate field distributions then take a closed analytic form. These are compared to the Monte Carlo results. We restrict ourselves to the cases $\gamma=1$, 2, and 4. The limiting case $\gamma\to 0$ (neutral point) is also considered.

A. Analytic computations

For simplicity of notation we place the test charge at the origin and put $\vec{r}_0=0$ in the following. By extending the computations of Jancovici[14] (which correspond to the case $\gamma=1$), it can be easily shown that for γ, a nonzero integer,

$$\rho_0(r_1)=\rho\left[1-\exp(-r_1^2)\sum_{p=0}^{\gamma-1}\frac{r_1^{2p}}{p!}\right] \tag{4.1}$$

and

$$\rho_0^{(2,T)}(\vec{r}_1,\vec{r}_2)=-\rho^2\exp(-r_1^2-r_2^2)\times\left|\exp(z_1z_2^*)-\sum_{p=0}^{\gamma-1}\frac{(z_1z_2^*)^p}{p!}\right|^2\ . \tag{4.2}$$

Here the distances are in units of the mean interparticle distance $r_j=r_j/a$ and $z_j=r_j\exp(i\theta_j)$, θ_j being the angle between $\vec{r}_j$ and a given reference direction. Choosing this direction along $\hat{k}$, we get from (3.14) and (4.1)

$$\hat{k}\cdot\vec{E}_{\rm MF}(\vec{r}_1)=\frac{e}{a}R(r_1)\cos\theta_1 \tag{4.3}$$

with

$$R(r_1)=-\frac{r_1\exp(-r_1^2)}{1-\exp(-r_1^2)} \tag{4.4a}$$

for $\gamma=1$,

$$R(r_1)=-\frac{r_1^3\exp(-r_1^2)}{2[1-\exp(-r_1^2)-r_1^2\exp(-r_1^2)]} \tag{4.4b}$$

for $\gamma=2$, and

$$R(r_1)=-\frac{r_1^7\exp(-r_1^2)}{24[1-\exp(-r_1^2)(1+r_1^2+\frac{1}{2}r_1^4+\frac{1}{6}r_1^6)]} \tag{4.4c}$$

for $\gamma=4$. When $r_1\to 0$, $R(r_1)$ behaves as

$$R(r_1)\sim -\frac{1}{r_1} \tag{4.5}$$

which means that $\vec{E}_{MF}(\vec{r}_1)$ reduces to the Coulomb field. When $r_1\to\infty$, $R(r_1)$ decays like a power times a Gaussian which means that $\vec{E}_{MF}(\vec{r}_1)$ is screened at large distances.

Using (4.1) and (4.2), the fluctuation term (3.17) is computed as

$$-\frac{e^2}{a^2}[S(r_1)+T(r_1)\cos^2\theta_1] \tag{4.6}$$

with

$$S(r_1)=\frac{r_1^2\exp(-r_1^2)}{4[1-\exp(-r_1^2)]}\ ,$$

$$T(r_1)=\frac{\exp(-r_1^2)\{r_1^2[1+\exp(-r_1^2)]+2[\exp(r_1^2)-1]\}}{2[1-\exp(-r_1^2)]^2} \tag{4.7a}$$

for $\gamma=1$,

$$S(r_1)=\frac{r_1^4\exp(-r_1^2)}{24[1-\exp(-r_1^2)-r_1^2\exp(-r_1^2)]}\ ,$$

$$T(r_1)=\frac{r_1^2\exp(-r_1^2)\{r_1^4\exp(-r_1^2)+2r_1^2[1+2\exp(-r_1^2)]+6[\exp(-r_1^2)-1]\}}{12[1-\exp(-r_1^2)-r_1^2\exp(-r_1^2)]^2} \tag{4.7b}$$

for $\gamma=2$, and

$$S(r_1)=\frac{r_1^8\exp(-r_1^2)}{960[1-\exp(-r_1^2)(1+r_1^2+\frac{1}{2}r_1^4+\frac{1}{6}r_1^6)]}\ ,$$

$$T(r_1)=\frac{r_1^6\exp(-r_1^2)[\exp(-r_1^2)(r_1^8+8r_1^6+36r_1^4+96r_1^2+120)+24r_1^2-120]}{2880\{1-\exp(-r_1^2)[1+r_1^2+\frac{1}{2}r_1^4+\frac{1}{6}r_1^6]\}^2} \tag{4.7c}$$

for $\gamma=4$. In the calculation we have first transformed the integral over $\vec{r}_2$ appearing in the right-hand side (rhs) of (3.17) as

$$\int d\vec{r}_2\rho_0^{(2,T)}(\vec{r}_1,\vec{r}_2)\frac{\partial^2 v}{\partial x_2^2}(r_2) = -\int d\vec{r}_2\frac{\partial\rho_0^{(2,T)}}{\partial x_2}(\vec{r}_1,\vec{r}_2)\frac{\partial v}{\partial x_2}(r_2)\ . \tag{4.8}$$

Expanding the integrand in power series with respect to $\exp(i\theta_2)$ and $\exp(-i\theta_2)$, the angular integral is then computed by using the orthogonality properties of the functions $\exp(in\theta_2)$. The remaining radial integral then involves a polynomial times a Gaussian and is easily performed. Contrary to $R(r_1)$, $S(r_1)$ and $T(r_1)$ remain finite when r_1 goes to zero. At large distances $S(r_1)$ and $T(r_1)$ have a similar behavior as $R(r_1)$ and decay like a power times a Gaussian.

Henceforth, E is redefined as the field in units of e/a, $E\equiv aE/e$, and k is also redefined as $k\equiv ke/a$. Using (4.3) and (4.6), $F_0(k)$ and $F_2(k)$ are rewritten as

$$F_0(k)=-\frac{i}{\pi}\int_0^\infty dr_1\,g_0(r_1)\int_0^k dt\int_0^{2\pi}d\theta_1\exp[itR(r_1)\cos\theta_1]\cos\theta_1 \tag{4.9}$$

and

$$F_2(k)=-\frac{i}{2\pi}\int_0^\infty dr_1\,g_0(r_1)\int_0^k dt\,t^2\int_0^{2\pi}d\theta_1[S(r_1)+T(r_1)\cos^2\theta_1]\exp[itR(r_1)\cos\theta_1]\cos\theta_1\ , \tag{4.10}$$

with

$$g_0(r_1)=\rho_0(r_1)/\rho\ . \tag{4.11}$$

The angular integrals involved in (4.9) and (4.10) lead to the Bessel functions J_n. Using functional relations between the J_n's, the integrals over the parameter t are then performed in terms of Bessel functions of the argument $kR(r_1)$. We find

$$F_0(k)=-2\int_0^\infty dr_1\frac{g_0(r_1)}{R(r_1)}[J_0(kR(r_1))-1] \tag{4.12}$$

and

$$F_2(k)=\int_0^\infty dr_1 \frac{g_0(r_1)}{R(r_1)}\left[\frac{kS(r_1)}{R(r_1)}[2J_1(kR(r_1))-kR(r_1)J_0(kR(r_1))] + \frac{T(R_1)}{R^2(r_1)}[2J_0(kR(r_1))-2+3kR(r_1)J_1(kR(r_1))-k^2R^2(r_1)J_0(kR(r_1))]\right] . \quad (4.13)$$

The Fourier transforms of the field distributions corresponding to MFFA and IMFF are then

$$A_{\text{MFFA}}(k)=\exp[F_0(k)] \quad (4.14)$$

and

$$A_{\text{IMFF}}(k)=\exp[F_0(k)+F_2(k)] , \quad (4.15)$$

where $F_0(k)$ and $F_2(k)$ are respectively given by (4.12) and (4.13).

We come now to APEX. For the case considered here, $A_{\text{APEX}}(k)$ takes the form

$$A_{\text{APEX}}(k)=\exp\left[2\int_0^\infty dr_1 \frac{g_0(r_1)}{\alpha K_1(\alpha r_1)} \times [J_0(k\alpha K_1(\alpha r_1))-1]\right] \quad (4.16)$$

which is very similar to $A_{\text{MFFA}}(k)$ with $-\alpha K_1(\alpha r_1)$ in place of $R(r_1)$ (α is redefined as α in units of $1/a$). Equation (2.18) determining the parameter α can be rewritten after some integration by parts as

$$\exp(\alpha^2/4)\text{Ei}(-\alpha^2/4)=-1 \quad (4.17\text{a})$$

for $\gamma=1$,

$$(1+\alpha^2/4)\exp(\alpha^2/4)\text{Ei}(-\alpha^2/4)=-\tfrac{3}{2} \quad (4.17\text{b})$$

for $\gamma=2$, and

$$(1+3\alpha^2/4+3\alpha^4/32+\alpha^6/384)\exp(\alpha^2/4)\text{Ei}(-\alpha^2/4) =-25/12-\alpha^2/3-\alpha^4/96 \quad (4.17\text{c})$$

for $\gamma=4$, where Ei is the exponential integral function defined in Ref. 13. By Newton's method we find

$$\begin{cases} 1.320 \text{ for } \gamma=1 & (4.18\text{a}) \\ 1.214 \text{ for } \gamma=2 & (4.18\text{b}) \\ 1.086 \text{ for } \gamma=4 . & (4.18\text{c}) \end{cases}$$

As shown in Sec. II, in the limiting case $\gamma\to0$ APEX gives a finite expression (2.21) for $A(k)$ which reads here

$$A_{\text{APEX}}(k)=\exp\left[2\int_0^\infty dr_1 \frac{[J_0(k\alpha K_1(\alpha r_1))-1]}{\alpha K_1(\alpha r_1)}\right] \quad (4.19)$$

with α computed from (2.19) as $\alpha=1.499$. When $\gamma\to0$, the mean-force potential can be computed from the linear response theory with the result

$$\beta V_{\text{MF}}(r_1)=-\gamma\text{Ei}(-r_1^2)+O(\gamma^2) . \quad (4.20)$$

Thus, the mean-force field given by (3.16) goes to a finite expression in this limit, and we can replace $g_0(r_1)$ by 1 in the computation of $F_0(k)$. This gives

$$A_{\text{MFFA}}(k)=\exp\left\{2\int_0^\infty dr_1\, r_1\exp(r_1^2) \times\left[J_0\left[\frac{k\exp(-r_1^2)}{r_1}\right]-1\right]\right\} . \quad (4.21)$$

Unlike $F_0(k)$, $F_2(k)$ diverges when $\gamma\to0$, in agreement with the discussion of Sec. II. Therefore IMFF is not applicable to the neutral-point case.

B. Numerical computations

The different integrals involved in the various approximations were numerically computed by the trapezoidal rule. The various electric field distributions, given by

$$P(E)=E\int_0^\infty dk\, kJ_0(kE)A(k) , \quad (4.22)$$

are compared to the exact Monte Carlo results in Figs. 1–4. APEX is accurate for both low and intermediate fields in the domain of studied γ; in particular the height and the position of the maximum of the field distribution are well reproduced by APEX. MFFA becomes more and more accurate as γ increases. For instance, at $\gamma=4$, the latter agrees quite well with the Monte Carlo results. For

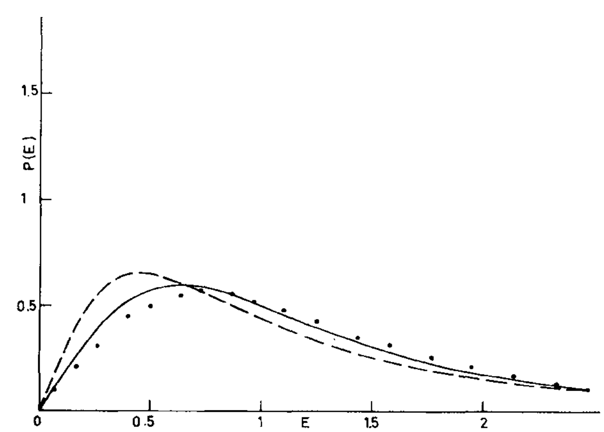

FIG. 1. Probability distribution $P(E)$ of the modulus E of the electric field at a test charge e and at $\Gamma=2$; E is expressed in units of e/a. Here, $\gamma=0$ (neutral-point case). Dashed line, MFFA; solid line, APEX; dots, Monte Carlo. IMFF is not applicable to the present case.

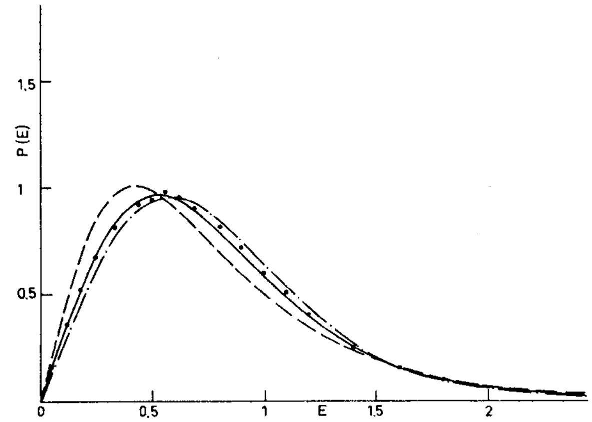

FIG. 2. The same as Fig. 1 at $\gamma=1$. Dashed-dotted line, IMFF.

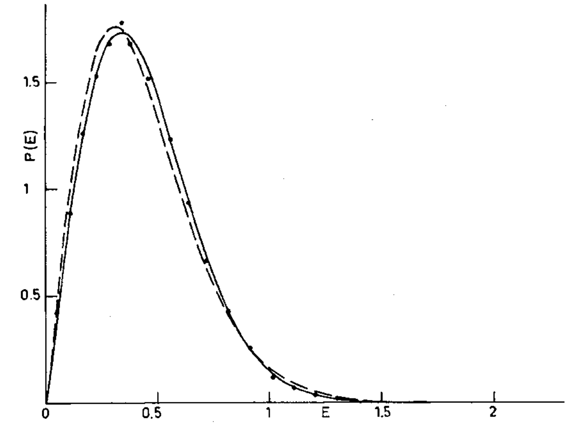

FIG. 4. The same as Fig. 2 at $\gamma=4$. The curve corresponding to IMFF almost coincides with the one corresponding to APEX and has not been drawn for clarity of the picture.

lower values of γ, the maximum of P_{MFFA} is too high and shifted to low fields. At $\gamma=1$, MFFA is still a reasonable approximation whereas its accuracy becomes rather poor at $\gamma=0$, i.e., for the neutral point. Except the case $\gamma=0$ where it is not applicable, IMFF appears to be a good approximation and improves MFFA as expected. At $\gamma=2$ or 4, IMFF differs slightly from MFFA and has an accuracy comparable to APEX. At $\gamma=1$, IMFF improves MFFA more appreciably and is slightly less accurate than APEX.

It appears from the numerical computations that for large fields the approximate distributions become identical. The large field asymptotic behavior is related to the singularities of $A(k)$ at the origin. In Appendix B we show that the small-k expansions of all the approximate $A(k)$ have the same singularity (with the same constant factors), namely,

$$\ln A(k)-\sum_{n=1}^{p} c_n k^{2n} \sim \mathrm{const}\times k^{2+\gamma\Gamma} \tag{4.23}$$

for $\gamma\Gamma$ noneven integer and

$$\ln A(k)-\sum_{n=1}^{p} c_n k^{2n} \sim \mathrm{const}\times k^{2+\gamma\Gamma}\ln k \tag{4.24}$$

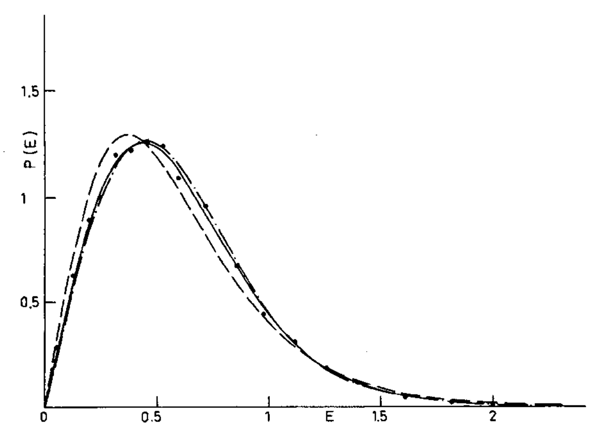

FIG. 3. The same as Fig. 2 at $\gamma=2$.

for $\gamma\Gamma$ even integer, where the coefficients c_n and the integer p are defined in Appendix B. Furthermore, on the basis of the systematic expansion (3.9), we give in this appendix plausible (but not completely rigorous) arguments indicating that the small-k expansion of $A_{\mathrm{exact}}(k)$ has also the singularity (4.23) or (4.24). Thus, the approximate field distributions should become equivalent to the exact one when $E\to\infty$. The exact behavior of $P(E)$ for small or intermediate E is, of course, much more difficult to describe.

C. General interpretation of the results

The improvement of MFFA when γ increases can be "explained" by essentially two general arguments. The first one is related to the singularity of the expansion (3.9) in 2D. For large $\gamma\Gamma$, this singularity arises in the high-order terms and does not play an essential role. For small $\gamma\Gamma$, the first corrections to MFFA diverge and it is not surprising that MFFA is then in poor agreement with the Monte Carlo results (in particular for the neutral point). The second argument is related to the hard-core effect induced by a large test charge. The Boltzmann factor behaves as const$\times r_j^{\gamma\Gamma}$ when $r_j\to 0$; thus the plasma particles are strongly repelled by the test charge when $\gamma\Gamma$ is large. The field distribution is shifted to low fields and the dominant contributions come from far away particles, namely, from large distances r_1 in (2.9). Since the differences between the fluctuation terms $\langle\ \rangle_1$ and $\langle\ \rangle$ are small for these large distances, they can be neglected in the computation of $A(k)$ as assumed by MFFA. We are not able to make this heuristic argument more rigorous, namely, we cannot show that the terms neglected in MFFA go to zero and become negligible compared to $F_0(k)$ when $\gamma\Gamma\to\infty$.

Compared to $P_{\mathrm{MFFA}}(E)$, the maximum of $P_{\mathrm{IMFF}}(E)$ is smaller and shifted to higher fields. This result should not depend on Γ and γ as suggested by the following qualitative argument. Looking at the expansion (3.8) of $\rho_\lambda(\vec{r}_1)$, we see that the density $\rho_0(r_1)$ involved in MFFA is replaced by

$$\rho_0(r_1)\{1-\tfrac{1}{2}\lambda^2[\langle(\hat{k}\cdot\vec{E}_0)^2\rangle_1-\langle\hat{k}\cdot\vec{E}_0\rangle_1^2+\langle\hat{k}\cdot\vec{E}_0\rangle^2-\langle(\hat{k}\cdot\vec{E}_0)^2\rangle]\} \tag{4.25}$$

in IMFF. The Coulomb field $-e\vec{r}_1/r_1^2$ created by particle 1 plays the role of a constant in the statistical averages $\langle\ \rangle_1$ and thus does not contribute to the fluctuation

$$\langle(\hat{k}\cdot\vec{E}_0)^2\rangle_1-\langle\hat{k}\cdot\vec{E}_0\rangle_1^2 . \tag{4.26}$$

Therefore, at $\vec{r}_1=0$, (4.26) is equal to the corresponding fluctuation computed with the statistical averages $\langle\ \rangle$ and a test charge $(\gamma+1)e$, i.e., to $1/(\gamma+1)\Gamma$ in our dimensionless units. The difference between the fluctuations $\langle\ \rangle_1$ and $\langle\ \rangle$ is then equal to $-1/\gamma(\gamma+1)\Gamma$ at $\vec{r}_1=0$. By continuity, for values of r_1 that are not too large, this difference is negative and the λ^2 term in (4.25) is positive. Thus IMFF enhances the nearest-neighbor contributions to the field distribution, and consequently $P_{\rm IMFF}(E)$ is shifted to higher fields compared to $P_{\rm MFFA}(E)$.

All the arguments described above hold also for 3D. However, the limiting case $\gamma\Gamma\to 0$ requires the following complementary remarks. In 3D, all the terms of (3.9) are finite for $\gamma\Gamma$ finite, but only $F_0(k)$ remains finite when $\gamma\to 0$. Therefore, MFFA should again become a rather crude approximation when $\gamma\to 0$, as in 2D. When $\Gamma\to 0$ (γ fixed), the 3D MFFA distribution goes to the Holtzmark distribution[2] corresponding to free particles. In the same limit the 2D MFFA distribution involves divergent terms, in agreement with the nonexistence of the field distribution of free particles in 2D.

V. SUMMARY AND CONCLUSION

We have derived a new systematic expansion for computing the electric field distribution at a test charge (with nonzero arbitrary magnitude) immersed in an infinite OCP. The lowest-order truncation of this expansion leads to a mean-field theory (MFFA) very similar to APEX. Both approximations require only the knowledge of the two-body correlations between the test charge and one plasma particle. The approximation (IMFF) taking into account the next corrections to MFFA requires much more information; it involves three-body correlations between the test charge and two plasma particles.

The approximate theories, MFFA, IMFF, and APEX have been applied to the 2D OCP at $\Gamma=2$ for various test charges. The practical computations corresponding to this specific model and the qualitative arguments given in Sec. IV strongly suggest that MFFA and IMFF have the following characteristics. First, MFFA is very accurate when $\gamma\Gamma$ is large, i.e., for large test charges or strongly coupled plasmas. (We have checked in 3D and for $\gamma=1$ that MFFA is in reasonable agreement with computer simulations but, as in 2D, they are not as good as the APEX results.) Then, IMFF slightly improves MFFA and can be regarded as a small correction to MFFA as assumed *a priori*. Furthermore, for intermediate values of γ and Γ, the whole MFFA distribution is shifted to small fields compared to the exact one. Although MFFA is not itself very accurate in this case, IMFF, which only takes into account the next corrections to MFFA, is still a good approximation and crucially improves MFFA. Finally, for small γ or $\gamma=0$ (neutral point), MFFA becomes a crude approximation and IMFF does not satisfactorily improve the latter. In the neutral-point case, IMFF is not even applicable since it involves divergent terms (our expansion is clearly not appropriate to this case).

Previous computations[11] in 3D have already illustrated the high accuracy of APEX for $\gamma=1$, especially in the strong-coupling regime. The present comparisons with Monte Carlo data in 2D show that APEX remains accurate when the test charge is not equal to the charge of the plasma particles, especially in the practical interesting case of the neutral point. This confirms APEX as a very good approximation. For intermediate or small values of γ and Γ, APEX is better than MFFA and then appears to be more reliable.

The systematic expansion (3.9) can be easily extended to multicomponent systems, as well as MFFA and IMFF. In this multicomponent case, MFFA will still include the exact second moment of the field distribution and the screening effects, which appear from our study and previous computations[11] as two essential physical ingredients. IMFF should again improve MFFA but will be rather difficult to handle. The corresponding generalization for APEX has been studied recently by Iglesias *et al*.:[15] the generalized version of APEX is still very accurate.

ACKNOWLEDGMENT

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APPENDIX A

In this appendix we show that the adjustable parameter α involved in APEX goes to a finite value when the test charge vanishes. First we consider the 2D case. Setting $x\equiv r_{01}/a$ and $g(x)\equiv\rho_0(\vec{r}_1)/\rho$, (2.18) becomes, after an integration by parts,

$$\int_0^\infty dx\, g'(x)K_0(\alpha x)=\frac{1}{\gamma\Gamma} \tag{A1}$$

(the prime means $\partial/\partial x$). Assuming that $g(x)$ is analytic with respect to γ, we may write

$$g(x)=\exp[\gamma\Gamma\ln x+\gamma G_1(x)]\left[1+\sum_{n=2}^{\infty}\gamma^n G_n(x)\right], \tag{A2}$$

where $G_n(x)$ are regular functions at the origin ($x=0$). Assuming strong clustering properties of $g(x)$, $G_1(x)+\Gamma\ln x$ and $G_n(x)$ for $n\geq 2$ decay faster than any inverse power of x when $x\to\infty$. Calling $\Psi(x)$ the term in large parentheses in (A2), the lhs of (A1) can be rewritten as

$$\gamma\int_0^\infty dx\left[\frac{\Gamma}{x}+G_1'(x)\right]\exp[\gamma\Gamma\ln x+\gamma G_1(x)]\Psi(x)K_0(\alpha x)+\int_0^\infty dx\exp[\gamma\Gamma\ln x+\gamma G_1(x)]\Psi'(x)K_0(\alpha x) . \tag{A3}$$

Let us assume *a priori* that α goes to a finite value $\alpha_0>0$ when $\gamma\to 0$. Since the integrals

$$\int_0^\infty dx\, G_n'(x)K_0(\alpha x) \tag{A4}$$

are finite, the second term of (A3) is of order γ^2. In the first term of (A3), we can replace $\Psi(x)$ by 1 neglecting terms of order γ. Thus we have

$$\int_0^\infty dx\, g'(x)K_0(\alpha x)=\gamma \int_0^\infty dx \left[\frac{\Gamma}{x}+G_1'(x)\right] \times\exp[\gamma\Gamma \ln x+\gamma G_1(x)] \times K_0(\alpha x)+O(\gamma)\ . \tag{A5}$$

Splitting the integral $\int_0^\infty dx \cdots$ of the rhs of (A5) into $\int_0^1 dx \cdots + \int_1^\infty dx \cdots$, we see that $\int_1^\infty dx \cdots$ is of order γ. In $\int_0^1 dx \cdots$, we make the variable change $u=x^\gamma$ and obtain

$$\int_0^\infty dx\, g'(x)K_0(\alpha x)= \int_0^1 du \left[\frac{\Gamma}{u}+\frac{G_1'(u^{1/\gamma})}{u^{1-1/\gamma}}\right] \times\exp[\Gamma \ln u+\gamma G_1(u^{1/\gamma})] \times K_0(\alpha u^{1/\gamma})+O(\gamma)\ . \tag{A6}$$

When $\gamma\to 0$, $u^{1/\gamma}\to 0$ for any $u<1$, thus we can replace in (A6) G_1, G_1', and K_0 by their small-argument expansions

$$\begin{aligned} G_1(u^{1/\gamma})&=G_1(0)+G_1'(0)u^{1/\gamma}+\cdots\ ,\\ G_1'(u^{1/\gamma})&=G_1'(0)+G_1''(0)u^{1/\gamma}+\cdots\ ,\\ K_0(\alpha u^{1/\gamma})&=-\ln(\alpha u^{1/\gamma}/2)-C+\cdots \end{aligned} \tag{A7}$$

(C is Euler's constant). That gives

$$\int_0^\infty dx\, g'(x)K_0(\alpha x)=\frac{1}{\gamma\Gamma}+\frac{G_1(0)}{\Gamma}-C-\ln(\alpha/2)+O(\gamma)\ . \tag{A8}$$

The leading term of (A8) is just equal to the required value (A1). The constant term of (A8) must then vanish; that implies

$$\alpha_0=2\exp\left[\frac{G_1(0)}{\Gamma}-C\right]\ . \tag{A9}$$

(The *a priori* assumption about α_0 is justified.) Using the linear response theory, it is easy to show that

$$G_1(0)=-2\beta U_{\text{exc}}(\Gamma)\ , \tag{A10}$$

where $U_{\text{exc}}(\Gamma)$ is the excess internal energy per particle of the homogeneous OCP without any test charge.

In 3D, with the same notations as in 2D, (2.18) becomes

$$\int_0^\infty dx\, g'(x)\frac{\exp(-\alpha x)}{x}=\frac{1}{\gamma\Gamma}\ . \tag{A11}$$

The equivalent form of (A2) here is

$$g(x)=\exp\left[-\frac{\gamma\Gamma}{x}+\gamma G_1(x)\right]\left[1+\sum_{n=2}^{\infty}\gamma^n G_n(x)\right]\ , \tag{A12}$$

where the functions $G_n(x)$ have similar properties to those introduced in the 2D case. Again, we assume *a priori* that α goes to a finite $\alpha_0>0$ when $\gamma\to 0$. Defining $\Psi(x)$ as before, we get

$$\int_0^\infty dx\, g'(x)\frac{\exp(-\alpha x)}{x} = \int_0^\infty dx\frac{\Psi(x)}{x}\left[\frac{\gamma\Gamma}{x^2}+\gamma G_1'(x)\right]\exp\left[-\frac{\gamma\Gamma}{x}+\gamma G_1(x)-\alpha x\right]+\int_0^\infty dx\frac{\Psi'(x)}{x}\exp\left[-\frac{\gamma\Gamma}{x}+\gamma G_1(x)-\alpha x\right]\ . \tag{A13}$$

Breaking up the integrals involved in the rhs of (A13) into $\int_0^1 dx \cdots + \int_1^\infty dx \cdots$, we see that the two integrals $\int_1^\infty dx \cdots$ are of order γ and γ^2, respectively. In $\int_0^1 dx \cdots$, $\Psi(x)$, $\Psi'(x)$, $G_1'(x)$, and $\varphi_1(x)\equiv\exp[\gamma G_1(x)]$, are replaced by their Taylor expansion with respect to x. We find

$$\begin{aligned}\int_0^\infty dx\, g'(x)\frac{\exp(-\alpha x)}{x}&=\gamma\Gamma\Psi(0)\varphi_1(0)\int_0^1 dx\frac{\exp(-\gamma\Gamma/x-\alpha x)}{x^3}\\ &\quad+\gamma\Gamma[\Psi(0)\varphi_1'(0)+\Psi'(0)\varphi_1(0)]\int_0^1 dx\frac{\exp(-\gamma\Gamma/x-\alpha x)}{x^2}\\ &\quad+\left[\gamma\Psi(0)G_1'(0)\varphi_1(0)+\frac{\gamma\Gamma}{2}\Psi(0)\varphi_1''(0)+\gamma\Gamma\Psi'(0)\varphi_1'(0)+\frac{\gamma\Gamma}{2}\Psi''(0)\varphi_1(0)\right.\\ &\qquad\left.+\Psi'(0)\varphi_1(0)\right]\int_0^1 dx\frac{\exp(-\gamma\Gamma/x-\alpha x)}{x}+O(\gamma)\ ,\end{aligned} \tag{A14}$$

where we have used that all the derivatives $\Psi^{(n)}(0)$ are of order γ^2. All the integrals involved in (A14) can be extended to infinity since their multiplicative factors are at least of order γ. The resulting integrals are related to Bessel functions of the third kind[13] as follows:

$$\int_0^\infty dx \frac{\exp(-\gamma\Gamma/x-\alpha x)}{x^3} = \frac{2\alpha}{\gamma\Gamma}K_2(2\sqrt{\alpha\gamma\Gamma}) ,$$

$$\int_0^\infty dx \frac{\exp(-\gamma\Gamma/x-\alpha x)}{x^2} = 2\sqrt{\alpha/\gamma\Gamma}K_1(2\sqrt{\alpha\gamma\Gamma}) , \tag{A15}$$

$$\int_0^\infty dx \frac{\exp(-\gamma\Gamma/x-\alpha x)}{x} = 2K_0(2\sqrt{\alpha\gamma\Gamma}) .$$

Using the small-argument expansions of the involved K_n and the small-γ expansions of $\Psi(0)$, $\Psi'(0)$, $\Psi''(0)$, $\varphi_1(0)$, $\varphi_1'(0)$, $\varphi_1''(0)$, and $G_1'(0)$, (A14) can be rewritten as

$$\int_0^\infty dx\, g'(x)\frac{\exp(-\alpha x)}{x} = \frac{1}{\gamma\Gamma} - \alpha + \frac{G_1(0)}{\Gamma} + O(\gamma \ln\gamma) \tag{A16}$$

[the $1/\gamma$ and constant terms of (A16) only come from the first integral of the rhs of (A14)]. As in 2D, the leading term of (A16) is just equal to the required value (A11). Thus, α goes to

$$\alpha_0 = G_1(0)/\Gamma , \tag{A17}$$

where $G_1(0)$ is given by (A10) [the *a priori* assumption on α_0 is justified since $U_{\text{exc}}(\Gamma)$ is strictly negative for any Γ].

In the weak-coupling limit ($\Gamma \to 0$), we have in 2D

$$\frac{G_1(0)}{\Gamma} = \tfrac{1}{2}\ln\Gamma + C - \tfrac{1}{2}\ln 2 + \mathscr{T} \tag{A18}$$

where $\mathscr{T}$ represents vanishing terms, and in 3D

$$\frac{G_1(0)}{\Gamma} \sim \sqrt{3\Gamma} . \tag{A19}$$

Using (A18) in (A9) and (A19) in (A17) we see that the 2D and 3D α_0 go to the Debye-Hückel wave numbers $\sqrt{2\Gamma}$ and $\sqrt{3\Gamma}$, respectively, when $\Gamma \to 0$.

APPENDIX B

In this appendix we show that the small-k expansions of $A_{\text{MFFA}}(k)$, $A_{\text{APEX}}(k)$, and $A_{\text{IMFF}}(k)$ have the singularities (4.23) and (4.24) in 2D. As before, we introduce the dimensionless quantities $x \equiv r_{01}/a$, $g(x) \equiv \rho_0(\vec{r}_1)/\rho$, and $k \equiv ke/a$. For both MFFA and APEX we have

$$\ln A(k) = 2\int_0^\infty dx\, x\frac{g(x)}{G(x)}\left[J_0\left[\frac{kG(x)}{x}\right]-1\right] , \tag{B1}$$

where $G(x)$ is a regular function at the origin $[G(0)=1]$ and decays faster than any inverse power of x when $x \to \infty$. The small-k expansion of (B1) is merely obtained by expanding $J_0[kG(x)/x]$ in Taylor's series. Since $g(x)$ only vanishes as $C_g x^{\gamma\Gamma}$ when $x \to 0$, the coefficients of k^{2n} diverges for any $n > p$, where p is equal to $\gamma\Gamma/2$ if $\gamma\Gamma$ is an even integer, and p is the entire part of $1+\gamma\Gamma/2$ in the other case. Defining (for $n \le p$)

$$c_n = \frac{2(-1)^n}{4^n(n!)^2}\int_0^\infty dx\, g(x)\left[\frac{G(x)}{x}\right]^{2n-1} , \tag{B2}$$

(B1) can be rewritten as

$$\ln A(k) - \sum_{n=1}^{p} c_n k^{2n} = 2\int_0^\infty dx\, x\frac{g(x)}{G(x)}D_p\left[\frac{kG(x)}{x}\right] , \tag{B3}$$

where

$$D_p(z) = J_0(z) - \sum_{n=0}^{p}\frac{(-1)^n}{4^n(n!)^2}z^{2n} . \tag{B4}$$

In the integral of the rhs of (B3), we make the variable change $x = ku_\infty$ and split the resulting integral into $\int_0^1 du \cdots + \int_1^\infty du \cdots$. That gives

$$\ln A(k) - \sum_{n=1}^{p} c_n k^{2n}$$
$$= 2k^2\int_0^1 du\, u\frac{g(ku)}{G(ku)}D_p\left[\frac{G(ku)}{u}\right]$$
$$+2k^2\int_1^\infty du\, u\frac{g(ku)}{G(ku)}D_p\left[\frac{G(ku)}{u}\right] . \tag{B5}$$

In $\int_0^1 du \cdots$, we can replace $g(ku)$ by $C_g(ku)^{\gamma\Gamma}$ and $G(ku)$ by $G(0)=1$ since

$$\int_0^1 du\, u^{1+\gamma\Gamma}D_p\left[\frac{1}{u}\right] \tag{B6}$$

is finite. Thus the first term of the rhs of (B5) is of order $k^{2+\gamma\Gamma}$ when $k \to 0$. In $\int_1^\infty du \cdots$, we make the variable change $t = ku$ and we integrate term to term the series expansion of D_p. The second term of the rhs of (B5) then becomes

$$2\sum_{n=p+1}^{\infty}\frac{(-1)^n}{4^n(n!)^2}k^{2n}\int_k^\infty dt\, g(t)\left[\frac{G(t)}{t}\right]^{2n-1} . \tag{B7}$$

Each integral involved in (B7) diverges when $k \to 0$. For $n > p+1$, we have

$$\int_k^\infty dt\, g(t)\left[\frac{G(t)}{t}\right]^{2n-1} \sim C_g\int_k^\infty dt\, t^{1+\gamma\Gamma-2n}$$
$$= \frac{C_g}{(2n-2-\gamma\Gamma)}k^{2+\gamma\Gamma-2n} . \tag{B8}$$

For $n = p+1$, we must distinguish the two following cases. For $\gamma\Gamma$ even integer, we have

$$\int_k^\infty dt\, g(t)\left[\frac{G(t)}{t}\right]^{2p+1} \sim C_g\int_k^1 dt\, t^{-1} = -C_g \ln k . \tag{B9}$$

For $\gamma\Gamma$ noneven integer, we have the same behavior as (B8) with $n = p+1$. Since the series

$$\sum_{n=p+2}^{\infty} \frac{(-1)^n}{4^n(n!)^2(2n-2-\gamma\Gamma)}\,, \qquad \sum_{n=p+1}^{\infty} \frac{(-1)^n}{4^n(n!)^2(2n-2-\gamma\Gamma)} \tag{B10}$$

for $\gamma\Gamma$ even integer and for $\gamma\Gamma$ noneven integer, respectively, are convergent, (B7) behaves when $k \to 0$ as $\text{const} \times k^{2+\gamma\Gamma}\ln k$ for $\gamma\Gamma$ even integer and $\text{const} \times k^{2+\gamma\Gamma}$ for the other case. Therefore, we find in the small-k limit

$$\ln A(k) - \sum_{n=1}^{p} c_n k^{2n} \sim \text{const} \times k^{2+\gamma\Gamma}\ln k\,, \qquad \ln A(k) - \sum_{n=1}^{p} c_n k^{2n} \sim \text{const} \times k^{2+\gamma\Gamma} \tag{B11}$$

for $\gamma\Gamma$ even integer and for $\gamma\Gamma$ noneven integer, respectively, for MFFA and APEX (the constants in the factor of the singularities are identical for both approximations).

Since the difference between the fluctuations (3.17) is finite at the origin, all the coefficients of k^{2n} of the small-k expansion of $F_2(k)$ are finite for $n \leq p+1$. The singularities arising in this expansion have a higher order than the ones appearing in (B11). Thus, $A_{\text{IMFF}}(k)$ has the same lowest-order singularities as $A_{\text{MFFA}}(k)$ and $A_{\text{APEX}}(k)$.

The previous argument can be easily extended to all the functions $F_n(k)$ with $n < 2+\gamma\Gamma$. This circumstance suggests that the lowest-order singularities of $A_{\text{exact}}(k)$ are also exactly given by (B11).

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