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Self-Diffusion for Particles with Stochastic Collisions in One Dimension

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Color diffusion in a classical fluid composed of two species differing *only* by color is intimately connected with the asymptotic behavior of trajectories of test particles in the equilibrium system. We investigate here such behavior in a one-dimensional system of "hard" points with density ρ and velocities ± 1. Colliding particles reflect each other with probability p and pass through each other with probability $1-p$. We show that for $p>0$ the appropriately scaled trajectories of n particles converge to $\rho^{-1}b(t)+(1-p)(\rho p)^{-1}b_j(t), j=1,\ldots,n$. The $b(t), b_j(t)$ are standard, independent Brownian motions. The common presence of $b(t)$ means that motions are not independent and hence that the macroscopic state of the colored system is *not* in local equilibrium with respect to color.

KEY WORDS: Self-diffusion; color profile; stochastic collisions; hydrodynamical limit; local equilibrium.

1. INTRODUCTION

In Ref. 1 we investigated the nonequilibrium phenomenon of self-diffusion in classical systems. We did this by studying the diffusion of the color profile of a binary mixture of black and white particles. Disregarding color the particles were mechanically identical and the system was in thermal equilibrium. Our main conclusions were the following: (1) If the motion of

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a marked particle (test particle) in the infinitely extended particle system in thermal equilibrium looks, on a "macroscopic scale," like Brownian motion, then time-dependent and steady state self-diffusion is governed on that scale by the diffusion equation. The diffusion coefficient, computed microscopically either through the steady state current or by the Einstein relation, coincide. (2) If in addition an arbitrary (finite) number of test particles move asymptotically independently, then the system is locally in equilibrium with respect to colors, i.e., locally particles are colored independently and the relative fraction of particles with a given color is determined through the diffusion equation.

The purpose of this paper is to investigate a model which is simple enough to allow for an analysis of the asymptotic motion of test particles. This is a system of hard rods moving in one dimension. It is known[2–4] that the properly scaled motion of a test particle in this system does behave like Brownian motion. It is also obvious that test particles never move independently, e.g., two initially neighboring test particles stay neighbors forever. Thus the first but not the second condition is satisfied. Starting with a state in which all particles to the left of the origin are painted black and the ones to the right are painted white, the time evolved state will also have a sharp line of demarcation between the two colors. The location of this line will behave, on the macroscopic scale, like Brownian motion and the average color density will therefore satisfy the diffusion equation. Locally, however, particles will be either all black or all white—which is very different from the local equilibrium state expected in real systems. To circumvent this problem we modify the dynamics of this system by stipulating that at a collision the particles have a probability $1-p$ to pass through each other and a probability p to be specularly reflected: $p<1$ permits intermingling of initially separated colors, while $p=0$ corresponds to ideal gas dynamics in which there is no diffusion.

There is little lost by letting the diameter of the hard rods shrink to zero—and we shall do so. We shall also assume, and this is a major simplification, that the velocity distribution of the particles is $h(v) = \frac{1}{2}[\delta(v+1)+\delta(v-1)]$, i.e., the speed of the particles is one. [Recall that any $h(v)$ is invariant under our dynamics whatever p is.] With this choice of $h(v)$ the motion of a single test particle is Markovian and governed by the linear Boltzmann equation as forward equation.[2] Standard results prove then convergence to Brownian motion for $p \neq 0$. The motion of several test particles is not Markovian because of time-delayed interactions transferred by the fluid particles. Asymptotically the joint motion of several test particles is diffusive. However, even when $p<1$ they do *not* move independently. The effects on the test particles of the random initial positions of the fluid particles is seen approximately simultaneously by all test particles.

This constitutes a common noise source for all test particles and prevents independence. It is the only randomness present when $p = 1$. The randomness due to the collisions, when $p < 1$, is approximately independent.

Since several test particles do not move independently, the system containing a nonuniform color density will on a macroscopic scale be locally in a nondegenerate mixture of equilibrium states. The usual hypothesis of local equilibrium is violated therefore even for $p < 1$, although not as badly or obviously as for $p = 1$. We expect "better mixing" for similar models in higher dimensions.

In Section 2 we define the model. In Section 3 we prove diffusive behavior of several test particles. This result is then used to study the global (Section 5) and local structure (Section 6) of the color density. In Section 4 we prove that the only time invariant states are the one where particles are colored independently. Section 7 contains some concluding remarks about the extension of these results to more realistic systems.

2. THE MODEL

We consider a system of point particles on the line. A particle has position $q \in R$, velocity $v \in \{-1, 1\}$, and color $\sigma \in \{0, 1\}$. 0 stands for white and 1 for black.

The phase space Ω of the infinite system of colored particles is the space of sequences $\{q_j, v_j, \sigma_j \mid j \in Z\}$ modulo permutations, with $q_j \in R$, $v_j \in \{-1, 1\}$, $\sigma_j \in \{0, 1\}$. Let $\Omega_c = \{0, 1\}^Z$ be the space of color sequences and Ω_p be the space of particle configurations, i.e., of sequences $\{q_j, v_j \mid j \in Z\}$ modulo permutations. We use the shorthand $x_j = (q_j, v_j)$, $x = \{x_j \mid j \in Z\}$ and $\sigma = \{\sigma_j \mid j \in Z\}$.

We consider states of the system, where the particles, with color disregarded, are in thermal equilibrium. Let μ be the Poisson measure on Ω_p with constant density ρ and independent velocities taking values ∓ 1 with probability $1/2$. For μ – a.a. $x \in \Omega_p$ let $P(d\sigma \mid x)$ be a probability measure on Ω_c. $P(d\sigma \mid x)$ is assumed to depend measurably on x. Given $\sigma \in \Omega_c$ we adopt the convention that σ_0 is the color of the first particle to the right of the origin and that σ_j is the color of the jth particle following their natural order. Then we admit as states of the system probability measures P on Ω such that

$$P(d\omega) = P(d\sigma \mid x)\,\mu(dx) \tag{2.1}$$

Conversely, a probability measure on Ω such that when integrated over colors results in the Poisson measure μ on Ω_p is of the form (2.1). $P(d\sigma \mid x)$ is the color distribution given the particle configuration x.

We assume the following dynamics. Particles move freely up to a collision. At a collision two particles with the same color pass through each other. (Since the particles have zero diameter this gives the same effect as reflection.) Two particles with different color specularly reflect each other with probability p and pass through each other with probability $1-p$, $0 \leqslant p \leqslant 1$.

Equivalently, all particles move freely, i.e., $q_j(t) = q_j + v_j t$, $v_j(t) = v_j$, $j \in Z$. At the coincidence of two particles they exchange their color with probability p and they retain their color with probability $1-p$. The case $p = 0$ corresponds to no interaction and the case $p = 1$ to hard rods of zero length.

For given particle configuration $x \in \Omega_p$ and color sequence $\sigma \in \Omega_c$ let $K_t(d\sigma' \mid \sigma, x)$ be the probability measure on Ω_c corresponding to the color distribution at time t obtained from the configuration $\omega = (\sigma, x)$ under the dynamics just described. By inspection $K_t(d\sigma' \mid \sigma, x)$ exists for $\mu -$ a.a. $x \in \Omega_p$ and all $\sigma \in \Omega_c$, $t \in R$. If the initial $(t = 0)$ measure is given by (2.1) then the time-evolved measure P_t at time t is

$$P_t(d\omega) = \int \left[K_t(d\sigma \mid \sigma', x) P(d\sigma' \mid x) \right] \mu(dx) \tag{2.2}$$

which is still the form (2.1).

3. ASYMPTOTIC DYNAMICS OF TEST PARTICLES

Following Ref. 1 we analyze the asymptotic motion of test particles in the fluid. For our model the n test particle process is defined in the following way. Configurations of the fluid are given by $x \in \Omega_p$. The Poisson measure μ describes the initial state of the fluid. One picks n fluid particles at positions $(q_1, \ldots, q_n)$ with velocities $(v_1, \ldots, v_n)$ as test particles. At the coincidence of two fluid particles they pass through each other. At the coincidence of either a fluid and a test particle or two test particles they pass through each other with probability $(1-p)$ and they specularly reflect each other with probability p. Let $(q_1(t), \ldots, q_n(t))$ be the positions of the n test particles at time t considered as random variables on $\{\Omega_p, \mu\}$. They still depend on $(q_1, v_1, \ldots, q_n, v_n)$. $(q_1(t), \ldots, q_n(t))$ defines a stochastic process with state space $\mathbb{R}^n$ and with continuous sample paths.

The scaled test particle process is defined by

$$q_j^\epsilon(t) = \epsilon q_j(\epsilon^{-2} t) \tag{3.1}$$

$j = 1, \ldots, n$, $\epsilon > 0$. Here $q_j(s)$ is the position at time s of the jth test particle with scaled initial conditions

$$q_j(0) = \epsilon^{-1} q_j + q_j'$$
$$v_j(0) = v_j$$

where q_j, q_j', and v_j are independent of ϵ.

Theorem 1. Let $b(t), b_1(t), \ldots, b_n(t)$ be standard independent Brownian motions. Then for $p \neq 0$

$$(q_1^\epsilon(t), \ldots, q_n^\epsilon(t))$$
$$\to \big(q_1 + (1-p)(p\rho)^{-1} b_1(t) + \rho^{-1} b(t), \ldots,$$
$$q_n + (1-p)(p\rho)^{-1} b_n(t) + \rho^{-1} b(t)\big) \tag{3.2}$$

as $\epsilon \to 0$ in the sense of weak convergence of the path measures on $C([0, \infty), R^n)$. In particular, the limit is independent of $(q_1', \ldots, q_n')$ and $(v_1, \ldots, v_n)$.

For $n = 1$ the proof is standard, since the velocity of the test particle, $v_1(t)$, is a Markov jump process on $\{-1, 1\}$ with jump rate $p\rho$ from v to $-v$ and since $q_1(t) = q_1 + \int_0^t ds\, v(s)$.

At first glance the nonindependence of several test particles may look surprising. For an intuitive understanding let us consider only two test particles. Assume that initially the fluid particles are regularly spaced and have velocity 1 to the left of the test particles and velocity -1 to the right of the test particles. Then the two test particles perform a random walk on a regular lattice where the only dependence comes from times when their positions coincide. For the scaling (3.1) such an interaction becomes negligible and the test particles move independently as $(b_1(t), b_2(t))$. In actual fact the fluid is Poisson distributed, which means that the lattice is randomly distorted. Typically the two test particles are at a distance of the order ϵ^{-1} apart. Since the fluid is noninteracting a distortion travels undisturbed in a time of the order ϵ^{-1} from one test particle to the other. On a time scale ϵ^{-2} this is instantaneous and the two test particles see an identical random distortion almost simultaneously. This phenomenon is the origin of $b(t)$; see also comments in Section 7.

Remark. As already noted this initial randomness is the only one present for $p = 1$, the case considered in Ref. 2. In that case, the convergence to Brownian motion is proven for a general even velocity distribution $h(v)$, with the diffusion coefficient given by $D = \rho^{-1} \int_{-\infty}^{\infty} |v| h(v)\, dv$. Our result for $p < 1$ presumably also carries over to the general $h(v)$ but we have not investigated this in detail. It may also be worth noting here that for the case $p = 1$ with an initial non-Poisson (hence also nonstationary) spatial distribution of the positions of the fluid particles $q_j^\epsilon(t)$ does not converge to Brownian motion.[3]

Proof. To keep the proof transparent, we consider only the case of two test particles. The convergence proof for several test particles follows the same lines.

Test particle one starts at $q_1 = 0$ and test particle two at $q_2 = q > 0$. To simplify notation we assume that initially there are no fluid particles in the interval $[0, q]$ and that the initial velocities are $v_1 = 1$, $v_2 = -1$. Notice that since the speed is one, all particles to the left of 0 with velocity -1 and all particles to the right of q with velocity 1 will never interact with the test particles and may therefore be ignored. In addition, if we disregard labeling, the system evolves freely. Hence the two-dimensional space-time diagram with gives the position at time t of all relevant particles forms a distorted lattice tilted at 45° relative to the coordinate axis (draw the picture!). The distortion originates in the random initial positions of the fluid particles. In order to keep track of this distortion we introduce a fictitious particle with label zero. It starts at 0 with velocity 1 and it reverses its velocity at each collision with another particle (therefore now and then coiniciding with one of the two test particles).

To a given configuration of particles we associate a new configuration where all particles with velocity pointing away from the test particles are suppressed and where the remaining particles are placed at equal distance of length two. To the original trajectory $q_i(t)$ of the ith test particle we associate a trajectory $y_i(t)$ on the regular lattice with the same rules of interaction as before ($i = 0, 1, 2$). Then clearly $y_0(t)$ oscillates between 0 and 1, while $q_0(t)$ diffuses. The $(y_1(t), y_2(t))$ perform an interacting random walk which will be shown (Lemma A) to approach two independent Brownian motions. We will now show that $(\epsilon q_1(\epsilon^{-2}t), \epsilon q_2(\epsilon^{-2}t))$ is well approximated by $(\epsilon(1/\rho)y_1(\epsilon^{-2}\rho t) + \epsilon q_0(\epsilon^{-2}t), \epsilon(1/\rho)y_2(\epsilon^{-2}\rho t) + \epsilon q_0(\epsilon^{-2}t))$, which is the desired result.

With this in mind we define the following quantities: Let T_1^i, $T_2^i, \ldots, T_n^i, \ldots$ be the successive instants of collisions for the test particle i. Call $v_1^i, \ldots, v_n^i, \ldots$ velocity of this particle immediately after T_n^i and define $y_i(n)$ inductively by $y_1(0) = 0 = y_0(0)$, $y_2(0) = 2$, $y_i(n+1) = y_i(n) + v_n^i, n = 0, 1, \ldots,\ i = 0, 1, 2$. $y_i(t)$ for all $t \geqslant 0$ is defined by linear interpolation. It follows from the construction that v_n^i is a discrete Markov chain with state space $\{-1, 1\}$ and transition matrix $Q(v' \mid v) = (1-p)\delta_{vv'} + p(1-\delta_{vv'})$ for $i = 1, 2$ and $Q_0(v' \mid v) = (1 - \delta_{vv'})$ for $i = 0$. Furthermore $((y_1(n), v_n^1), (y_2(n), v_n^2))^{vv'}$ is a Markov chain, where the transition matrix for v_n^1 and v_n^2 is $Q(v' \mid v)$ in the case $y_1(n) \neq y_2(n)$ and $Q(v', u' \mid v, u) = (1 - \delta_{vu})(1 - \delta_{v'u'})[(1-p)\delta_{vv'}\delta_{uu'} + p\delta_{uv'}\delta_{vu'}]$ in the case $y_1(n) = y_2(n)$.

Let $U_n^i = T_n^i - T_{n-1}^i$ with $T_0^i = 0$. Let $B_0 = q$ and denote by $B_1, \ldots, B_k, \ldots$ the successive spacings between the particles located to the right of q, and by $A_1, \ldots, A_k \ldots$ those to the left of 0. Note that $(A_1, A_2, \ldots, B_1, B_2, \ldots)$ are independent and are all exponentially distributed with mean $2/\rho$. The U^i for $i = 0, 1, 2$ are related to the A's and B's by

$2U_1^i = B_0$ and

$$2U_{n+1}^i = B_{(n+k)/2} \tag{3.3}$$

if $y_i(n) = k$ and $v_n^i = 1$, whereas if $v_n^i = -1$:

$$2U_{n+1}^i = A_{[(n-k)/2]+1} \tag{3.4}$$

Hence it follows that $(U_1^i, U_2^i, \ldots)$ are independent although (U_n^1) and (U_n^2) are not jointly independent, since they are built out of the same A's and B's.

Let N_t^i be the number of collisions of test particle i up to time t. We can now reconstruct $q_i(t)$ from $(y_i(n), v_n^i)$ and the A's and B's by (3.3), (3.4). If $N_t^i = k_i$, then

$$q_i(t) = q_i = \sum_{p=1}^{k_i} v_{p-1}^i U_p^i + \left(t - \sum_{p=1}^{k_i} U_p^i \right) v_k^i \tag{3.5}$$

Since for a given i the T_n^i form a Poisson process with intensity ρ, $N_{\epsilon^{-2}t}^i \sim \rho\epsilon^{-2}t$. Therefore for $i = 1, 2$ we make the following decomposition:

$$\begin{aligned} &\epsilon\left[q_i(\epsilon^{-2}t) - q_0(\epsilon^{-2}t) \right] \\ &= \epsilon \sum_{p=1}^{[\rho\epsilon^{-2}t]} v_{p-1}^i U_p^i - \epsilon \sum_{p=1}^{[\rho\epsilon^{-2}t]} v_{p-1}^0 U_p^0 \\ &= \epsilon \sum_{p=[\rho\epsilon^{-2}t]+1}^{k_i} v_{p-1}^i U_p^i + v_{k_i}\left(\epsilon^{-2}t - \sum_{p=1}^{k_i} U_p^i \right) + \epsilon q_i \\ &\quad - \epsilon \sum_{p=[\rho\epsilon^{-2}t]+1}^{k_0} v_{p-1}^0 U_p^0 - \epsilon v_{k_0}\left(\epsilon^{-2}t - \sum_{p=1}^{k_0} U_p^0 \right) \end{aligned} \tag{3.6}$$

where $N_t^i = k_i$, $i = 0, 1, 2$. Here [ ] denotes the integer part and the summation $\sum_{p=m}^{k}$ is understood algebraically, i.e., with negative sign if $k < m$. For $i = 1, 2$ we set

$$z_i^\epsilon(t) = \epsilon \sum_{p=1}^{\rho\epsilon^{-2}t} v_{p-1}^i U_p^i = \epsilon \sum_{p=1}^{\rho\epsilon^{-2}t} v_{p-1}^0 U_p^0 \tag{3.7}$$

for integer values of $\rho\epsilon^{-2}t$ and define $z_i^\epsilon(t)$ for all $t \geqslant 0$ by linear interpolation. Then for any $\delta > 0$

$$\lim_{\epsilon\to 0} P\left(\sup_{0 \leqslant s \leqslant t} \sum_{i=1}^{2} |q_i^\epsilon(s) - q_0^\epsilon(s) - z_i^\epsilon(s)| \geqslant \delta \right) = 0 \tag{3.8}$$

Clearly

$$P\left(\sup_{0\leqslant s\leqslant t}\sum_{i=1}^{2}|q_i^\epsilon(s)-q_0^\epsilon(s)-z_i^\epsilon(s)|\geqslant\delta\right)$$

$$\leqslant\sum_{i=0}^{2}P\left(|N^i_{\epsilon^{-2}s}-\rho\epsilon^{-2}s|\geqslant\lambda\epsilon^{-1}\sqrt{s}\,,\,0\leqslant s\leqslant t\right)$$

$$+2\sum_{i=0}^{2}P\left(\sup_{0\leqslant s\leqslant t}\sup_{|k-\rho\epsilon^{-2}|<\lambda\epsilon^{-1}\sqrt{s}}\left|\epsilon\sum_{p=\rho\epsilon^{-2}s}^{k}v^i_{p-1}U^i_p\right|\geqslant\delta\right)\tag{3.9}$$

The first term vanishes as $\rho\to 0$, since N_t^i is a Poisson process $\epsilon\sum_{p=1}^{[\rho\epsilon^{-2}t]}v^i_{p-1}U^i_p$ tends in distribution to Brownian motion as $\epsilon\to 0$. Therefore the second term in (3.9) vanishes for arbitrary δ.

Since $U^0_{2p}=\frac{1}{2}A_{p+1}$ and $U^0_{2p+1}=\frac{1}{2}B_p$, (3.7) for integer values of $\rho\epsilon^{-2}t$ can be rewritten as

$$z_i^\epsilon(t)=\epsilon\sum_{p=1}^{(1/2)y_i(\rho\epsilon^{-2}t)}\frac{1}{2}\left(A_{\frac{1}{2}\rho\epsilon^{-2}t-p+1}+B_{\frac{1}{2}\rho\epsilon^{-2}t+p}\right)\tag{3.10}$$

where the sum is again in the algebraic sense. $z_i^\epsilon(t)$ is the sum of $\frac{1}{2}y_i(\rho\epsilon^{-2}t)$ independent random variables with mean $2\epsilon/\rho$. $\epsilon y_i(\epsilon^{-2}t)$ tends to Brownian motion as $\epsilon\to 0$. As before, we break the sum (3.10) into a term with $y_i(\epsilon^{-2}t)\sim\epsilon^{-1}$ and a remainder. The same argument used before shows then that any $\delta>0$

$$\lim_{\epsilon\to 0}P\left(\sup_{0\leqslant s\leqslant t}\sum_{i=1}^{2}\left|z_i^\epsilon(t)-\frac{1}{\rho}\epsilon y_i(\rho\epsilon^{-2}t)\right|\geqslant\delta\right)=0\tag{3.11}$$

$q_0^\epsilon(t)$ and $(\epsilon y_1(\epsilon^{-2}t),\epsilon y_2(\epsilon^{-2}t))$ are independent. $q_0^\epsilon(t)$ tends in distribution to $\rho^{-1}b(t)$ as $\epsilon\to 0$. $(\epsilon y_1(\epsilon^{-2}t),\epsilon y_2(\epsilon^{-2}t))$ tends to two independent Brownian motions with variance $[(1-p)/p]t$ as $\epsilon\to 0$, c.f. Lemma A in the Appendix. This, together with (3.8) and (3.11), concludes the proof of the theorem. ■

4. TIME INVARIANT MEASURES

Let $B_w(d\sigma)$ denote the independent (Bernoulli) measure on Ω_c with the probability that $\sigma_j=1$, B_w $(\sigma_j=1)$, equal to w for all j.

Theorem 2. Let $P(d\omega)$ be a measure on Ω which is of the form (2.1) and which is extremal time invariant. Then for $p\neq 0,1$

$$P(d\omega)=B_w(d\sigma)\,\mu(dx)\tag{4.1}$$

for some w, $0\leqslant w\leqslant 1$.

Proof. For $x \in \Omega_p$ let $x_t = \{q_j + v_j t, v_j \,|\, j \in Z\}$ be the configuration obtained through free motion. Let P be an invariant measure. Then

$$P(d\omega) = P(d\sigma \,|\, x)\,\mu(dx) = P_t(d\omega) = P_t(d\sigma \,|\, x)\,\mu(dx) \tag{4.2}$$

which implies

$$P_t(d\sigma \,|\, x) = P(d\sigma \,|\, x) \qquad \mu - \text{a.s.} \tag{4.3}$$

On the other hand

$$P_t(d\sigma \,|\, x) = \int K_t(d\sigma \,|\, \sigma', x_{-t}) P(d\sigma' \,|\, x_{-t}) \tag{4.4}$$

We show that (4.3) and (4.4) imply the "exchangeability" of $P(d\sigma \,|\, x)$. For $i \neq j$

$$\begin{aligned} &P(\sigma_i = 0, \sigma_j = 1 \,|\, x) - P(\sigma_i = 1, \sigma = 0 \,|\, x) \\ &\quad = \int \left[K_t(\sigma_i = 0, \sigma_j = 1 \,|\, \sigma', x_{-t}) - K(\sigma_i = 1, \sigma_j = 0 \,|\, \sigma', x_{-t}) \right] \\ &\quad \times P(d\sigma' \,|\, x_{-t}) \end{aligned} \tag{4.5}$$

To compute (4.5) one places a test particle at site i and a test particle at site j of the configuration x. The test particles move backward in time. Their final sites on the configuration x_{-t} determine then their color through σ'. At points of coincidence of the two test particles their roles may be interchanged. Therefore, since with probability one the two test particles meet infinitely often during the time interval $(-\infty, 0)$, the right-hand side of (4.5) has to vanish as $t \to \infty$. Repeating the argument for any finite number of points we conclude that $P(d\sigma \,|\, x)$ is exchangeable and therefore by De Finetti's theorem

$$P(d\sigma \,|\, x) = \int \nu(dw \,|\, x) B_w(d\sigma) \tag{4.6}$$

for some probability measure $\nu(dw \,|\, x)$. Since the Bernoulli measure is invariant for the stochastic kernel $K_t(d\sigma \,|\, \sigma', x)$, (4.3) and the uniqueness of the decomposition imply

$$\nu(dw \,|\, x) = \nu(dw \,|\, x_t) \tag{4.7}$$

Since $\mu(dx)$ is ergodic for the free evolution, $\nu(dw \,|\, x)$ has to be independent of x. By extremality $\nu(dw \,|\, x)$ has to be concentrated on a single point.

For $p = 0$ and $p = 1$ there exist invariant measures other than (4.1). ■

5. TIME-DEPENDENT AND STEADY STATES IN THE HYDRODYNAMIC LIMIT

We consider time-dependent phenomena for a particular class of initial states. Let $g : R \to [0,1]$ be a measurable function. Let $P_g(d\sigma \mid x)$ be the product measure on Ω_c such that

$$P_g(\sigma_j = 1 \mid x) = g(q_j) \tag{5.1}$$

where x is labeled in its natural order such that q_0 is the position of the first particle to the right of the origin. We assume that the initial measure is

$$P(d\omega) = P_g(d\sigma \mid x)\,\mu(dx) \tag{5.2}$$

g is then the profile of black particles. We denote by $\rho_n(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n, t)$ the correlation functions of the time evolved measure at time t. A slowly varying color profile is assumed by setting

$$g_\epsilon(q) = g(\epsilon q) \tag{5.3}$$

Then the scaled correlation functions are defined by

$$\begin{aligned}\rho_n^\epsilon&(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n, t)\\ &= \rho_n(\epsilon^{-1}q_1, v_1, \sigma_1, \ldots, \epsilon^{-1}q_n, v_n, \sigma_n, \epsilon^{-2}t)\end{aligned} \tag{5.4}$$

$$\rho_n^\epsilon(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n, 0) = \prod_{j=1}^{n} \frac{1}{2}\rho\big[(1-\sigma_j) - (1-2\sigma_j)g(q_j)\big] \tag{5.5}$$

The nth correlation function is related to the n test particle process by

$$\begin{aligned}\rho_n^\epsilon&(q_1, v_1, \sigma_1, \ldots, q_n, v_n\sigma_n, t)\\ &= E^\epsilon_{(q_1, -v_1, \ldots, q_n, -v_n)}\left(\prod_{j=1}^{n} \frac{\rho}{2}\big[(1-\sigma_j) - (1-2\sigma_j)g(q_j^\epsilon(t))\big]\right)\end{aligned} \tag{5.6}$$

Here $(q_1^\epsilon(t), \ldots, q_n^\epsilon(t))$ is the scaled n test particle process as defined in Section 3 and $E^\epsilon_{(\ldots)}$ denotes expectation conditioned that the test particles start at $(\epsilon^{-1}q_1, -v_1, \ldots, \epsilon^{-1}q_n, -v_n)$. As an immediate consequence of Theorem 1

$$\begin{aligned}\lim_{\epsilon\to 0}\rho_n^\epsilon&(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n, t)\\ &= \int dx \frac{1}{(2\pi D t)^{1/2}} \exp(-x^2/2Dt) \prod_{j=1}^{n}\\ &\quad\times\left\{\int dy \frac{1}{(2\pi D' t)^{1/2}} \exp\big[-(x + q_j - y)^2/2D't\big]\right.\\ &\quad\left.\times \frac{1}{2}\rho\big[(1-\sigma_j) - (1-2\sigma_j)g(y)\big]\right\}\end{aligned} \tag{5.7}$$

pointwise, $D = 1/\rho$, $D' = (1/\rho)(1-p)/p$. On a macroscopic scale the color random fields are a Gaussian superposition of deterministic fields which are obtained from the solution of the diffusion equation.

For the steady state we impose boundary conditions on the colors at the points $-L$ and L.[1] Particles to the left of $-L$ are black and all particles to the right of L are white. If a particle exits $[-L, L]$ at $-L$, then its color is changed to (or remains) black, and if a particle exits $[-L, L]$ at L, then its color is changed to (or remains) white. Under these boundary conditions there is a unique stationary measure $P_L(d\omega)$. Let $\rho_n(\cdot\,; L)$ be its nth correlation functions. Then the scaled correlation functions are

$$\rho_n^\epsilon(q_1, \sigma_1, \ldots, q_n, v_n, \sigma_n; L) = \rho_n\left(\epsilon^{-1}q_1, v_1, \sigma_1, \ldots, \epsilon^{-1}q_n, v_n, \sigma_n; \epsilon^{-1}L\right) \tag{5.8}$$

They are related to the scaled n test particle process by

$$\begin{aligned}\rho_n^\epsilon(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n; L) &= \left[(\rho/2)\right]^n P^\epsilon_{(q_1, -v_1, \ldots, q_n, -v_n)} \\ &\quad \left(q_j(t) \text{ exits } \left[-L, L\right] \text{ first at } (1-2\sigma_j)L, j = 1, \ldots, n\right)\end{aligned} \tag{5.9}$$

Let db be the path measure on $C(R)$ of the Weiner process with variance ρ^{-1}. Let $t \to \gamma(t)$ be a continuous function such that $\gamma(0) = 0$. Then $P(q, \gamma)$ is defined as the probability that Brownian motion which starts at q and has variance $(1-p)t/\rho p$ exits $[-L + \gamma(t), L + \gamma(t)]$ at $-L + \gamma(t)$. As a consequence of Theorem 1 and (5.9),

$$\begin{aligned}&\lim_{\epsilon \to 0} \rho_n^\epsilon(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n; L) \\ &\quad = \int db(\gamma) \prod_{j=1}^{n} \left\{ \frac{1}{2}\rho\left[(1-\sigma_j) - (1-2\sigma_j)P(q_j, \gamma)\right]\right\}\end{aligned} \tag{5.10}$$

As $p \to 1$, $D' \to 0$. In this case one sees at each macroscopic point either only black or only white particles. Their relative weight is such that on the *average* the diffusion equation is valid and that on the average the steady state color profile is linear.

We remark that, since the motion of a single test particle is Markovian, the first correlation function can be computed explicitly. In the time-dependent case it satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\rho_1(q, v, 1, t) = -v\frac{\partial}{\partial q}\rho_1(q, v, 1, t) &+ 2p\rho_1(q, -v, 1, t) \\ &- p\rho_1(q, -v, 1, t)\end{aligned} \tag{5.11}$$

with initial conditions $\rho_1(q, v, 1, 0) = \frac{1}{2}\rho g(q)$. Equation (5.11) can be solved in terms of Bessel functions.[4]

In the steady state (5.11) has to be satisfied with boundary conditions

$$\rho_{L,1}(-L,1,1)=\tfrac{1}{2}\rho, \qquad \rho_{L,1}(L,-1,1)=0 \tag{5.12}$$

The solution is

$$\begin{aligned}\rho_{L,1}(q,1,1)&=\frac{1}{2}\rho\left\{1-(q+L)\frac{2p}{1+4pL}\right\}\\ \rho_{L,1}(q,1,1)&=\frac{1}{2}\rho(L-q)\frac{2p}{1+4pL}\end{aligned} \tag{5.13}$$

The steady state current is

$$j_L=\frac{1}{2}\rho\frac{1}{1+4pL} \tag{5.14}$$

The average color profile is linear, although with a jump of size $1/L$ at the boundary. This is a particularly simple form of the boundary layer of size $1/L$ to be expected in general. The steady state current is of the order $1/L$ except for $p=0$ when it is independent of L as it should be for an ideal gas.(5)

6. LOCAL EQUILIBRIUM STATES

The local state $P_{(q,t)}$ of the system at the macroscopic point (q,t) may be defined through its correlation functions

$$\begin{aligned}&\rho_n^\epsilon(q_1,v_1,\sigma_1,\ldots,q_n,v_n,\sigma_n\mid q,t)\\ &\quad=\rho_n(\epsilon^{-1}q+q_1,v_1,\sigma_1,\ldots,\epsilon^{-1}q+q_n,v_n,\sigma_n,\epsilon^{-2}t)\end{aligned} \tag{6.1}$$

From Theorem 1 and (5.6) it follows that in the hydrodynamic limit

$$\begin{aligned}&\lim_{\epsilon\to 0}\rho_n^\epsilon(q_1,v_1,\sigma_1,\ldots,q_n,v_n,\sigma_n)\\ &\quad=\int dx\frac{1}{(2\pi Dt)^{1/2}}\exp(-x^2/Dt)\\ &\qquad\times\prod_{j=1}^{n}\left\{\int dy\frac{1}{(2\pi D't)^{1/2}}\exp\left[-(x+q-y)^2/2D't\right]\right.\\ &\qquad\qquad\left.\times\frac{1}{2}\rho\left[(1-\sigma_j)-(1-2\sigma_j)g(y)\right]\right\}\end{aligned} \tag{6.2}$$

Therefore the local state is a mixture of equilibrium states,

$$P_{(q,t)}(d\omega)=\int\nu(dw\mid q,t)B_w(d\sigma)\mu(dx) \tag{6.3}$$

Similarly, in the steady state the local state P_q at the macroscopic point

q may be defined through its correlation functions

$$\rho_n^\epsilon(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n \mid q; L) = \rho_n(\epsilon^{-1}q + q_1, v_1, \sigma_1, \ldots, \epsilon^{-1}q + q_n, v_n, \sigma_n; \epsilon^{-1}L) \tag{6.4}$$

From Theorem 1 and (5.9) it follows that in the hydrodynamic limit

$$\lim_{\epsilon \to 0} \rho_n^\epsilon(q_1, v_1, \sigma_1, \ldots, q_n, v_n, \sigma_n) = \int db(\gamma) \prod_{j=1}^{n} \left\{ \frac{1}{2}\rho\left[(1-\sigma_j) - (1-2\sigma_j)P(q,\gamma)\right]\right\} \tag{6.5}$$

Again, the local state is a mixture of equilibrium states,

$$P_q(d\omega) = \int \nu(dw \mid q) B_w(d\sigma)\ \mu(dx) \tag{6.6}$$

As $p \to 1$ the mixtures become degenerate, in the sense that $\nu(dw \mid q, t)$ and $\nu(dw \mid q)$ are concentrated on 0 and 1.

7. CONCLUDING REMARKS

(i) Our analysis has been confined to the case $h(v) = \frac{1}{2}[\delta(v-1) + \delta(v+1)]$. For a general velocity distribution the space-time diagram has a more complicated geometry. If one conditions on the initial configuration of particles, one obtains a random walk ($0 < p < 1$) on a kind of random network. We expect a qualitative similar behavior as for speed one. This is in fact what one finds for $p = 1$.[2]

(ii) We note in (5.7) that for $\rho \to \infty$, $p \to 0$, $\rho p = \text{const}$ the limiting correlation functions factorize which implies local equilibrium. This corresponds to the one-dimensional Boltzmann–Grad limit. Using the same technique as in Ref. 7, one can prove, for a general velocity distribution, that in this limit the motion of a single test particle is governed by the one-dimensional linear Boltzmann equation and that several test particles move independently.

(iii) A two-dimensional version of our model consists of replicas of the one-dimensional system arranged in parallel lines. Particles are allowed to jump with a certain rate between neighboring lines. In the case of N lines it has been shown[8] that the common diffusion constant is proportional to $1/N$. This suggests that for two and more dimensions the common noise source becomes incoherent and several test particles move independently in the scaling limit.

(iv) A mechanical model which comes close to our model is a ν-dimensional channel with "matchsticks" (or rectangles) oriented perpendicular to the boundary. The sticks have a velocity with an angle of 45° relative to the boundary. The ratio of the stick length and the width of the channel corresponds to $1-p$. However, in this mechanical model reflections and transmissions are correlated. We still would expect a behavior comparable to the one we found for stochastic uncorrelated collisions. If one considers hard spheres in a channel then the collisions between the spheres randomize the velocity and the common noise source is, at least partially, suppressed. For this model local equilibrium seems to be a possibility.

We certainly expect that in the full three-dimensional system several test particles will diffuse independently asymptotically and the system establishes local equilibrium for the colors. For hard disks in two dimensions even a single test particle may not diffuse due to the long time tails.[9]

(v) Typical stochastic particle models are interacting Brownian particles and stochastic lattice gases with Kawasaki dynamics. For these models it has been shown recently[10,11] that if either $d \geqslant 2$ or $d=1$ but interchange in ordering is allowed (i.e., $0 \leqslant p < 1$), a single test particle tends to Brownian motion and several test particles become independent in the scaling limit.

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APPENDIX

We still have to prove that the process $(\epsilon y_1(\epsilon^{-2}t), \epsilon y_2(\epsilon^{-2}t))$ tends to two independent Brownian motions as $\epsilon \to 0$. This discrete time problem is essentially equivalent to the corresponding one continuous in time. Since the proof is more easily written for continuous time, we choose this setup here.

The proof consists of a standard martingale argument together with an additional trick as used in Ref. 3. We therefore indicate only the modifications in the proof of the theorem in Ref. 3 needed here.

Let $(x(t), v(t), y(t), w(t))$ be a Markov jump process on $(Z \times \{-1, 1\})^2$ with generator

$$(Lf)(x, v, y, w) = \sum_{v', w' = \mp 1} Q(v \mid v') Q(w \mid w') \left[f(x+v, v', y+w, w') - f(x, v, y, w) \right]$$

if $x + v \neq y + w$, and

$$(Lf)(x, v, y, w) = \sum_{v', w' = \mp 1} Q(v, w \mid v', w') \left[f(x+v, v', y+w, w') - f(x, v, y, w) \right] \tag{A.1}$$

if $x + v = y + w$, where $Q(v \mid v') = (1-p)\delta_{vv'} + p(1-\delta_{vv'})$ and $Q(v, w \mid v', w') = (1-\delta_{vw})(1-\delta_{v'w'})[(1-p)\delta_{vv'}\delta_{ww'} + p\delta_{vw'}\delta_{v'w})$ with $p \neq 0, 1$. Note that the subset $\{x \neq y \text{ or } v \neq w\}$ is closed.

Lemma A. The process $(\epsilon x(\epsilon^{-2}t), \epsilon y(\epsilon^{-2}t))$ tends in distribution to two independent Brownian motions with covariance $[(1-p/p)]t$ as $\epsilon \to 0$.

Proof. Let L^ϵ be the generator of the scaled process. Let

$$\phi^\epsilon(x, v, y, w) = f(x, y) + \frac{\epsilon}{2p}\left[v\frac{\partial f}{\partial x}(x, y) + w\frac{\partial f}{\partial y}(x, y)\right] + \left(\frac{\epsilon}{2p}\right)^2\left[v\frac{\partial^2 f}{\partial x^2}(x, y) + w\frac{\partial^2 f}{\partial y^2}(x, y) + vw\frac{\partial^2 f}{\partial x\,\partial y}(x, y)\right] \tag{A.2}$$

Then expanding to third order in ϵ one obtains for $|xy| \neq \epsilon$

$$(L^\epsilon\phi^\epsilon)(x, v, y, w) = \frac{1-p}{2p}\ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} f(x, y) + O(\epsilon) \tag{A.3}$$

We would be finished were it not for the term close to the diagonal. Notice that the very choice of the function ϕ^ϵ eliminates the term in $1/\epsilon$ of the Taylor expansion and leaves us with a bounded term which differs from zero only on the set $\{x = y\}$. We now argue as in the theorem of Ref. 3, to conclude that, since the time spent in this subset is of order ϵ^{-1}, the contribution of this term is negligible in the limit.

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