Commun. Math. Phys. 78, 117–135 (1980)

Communications in Mathematical Physics

Low Temperature Expansion for Continuous-Spin Ising Models*

J. Bricmont[1]**, J. L. Lebowitz[2], and C. E. Pfister[3]***

1 Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
2 Department of Mathematics and Physics, Rutgers University, New Brunswick, NJ 08903, USA
3 Departement de Mathematiques, Ecole Polytechnique Fédérale, 61, Av. de Cour, CH-1007 Lausanne, Switzerland

Abstract. We consider general ferromagnetic spin systems with finite range interactions and an even single-spin distribution of compact support on $\mathbb{R}$. It is shown under mild assumptions on the single-spin distribution that a low temperature expansion, in powers of T, for the free energy and the correlation functions is asymptotic. We also prove exponential clustering in the pure phases and analyticity of the free energy and of the correlation functions in the reciprocal temperature β for $\mathrm{Re}\,\beta$ large.

I. Introduction

In this paper we develop a low temperature expansion for bounded spins on $\mathbb{R}$ distributed with a continuous measure e.g. a uniform distribution on $[-1, +1]$ and nearest neighbour interactions Js_is_j, $J>0$. As a consequence, we obtain analyticity and exponential clustering of the correlation functions at low temperatures. We also obtain asymptotic series in powers of T for various quantities.

The bounded continuous spin system is somewhat intermediate between the Ising model and the Field Theory case studied respectively in [1, 2]. In the Ising model the low temperature expansion is in terms of contours: to each configuration one associates a family of connected contours i.e. lines separating + and − regions. These are pairwise disconnected and summing over configurations amounts to considering all possible arrangements of connected contours. In Field Theory one deals with continuous variables and it is impossible to associate uniquely configurations and contours. Instead, one first uses a contour expansion in order to isolate regions of pure phases (separated by contours) where the field sits in one of the "potential wells". It is then necessary to supplement this contour expansion by an expansion in the pure phases (away from the contours) which is a

* Supported in part by N.S.F. Grant No. Phy. 78-15920 and MCS78-01885
** On leave from: Département de Physique Théorique, Université de Louvin, Belgium
*** Supported in part by the Swiss National Foundation for Scientific Research

0010-3616/80/0078/0117/$03.80

weak coupling expansion (the cluster expansion) about the Gaussian near the minimum of the appropriate well.

In the case we shall consider here, e.g. a uniform distribution on $[-1,+1]$, there is no Gaussian around which to perturb in the pure phases. What we do is first define the contours in the usual way, separating sites where s_i is positive from sites where s_i is negative. Then we perform a change of variables $s_i \to 1 - \frac{s_i'}{\mathrm{Re}\beta}$; s_i' measures the deviation from the ground state, $s_i = 1$, imposed by the boundary conditions. After that, the interaction between the s' variables looks similar to a high-temperature situation. This permits us to do, for the nearest neighbour pairs outside the contours, a high-temperature Mayer expansion which provides an exponential decoupling of distant regions. Once we have that, the estimates are simple and we can use, for example, the algebraic formalism [3] to obtain the usual consequences of the expansion.

We state our results for the simplest case in Sect. 2. Section 3 explains the contour and the high-temperature expansion. Section 4 contains the proof of Theorem 1 using the algebraic formalism. In Sect. 5 we extend our result to general finite range ferromagnetic interactions and a large class of single-spin distributions. This gives, using the results of [4] based on the inequalities of [5, 6] a complete description of the set of periodic Gibbs states at low temperatures for these systems.

II. The Main Result

We consider for definiteness a two dimensional square lattice (for extensions, see Sect. V). At each $i \in \mathbb{Z}^2$ there is a spin variable s_i, with the Lebesgue measure on $[-1,+1]$. The interaction $s_i s_j$, $J=1$, is nearest neighbour.

For $\beta = T^{-1}$ large enough, this model has (at least) two extremal translation invariant Gibbs states related by the $s_i \leftrightarrow -s_i$ symmetry [7]. These Gibbs states can be obtained by taking a finite box $\Lambda \subset \mathbb{Z}^2$ with $+(-)$ boundary conditions (b.c.) $s_i = +1$, $i \in \Lambda$.

The Gibbs measure for the + b.c. is

$$d\mu_\Lambda = Z_\Lambda^{-1} \exp\Big(\beta \sum_{\langle ij\rangle \subset \Lambda} s_i s_j + \sum_{i \in \partial\Lambda} h_i s_i\Big) \prod_{i \in \Lambda} ds_i\,, \tag{1}$$

$$Z_\Lambda = \int_{-1}^{1} \exp\Big(\beta \sum_{\langle ij\rangle \subset \Lambda} s_i s_j + \sum_{i \in \partial\Lambda} h_i s_i\Big) \prod_{i \in \Lambda} ds_i\,. \tag{2}$$

$\langle ij\rangle$ means that i and j are nearest neighbour, i.e. $\|i-j\|=1$

$$\partial\Lambda = \{i \in \Lambda \mid \mathrm{dist}(i,\Lambda^c)=1\}$$

$$h_i = \#\{j \notin \Lambda \mid \|i-j\|=1\}\,.$$

A multiplicity function (m.f.) is a function from $\mathbb{Z}^d$ into $\mathbb{N}$ of finite support. We write, for any m.f. A,

$$s_A = \prod_i s_i^{A(i)}$$

$$\mathrm{supp}\,A = \{i \mid A(i) \neq 0\}$$

$$\bar{A} = \{i \mid A(i) \text{ is odd}\}\,.$$

We define the correlation functions with + boundary conditions as

$$\langle s_A \rangle_A = \int_{-1}^{+1} s_A d\mu_A$$

One knows (see e.g. [8]) that, for all m.f. A, $\lim_{A \to \mathbb{Z}^2} \langle s_A \rangle_A$ exists and defines the correlation functions $\varrho^+(s_A)$ of an extremal translation invariant Gibbs state. For β large, $\varrho^+(s_0) \neq 0$ [7] (spontaneous magnetization) and, by symmetry, we obtain, with $-$ b.c., a state $\varrho^- \neq \varrho^+$.

The following limit also exists and defines (β times) the free energy per unit volume:

$$\psi(\beta) = \lim_{A \to \infty} \frac{1}{|A|} \log Z_A. \tag{3}$$

Theorem 1. 1) *There exists a complex domain*

$$\mathscr{D} = \left\{ \beta \in \mathbb{C} | \operatorname{Re}\beta > c, \frac{|\operatorname{Im}\beta|}{\operatorname{Re}\beta} \leqq c' \right\},$$

where $\psi(\beta)$ and $\varrho^+(s_A)$ are analytic (for all A's).

2) *$\psi(\beta) - 2\beta + \log\beta$ and $\varrho^+(s_A)$ (all m.f.) have an asymptotic expansion in powers of $T = \beta^{-1}$ around $T = 0$.*

3) *For all m.f. A, B, there exists a constant $c(A, B)$ and for all β large enough there exists a $m(\beta) > 0$ such that*

$$0 \leqq \varrho^+(s_A \tau^j s_B) - \varrho^+(s_A)\varrho^+(s_B) \leqq c(A, B) \exp(-m(\beta)|j|)$$

for all $j \in \mathbb{Z}^2$ $|j| = \max_{\alpha=1,2} |j_\alpha| : \tau^j, j \in \mathbb{Z}^2$, represents the natural action of the group $\mathbb{Z}^2$.

One may choose $m(\beta)$ such that $\lim_{\beta \to \infty} \frac{m(\beta)}{\log(\beta)} = 1.$

Remark. Part 1) together with the correlation inequalities of [5] imply that there are exactly two extremal translation invariant Gibbs states for this model when $\beta \in \mathscr{D} \cap \mathbb{R}$ (see [4]).

III. The Expansion

We start with the partition function. Once we have expanded Z_A we obtain the expansion for the free energy and the correlation functions in a standard way (Sect. 4).

A. The Contour Expansion

We write $s_i = \sigma_i r_i$, $\sigma_i = \pm 1$, $r_i \in [0, 1]$ and

$$Z_A = \sum_{\sigma_i = \pm 1} \int_{-1}^{+1} \exp\left(\beta \sum_{\langle ij \rangle \subset A} \sigma_i \sigma_j r_i r_j + \sum_{i \in \partial A} h_i \sigma_i r_i\right) \prod_{i \in A} dr_i. \tag{4}$$

For each term in (4) we draw a unit line perpendicular to the pairs $\langle ij \rangle$ for $\langle ij \rangle \cap A \neq \emptyset$ and $\sigma_i \sigma_j = -1$, and we obtain the same sets of lines, indexing the terms in (4) as in the usual spin $\frac{1}{2}$ Ising model. We decompose these sets of lines into connected components and we define a *contour* (in Λ) as a connected line occurring in one of the terms in (4). An *admissible set of contours* $\Gamma=(\gamma_1,\dots,\gamma_n)$ is a set of contours such that γ_i and γ_j are disconnected if $i\neq j$. Then, as in the Ising model, there is a one-to-one correspondence between the terms in (4) and admissible families of contours (in Λ). We write

$$Z_\Lambda=\exp(\beta c_\Lambda)\sum_{\Gamma=(\gamma_1\dots\gamma_n)} Z_\Gamma, \tag{5}$$

where the sum is over all admissible families of contours (in Λ),

$$c_\Lambda=\#\{\langle ij\rangle|\langle ij\rangle\cap\Lambda\neq\emptyset\}$$

and

$$Z_\Gamma=\int_{-1}^{+1}\prod_{\langle ij\rangle\in\Gamma}\exp(-\beta(r_ir_j+1))\prod_{\langle ij\rangle\notin\Gamma}\exp(\beta(r_ir_j-1))\prod_{i\in\Lambda}dr_i,$$

where $\langle ij\rangle\in\Gamma$ means that $\langle ij\rangle$ is intersected by a contour in Γ.

For a contour γ, let $|\gamma|$ be its length:

$$|\gamma|=\#\{\langle ij\rangle\in\gamma\}$$

and let

$$\boldsymbol{\gamma}=\{i\in\Lambda|\exists j,\ \langle ij\rangle\in\gamma\}$$

$$|\Gamma|=\sum_{i=1}^{n}|\gamma_i|$$

$$\boldsymbol{\Gamma}=\bigcup_{i=1}^{n}\boldsymbol{\gamma}_i.$$

B. The Expansion in the Pure Phases (High Temperature Expansion)

We define new variables r_i' by $r_i=\left(1-\frac{r_i'}{\mathrm{Re}\beta}\right)$ or $r_i'=\mathrm{Re}\beta(1-r_i)$.

For $\langle ij\rangle\notin\Gamma$, we write

$$r_ir_j-1=\frac{r_i'r_j'}{(\mathrm{Re}\beta)^2}-\frac{r_i'}{\mathrm{Re}\beta}-\frac{r_j'}{\mathrm{Re}\beta}$$

so that Z_Γ becomes:

$$Z_\Gamma=\int\prod_{\substack{\langle ij\rangle\in\Gamma\\ \langle ij\rangle\subset\Lambda}}\exp(-\beta(r_ir_j+1))\prod_{\substack{i\in\partial\Lambda\\ \sigma_i=-1}}\exp(-\beta h_i(r_i+1))$$
$$\cdot\prod_{\langle ij\rangle\notin\Gamma}\exp\left(\beta\frac{r_i'r_j'}{(\mathrm{Re}\beta)^2}\right)\prod_{i\in\Lambda}\exp\left(-\frac{\beta c_ir_i'}{\mathrm{Re}\beta}\right)\prod_{i\in\boldsymbol{\Gamma}}dr_i\prod_{i\notin\boldsymbol{\Gamma}}\frac{dr_i'}{\mathrm{Re}\beta},$$

where the integral is over $r_i\in[0,1]$, $i\in\boldsymbol{\Gamma}$, $r_i'\in[0,\mathrm{Re}\beta]$, $i\notin\boldsymbol{\Gamma}$,

$$c_i=\#\{j\in\mathbb{Z}^2\,|\,\|i-j\|=1 \text{ and } \langle ij\rangle\notin\Gamma\}$$
$$\cdot(c_i=4 \text{ if } i\notin\boldsymbol{\Gamma}).$$

For $\langle ij\rangle \notin \Gamma$, we write

$$\exp\left(\frac{\beta r_i' r_j'}{(\mathrm{Re}\beta)^2}\right) = \left(\exp\left(\frac{\beta r_i' r_j'}{(\mathrm{Re}\beta)^2}\right) - 1 + 1\right)$$

and we expand the product over $\langle ij\rangle \notin \Gamma$. We obtain a sum over sets X of nearest-neighbour pairs contained in $\Gamma^c = \{\langle ij\rangle \notin \Gamma\}$:

$$Z_\Gamma = \sum_{X \subset \Gamma^c} \tilde{\phi}(\Gamma, X) \prod_{i \notin \mathbf{\Gamma} \cup \mathbf{X}} \int_0^{\mathrm{Re}\beta} \exp\left(-\frac{4\beta r_i'}{\mathrm{Re}\beta}\right) \frac{dr_i'}{\mathrm{Re}\beta} \tag{7}$$

with $\mathbf{X} = \{i \in \Lambda | \exists j, \langle ij\rangle \in X\}$ and

$$\begin{aligned}\tilde{\phi}(\Gamma, X) = \int & \prod_{\substack{\langle ij\rangle \in \Gamma \\ \langle ij\rangle \subset \Lambda}} \exp(-\beta(r_i r_j + 1)) \prod_{\substack{i \in \partial\Lambda \\ \sigma_i = -1}} \exp(-\beta h_i(r_i + 1)) \\ & \cdot \prod_{\langle ij\rangle \in X} \left(\exp\left(\frac{\beta r_i' r_j'}{(\mathrm{Re}\beta)^2}\right) - 1\right) \prod_{i \in \mathbf{X} \cup \mathbf{\Gamma}} \exp\left(-\frac{\beta c_i r_i'}{\mathrm{Re}\beta}\right) \\ & \cdot \prod_{i \in \mathbf{\Gamma}} dr_i \prod_{i \in \mathbf{X} \setminus \mathbf{\Gamma}} (dr_i' / \mathrm{Re}\beta),\end{aligned} \tag{8}$$

where the integral is as in (6).
So

$$Z_\Gamma = \exp(\beta c_\Lambda) \sum_{\Gamma, X} \tilde{\phi}(\Gamma, X) \prod_{i \notin \mathbf{\Gamma} \cup \mathbf{X}} \int_0^{\mathrm{Re}\beta} \exp\left(-\frac{4\beta r_i'}{\mathrm{Re}\beta}\right) \frac{dr_i'}{\mathrm{Re}\beta}. \tag{9}$$

X can be viewed as a set of Mayer (high-temperature) graphs.

We say that (Γ, X) is connected it any two points in $\mathbf{\Gamma} \cup \mathbf{X}$ can be joined by a path of nearest-neighbour points in $\mathbf{\Gamma} \cup \mathbf{X}$. We decompose $\Gamma \cup X$ into connected components and we notice that

a) the sum (9) runs over all admissible families (i.e. pairwise disconnected) of connected (Γ, X).

b) $\tilde{\phi}(\Gamma, X)$ factorizes over the connected components of (Γ, X).

The term with $\Gamma = X = \emptyset$ in (9) is $g(\beta)^{|\Lambda|}$ with

$$g(\beta) = \int_0^{\mathrm{Re}\beta} \exp\left(-\frac{4\beta r}{\mathrm{Re}\beta}\right) \frac{dr}{\mathrm{Re}\beta}. \tag{10}$$

We divide both sides of (9) by $\exp(\beta c_\Lambda) g(\beta)^{|\Lambda|}$ and we obtain

$$\begin{aligned}\tilde{Z}_\Lambda &= Z_\Lambda \exp(-\beta c_\Lambda) g(\beta)^{-|\Lambda|} \\ &= \sum_{(\Gamma, X)} \prod_i \phi(\Gamma_i, X_i),\end{aligned} \tag{11}$$

where the sum is over all admissible families of (Γ, X) and the product is over the connected components

$$\phi(\Gamma, X) = \frac{\tilde{\phi}(\Gamma, X)}{g(\beta)^{|\mathbf{\Gamma} \cup \mathbf{X}|}}. \tag{12}$$

(11) is the expression used for our expansion. Now we state and prove the basic estimate, which controls the factors $\phi(\Gamma, X)$ in (11).

Lemma 1. *There exist constants c, c′, and K such that for*

$$\beta \in \mathscr{D} = \left\{ \beta \in \mathbb{C} | \mathrm{Re}\beta > c, \frac{|\mathrm{Im}\beta|}{\mathrm{Re}\beta} \leqq c' \right\}$$

$$|\phi(\Gamma, X)| \leqq \exp\left(-\frac{\mathrm{Re}\beta}{2} |\Gamma| \right) (K\,\mathrm{Re}\beta)^{-|X|}. \tag{13}$$

Proof. We start by estimating the contribution of the contours in $\tilde{\phi}(\Gamma,X)$ (12). Since $r_i \geqq 0$.

$$\left| \prod_{\langle ij \rangle \in \Gamma} \exp(-\beta(r_i r_j + 1)) \prod_{\substack{i \in \partial\Lambda \\ \sigma_i = -1}} \exp(-\beta h_i(r_i + 1)) \right| \leqq \exp(-\beta|\Gamma|). \tag{14}$$

We also need a lower bound on $g(\beta)$:

$$\begin{aligned} (\mathrm{Re}\beta) g(\beta) &= \int_0^{\mathrm{Re}\beta} \exp\left(-\frac{4i\,\mathrm{Im}\beta r}{\mathrm{Re}\beta} \right) \exp(-4r) dr \\ &= \frac{1 - \exp(-4(\mathrm{Re}\beta + i\,\mathrm{Im}\beta))}{4\left(1 + i\dfrac{\mathrm{Im}\beta}{\mathrm{Re}\beta}\right)} \end{aligned} \tag{15}$$

so that $(\mathrm{Re}\beta) g(\beta) \geqq \frac{1}{c_1}$ for $\beta \in \mathscr{D}$, with c_1 a constant.

Then using $|\mathbf{X} \cup \mathbf{\Gamma}| \leqq |\mathbf{X}| + |\mathbf{\Gamma}| \leqq 2(|X| + |\Gamma|)$,

$$\begin{aligned} |\phi(\Gamma,X)| &\leqq ((c_1\,\mathrm{Re}\beta)^2 \exp(-\mathrm{Re}\beta))^{|\Gamma|} (c_1)^{2|X|} \\ &\cdot \int_0^{\mathrm{Re}\beta} \prod_{\langle ij \rangle \in X} \left| \exp\left(\frac{\beta r_i' r_j'}{(\mathrm{Re}\beta)^2} \right) - 1 \right| \prod_{i \in \mathbf{X}} \exp(-c_i r_i') dr_i'. \end{aligned} \tag{16}$$

[We have cancelled the factor $(\mathrm{Re}\beta)^{|\mathbf{X}|}$ in the numerator and denominator of $\phi(\Gamma,X)$.] We estimate

$$\left| \exp\left(\frac{\beta r_i' r_j'}{(\mathrm{Re}\beta)^2} \right) - 1 \right| \leqq \left(\frac{\sqrt{1 + c'^2}}{\mathrm{Re}\beta} r_i' r_j' \right) \exp\left(\frac{|\beta| r_i' r_j'}{(\mathrm{Re}\beta)^2} \right).$$

But,

$$\frac{r_i' r_j'}{\mathrm{Re}\beta} \leqq \tfrac{1}{2}(r_i' + r_j') \quad \text{if} \quad r_i', r_j' \in [0, \mathrm{Re}\beta]$$

and

$$\sum_{\langle ij \rangle \in X} \frac{r_i' r_j'}{\mathrm{Re}\beta} \leqq \tfrac{1}{2} \sum_{i \in \mathbf{X}} d_i r_i'$$

with

$$d_i = \# \{ j | \langle ij \rangle \in X \} \leqq c_i.$$

Hence, taking $\frac{|\beta|}{\mathrm{Re}\beta} \leqq \sqrt{1 \pm c'^2} \leqq \frac{3}{2}$, we obtain a bound on the integral (16) of the form

$$\left(\frac{2\,\mathrm{Re}\beta}{3}\right)^{-|X|} \int\limits_0^{\mathrm{Re}\beta} \prod_{i\in \mathbf{X}} (r_i')^{d_i} \exp\left(-\frac{c_i}{4} r_i'\right) dr_i \leqq (\mathrm{const})^{|\mathbf{X}|} \leqq (\mathrm{const})^{2|X|} \tag{17}$$

because $c_i \geqq 1$, $d_i \leqq 4$ and $\mathbf{X} \leqq 2|X|$. Using (16) and (17), and $c(\mathrm{Re}\beta)^2 \exp(-\mathrm{Re}\beta) \leqq \exp\left(-\frac{\mathrm{Re}\beta}{2}\right)$ for $\mathrm{Re}\beta$ large, finishes the proof.

IV. Proof of Theorem 1

A. The Algebraic Formalism

We first recall what we need from the algebraic formalism; see [3] for details. Let us call a polymer any connected component of $\Gamma \cup X$ and let $\mathscr{P}$ be the set of all possible such polymers contained in any $\Lambda \subseteqq \mathbb{Z}^2$. We write $p_1, p_2 \ldots$ for the elements of $\mathscr{P}$. Let $\mathscr{F}$ be the set of functions from $\mathscr{P}$ to $\mathbb{N}$ which are zero except on a finite set. For $P \in \mathscr{F}$ we write

$$\begin{aligned} &\phi(P)=0 \quad \text{if} \quad P(p)>1 \quad \text{for some } p \\ &\phi(P)=\prod_{i=1}^{n} \phi(p_i) \prod_{i,j=1}^{n} (1+g(p_i,p_j)) \end{aligned} \tag{18}$$

if $P(p)=0$ or 1 for all p.

Calling $p_1, \ldots, p_n$ the elements of P for which $P(p)=1$, we define $\phi(p)$ by (12) and set

$$g(p,p')\begin{cases} =0 & \text{if } p \text{ and } p' \text{ are disconnected} \\ =1 & \text{otherwise.} \end{cases}$$

This allows us to rewrite our modified partition function $\tilde{Z}_\Lambda$ as:

$$\tilde{Z}_\Lambda = \sum_{P\subset \Lambda} \phi(P) \tag{19}$$

where $P \subset \Lambda$ means $P(p)=0$ if $p \not\subset \Lambda$.

We next define $\phi^T(P)$ as the coefficients of the formal power series $\log\left(\sum_P \phi(P) X_P\right)$, with $X_P = \prod_p X_p^{P(p)}$, explicitly,

$$\phi^T(P) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum{}' \prod_{i=1}^{n} \phi(P_i), \tag{20}$$

where the sum $\sum'$ runs over all $P_1, \ldots, P_n$, such that $P_i \neq 0$ and $\sum_{i=1}^{n} P_i = P$ (the addition being defined pointwise). Thus, formally at least,

$$\tilde{Z}_\Lambda = \exp \sum_{P\subset \Lambda} \phi^T(P). \tag{21}$$

Lemma 2 below gives an estimate on the $\phi^T(P)$ which justifies (21). To prove the lemma, we shall use two simple "entropy" estimates. Let i be a point in $\mathbb{Z}^2$, then

$$\#\{p \,|\, |p| = |\Gamma| + |X| = n,\ \mathbf{p} = \mathbf{\Gamma} \cup \mathbf{X} \ni i\} \leqq 8^{2n}, \tag{22}$$

$$\#\{p \,|\, |p| = n,\ \mathrm{Int}\, p \ni i\} \leqq 8^{2n} n^2, \tag{23}$$

where $\mathrm{Int}\, p = (\mathrm{Int}\, \Gamma) \cup \mathbf{X}$ and the interior of a contour is the set of sites where $\sigma_i = -1$ in the configuration where this contour is the unique contour. The interior of a family of contours is the union of their interiors.

Lemma 2. *Let* $\phi(P)$, $\phi^T(P)$ *be as defined in* (18), (20) *but with* $\phi(p)$ *any complex number not necessarily of the form* (12). *Then*

a) $\phi^T(P) = 0$ *if* $P = P_1 + P_2$ *with the property that for each* p *with* $P_1(p) \neq 0$, *and each* p' *with* $P_2(p') \neq 0$, p *is disconnected from* p'.

b) *There exists a constant* c *such that, for all* $K > 28 \ln 2$, *if* $|\phi(p)| \leqq \exp(-K|p|)$, *then*

$$\sum_{\mathrm{Int}\, P \ni i} |\phi^T(P)| \exp\left(\frac{K}{4}|P|\right) \leqq c, \quad \mathrm{Int}\, P = \bigcup_{P(p) \neq 0} \mathrm{Int}\, p \quad \text{and} \quad |P| = \sum_{P(p) \neq 0}.$$

Proof. This is similar to the proof in [3] where contours play the role of polymers. Part a) is explained in Eq. (4.21) of [3] and part b) is essentially Eq. (4.33) with the following modifications: one defines

$$I_m = \sup_P \sum_{P'} |\Delta_P(P')| \exp\left(\frac{K}{2}|P| + \frac{K}{4}|P'|\right), \quad N(P) + N(P') = m$$

instead of (4.20) in [3]. $N(P) = \sum_p P(p)$. The $\Delta_P(P')$ satisfy recursion relations like (4.25) and, for K large enough, one concludes as in [3] that $\sum_{m=1}^{\infty} I_m < \infty$ uniformly in K. Now, $\phi^T(p+P) = \Delta_p(P) \dfrac{P'}{(P+p')!}$ where p is to be identified with the function $\delta_{pp'}$ (see (4.31) in [3]) and therefore

$$\sum_{\substack{P \\ i \in \mathrm{Int}\, P}} |\phi^T(P)| \exp\left(\frac{K}{4}|P|\right) \leqq \sum_{\substack{p: \\ i \in \mathrm{Int}\, p}} \exp\left(-\frac{K}{4}|p|\right) \sum_P |\phi^T(p+P)| \exp\left(\frac{K}{2}|p| + \frac{K}{4}|P|\right)$$

$$\leqq \sum_{\substack{p: \\ i \in \mathrm{Int}\, p}} \exp\left(-\frac{K}{4}|p|\right) \sum_{m=1}^{\infty} I_m < \infty \text{ uniformly in } K.$$

B. Proof of Theorem 1

a) The free Energy. 1) It is clear that Lemma 2 is applicable to $\phi(p)$ defined in (12) for $\beta \in \mathscr{D}$ because of Lemma 1 [K may be taken of the order of $\log(\mathrm{Re}\, \beta)$]. This shows that (21) holds and hence we have

$$\log Z_\Lambda = \beta c_\Lambda + |\Lambda| \log g(\beta) + \sum_{P \subset \Lambda} \phi^T(P). \tag{24}$$

From (24), (15), and Lemma 2, it is clear that $\frac{1}{|\Lambda|}\log Z_\Lambda$ is uniformly bounded on compact subsets of $\mathcal{D}$. Then since Z_Λ is analytic in β, Vitali's theorem implies that $\psi(\beta)$ is analytic in β [one may also prove the convergence in $\mathcal{D}$ directly using (24) and Lemma 2].

2) Concerning the asymptotic expansion (point 2) we take β real; it is clear that $g(\beta)\simeq 1/4\beta$ up to an exponentially small error [see (15)]. So we need only bound the remainder of the asymptotic expansion of $|\Lambda|^{-1}\sum_{P\subset\Lambda}\phi^T(P)$ uniformly in Λ. By Lemma 1, we may take K in Lemma 2 of the order of $\log\beta$ and therefore by Lemma 2, we can disregard all terms with $|P|>4n$ when considering the expansion up to order n. Also, since $\phi^T(P)$ is a sum of products of $\phi(P)$ all terms where $P(p)\neq 0$ for some $p=(\Gamma,X)$ with $\Gamma\neq\emptyset$ are exponentially small as $T\to 0$ [see (13)]. Thus, it is only necessary to expand $\phi^T(P)$'s which are linear combinations of products of the form $\prod_p \phi(p)$ with $p=(\emptyset,X)$:

$$\phi(p)=\tilde{\phi}(p)/g(\beta)^{|X|}=4^{|\mathbf{X}|}\int_0^\beta \prod_{\langle ij\rangle\in X}\left(\exp\left(\frac{r_i'r_j'}{\beta}\right)-1\right)\prod_{i\in\mathbf{X}}\exp(-4r_i')dr_i'$$
$$+\text{exponentially small terms}.$$

We do this simply by expanding $\left(\exp\left(\frac{r_i'r_j'}{\beta}\right)-1\right)$ in a series up to the desired order. By estimates similar to those used in the proof of Lemma 1, one shows that the remainder is of higher order. We then let, in each term, the integration run from 0 to ∞: this produces again an exponentially small error.

b) The Correlation Functions. We expand the numerator and the denominator of $\langle s_A\rangle_\Lambda$ in the same way as we did the partition function i.e. for the denominator. We obtain

$$\langle s_A\rangle_\Lambda=\left(\sum_{P\subset\Lambda}\phi_A(P)\Big/\sum_{P\subset\Lambda}\phi(P)\right)\prod_i(g_i(\beta)/g(\beta)),$$

where

$$\phi_A(P)\begin{cases}=0 \quad \text{if} \quad P(p)>1 & \text{for some } p\\ =\sigma_A(\Gamma)\prod_p\phi_A(p), \quad \text{if} \quad P(p)=0 \text{ or } 1 & \text{for all } p\end{cases}$$

with

$$\sigma_A(\Gamma)=\prod_{\gamma\in\Gamma}\prod_i\sigma_i(\gamma)^{A(i)}$$

$$\sigma_i(\gamma)=\begin{cases}+1, & \text{if} \quad i\notin\operatorname{Int}\gamma\\ -1, & \text{if} \quad i\in\operatorname{Int}\gamma\end{cases} \tag{25}$$

$$\phi_A(p)=\tilde{\phi}_A(p)\Big/\prod_{i\in\Lambda}g_i(\beta),$$

where $\tilde{\phi}_A(p)$ is defined as $\tilde{\phi}(p)$ in (8) but with factors of $\prod_{i\in\Gamma} r_i'^{A(i)} \prod_{i\notin\Gamma}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}$ in the integrand of (8)

$$g_i(\beta)=\int_0^{\operatorname{Re}\beta}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp\left(-\frac{4\beta r_i'}{\operatorname{Re}\beta}\right)\frac{dr_i'}{\operatorname{Re}\beta}. \tag{26}$$

Formula (25) was first introduced, for Ising spin $\frac{1}{2}$ systems, in [3].

The difference between $\phi_A(P)$ and $\phi(P)$ comes from the fact that the term with $\Gamma=X=\emptyset$ is different in the numerator and in the denominator. Note however that

$$\begin{aligned}&\phi_A(p)=\phi(p) \text{ unless } \operatorname{Int} p\cap \operatorname{supp} A\neq\emptyset\\ &\operatorname{supp} A=\{i|A(i)\neq 0\}.\end{aligned} \tag{27}$$

We also have the analogue of Lemma 1 for $\phi_A(P)$:

Lemma 3. *There exist constants c, c′, and K such that for*

$$\beta\in\mathscr{D}=\left\{\beta\in\mathbb{C}|\operatorname{Re}\beta>c, \frac{|\operatorname{Im}\beta|}{\operatorname{Re}\beta}\leqq c'\right\}$$

and all m.f. A,

$$|\phi_A(\Gamma,X)|\leqq\exp\left(-\frac{\operatorname{Re}\beta}{2}|\Gamma|\right)(K\operatorname{Re}\beta)^{-|X|}. \tag{28}$$

We prove Lemma 3 after finishing the proof of Theorem 1.

By Lemma 3, we may define $\phi_A^T(P)$ which satisfy the same estimates as $\phi^T(P)$ (by Lemma 2). Moreover, by (27)

$$\phi_A^T(P)=\phi^T(P) \text{ unless } \operatorname{Int} P\cap\operatorname{supp} A\neq\emptyset \tag{29}$$

we therefore have

$$\langle s_A\rangle_\Lambda=\prod_{i\in A}\frac{g_i(\beta)}{g(\beta)}\exp\left(\sum_{A\subset\Lambda}\phi_A^T(P)-\phi^T(P)\right) \tag{30}$$

and the exponent is bounded by

$$\sum_{\substack{P:\\ \operatorname{Int} P\cap\operatorname{supp} A\neq\emptyset}}[|\phi_A^T(P)|+|\phi^T(P)|]<\text{const}$$

uniformly in $\beta\in\mathscr{D}$ by Lemma 2.

This, and Vitali's theorem, proves analyticity for $\varrho^+(s_A)$. The asymptotic expansion is obtained in a similar way as for $\psi(\beta)$.

c) Exponential Clustering. We know from (30) that $\langle s_A\rangle_\Lambda$ and $\langle s_B\rangle_\Lambda$ are nonzero, we may therefore consider the quantity

$$\frac{\langle s_A s_B\rangle_\Lambda}{\langle s_A\rangle_\Lambda\langle s_B\rangle_\Lambda}-1=\exp\left(\sum_{P\subset\Lambda}\phi_{A\cup B}^T(P)-\phi_A^T(P)-\phi_B^T(P)+\phi^T(P)\right)-1. \tag{31}$$

The terms in the exponent vanish unless

$$\operatorname{Int} P\cap\operatorname{supp} A\neq\emptyset \quad\text{and}\quad \operatorname{Int} P\cap\operatorname{supp} B\neq\emptyset.$$

Therefore, by Lemma 2, the exponent is of order $\exp\left(-\frac{K}{4}\operatorname{dist}(A,B)\right)$. Using $|e^x-1|\leqq|x|e^{|x|}$ proves the exponential cluster property with $m(\beta)>0$.

In order to show that $\lim_{\beta\to\infty}\frac{m(\beta)}{\log\beta}=1$ we use the F.K.G. inequalities as in [10] to reduce ourselves to the case of the two point correlation function, $\varrho^+(s_0s_i)-(\varrho^+(s_0))^2$. Since we take as metric $|i|=\max_\alpha|i_\alpha|$, it is enough by reflection positivity to consider i along one coordinate axis. Then we can compute the behaviour of $m(\beta)$ as $\beta\to\infty$ from (31) with $s_A=s_0$ and $s_B=s_i$; we look at the largest term in the exponent of (31) and we find that it corresponds to a graph connecting 0 and i with $|i|$ nearest-neighbour pairs. This graph is of order $T^{|i|}$. Using the estimates on ϕ^T that we have, we obtain a bound of the form $T^{|i|}c^{|i|}$ for the exponent in (31) and this proves the behaviour of $m(\beta)$ as $\beta\to\infty$.

Proof of Lemma 3. The proof is very similar to the proof of Lemma 1 except that $g_i(\beta)$ in the denominator of $\phi_A(P)$ is more difficult to bound from below. We write (26) as

$$\begin{aligned}(\operatorname{Re}\beta)g_i(\beta)&=\int_0^{\operatorname{Re}\beta}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp(-4r_i')\exp\left(-4i\frac{\operatorname{Im}\beta}{\operatorname{Re}\beta}r_i'\right)dr_i'\\&=\int_0^{\operatorname{Re}\beta}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp(-4r_i')dr_i'\\&\quad+\int_0^{\operatorname{Re}\beta}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp(-4r_i')\left(\exp\left(-\frac{4i\operatorname{Im}\beta}{\operatorname{Re}\beta}r_i'\right)-1\right)dr_i'.\end{aligned}$$

The second term is estimated by $|e^{ix}-1|\leqq|x|e^{|x|}$ and is smaller than the first for $\frac{|\operatorname{Im}\beta|}{\operatorname{Re}\beta}\leqq c'$ small enough. The first term goes to zero as $A(i)$ becomes large but after estimating the product $\prod_{\langle ij\rangle\in X}\left(\exp\left(\frac{\beta r_i'r_j'}{(\operatorname{Re}\beta)^2}\right)-1\right)$ in the numerator as in Lemma 1, we are left with a ratio of the form [see the l.h.s. of (17)]

$$\prod_{i\in\mathbf{X}}\frac{\int_0^{\operatorname{Re}\beta}r_i'^{d_i}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp\left(-\frac{c_i}{4}r_i'\right)dr_i'}{\int_0^{\operatorname{Re}\beta}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp(-4r_i')dr_i'}. \tag{32}$$

We bound $r_i'^{d_i}\exp\left(-\frac{c_i}{4}r_i'\right)\leqq\text{const}\exp(-\delta r_i')$ for some $\delta>0$ and we change variables in the numerator $r_i''=\frac{\delta r_i'}{4}$. (32) is then bounded by

$$\left(\frac{4}{\delta}\right)^{|X|}\prod_{i\in X}\frac{\int_0^{\delta\operatorname{Re}\beta/4}\left(1-\frac{4r_i''}{\delta\operatorname{Re}\beta}\right)^{A(i)}\exp(-4r_i'')dr_i''}{\int_0^{\operatorname{Re}\beta}\left(1-\frac{r_i'}{\operatorname{Re}\beta}\right)^{A(i)}\exp(-4r_i')dr_i'}.$$

The numerator in this last expression has the same form as the denominator with ($\delta\,\mathrm{Re}\,\beta/4$) instead of $\mathrm{Re}\,\beta$. Since the integral increases with $\mathrm{Re}\,\beta$, the ratio is bounded by one.

Remarks. 1. If we define $\tilde{m}(\beta)$, the "true" exponential decay rate as

$$\tilde{m}(\beta) = -\lim_{|i|\to\infty} \frac{1}{|i|} \log(\varrho^+(s_0 s_i) - \varrho^+(s_0)^2) \tag{33}$$

with i along a coordinate axis, then one can use an argument similar to the one used in [11] for the Ising model to show $\lim_{\beta\to\infty} \frac{m(\beta)}{\log\beta} = 1$. The argument is as follows:

By reflection positivity, there exists a self adjoint transfer matrix and we can write [11],

$$\varrho^+(s_0 s_i) - (\varrho^+(s_0))^2 = \int_0^1 \lambda^{|i|} d\mu(\lambda).$$

Then Hölder's inequality shows that (33) exists and moreover that

$$0 \leqq \varrho^+(s_0 s_i) - (\varrho^+(s_0))^2 \leqq \exp(-\tilde{m}(\beta)|i|). \tag{34}$$

From the proof of the Theorem, we know that $\lim_{\beta\to\infty} \frac{\tilde{m}(\beta)}{\log\beta} \geqq 1$. To get an upper bound on $\tilde{m}(\beta)$ it is enough, because of (34), to analyse the limit of $\varrho^+(s_0 s_i) - (\varrho^+(s_0))^2$ as $\beta\to\infty$ for fixed $|i|$. Since the largest contribution to that quantity is of order $T^{|i|}$ (see the proof of Theorem 1) we obtain the exact asymptotic behaviour of $\tilde{m}(\beta)$ as $\beta\to\infty$.

It is worthwhile noting that in the Ising model, $\tilde{m}(\beta)$ is of the order of β at low temperature [11], because only contours ($\sim\exp(-\beta)$) contribute to the expansion and that in the field theory case the mass gap tends to a finite value [2] (the curvature of the potential at the minimum of the well) while in this case $\tilde{m}(\beta)$ behave "like" it would in the high-temperature limit, $m(\beta) \sim |\log\beta|$ as $\beta\to 0$.

2. One may ask: does our asymptotic expansion completely determine the free energy and the correlation functions of our model at some fixed T? First, it is clear that the coefficients of our expansion depend only on the behaviour of the single-spin distribution around $+1$ and -1. That is, if instead of the Lebesgue measure we had a single-spin distribution $\chi_a(s_i)ds_i$ where χ_a is the characteristic function of $[-1, -1+a] \cup [1-a, 1]$ for any $a \leqq 1$, we would obtain the same asymptotic expansions. However, we do not know whether for some value of a ($a=0$?) the free energy and the correlation functions would be Borel summable.

V. General Ferromagnetic Systems

In this section we extend Theorem 1 to arbitrary finite range ferromagnetic interactions on a lattice $\mathbb{Z}^d$ and a large class of single-spin distributions. We state our results for the free energy only, but in some cases the correlation functions can also be treated (see the remarks).

The Hamiltonian, in the notation used before, is

$$-H_\Lambda = \sum_A J(A) s_{A \cap \Lambda}, \tag{35}$$

where the sum is over all multiplicity functions and

$$\begin{aligned}(A \cap \Lambda)(i) &= A(i) \quad \text{if} \quad i \in \Lambda \\ &= 0 \qquad \text{if} \quad i \notin \Lambda .\end{aligned}$$

J is a real-valued function defined on m.f. which is translation invariant. $J(A) = J(\tau^i A)$; ferromagnetic, $J(A) \geqq 0$; and with a finite fundamental family: $\operatorname{supp} J = \{A | J(A) \neq 0\}$ contains a finite subset $\mathscr{B}_0$ such that every element of $\operatorname{supp} J$ is the translate of exactly one element of $\mathscr{B}_2$. The Hamiltonian (35) corresponds to + b.c.

We put a somewhat technical restriction on the even single-spin distribution $v(s_i)$ which insures that it does not give too little weight to the neighbourhood of the ground states $+1$ and -1: either

$$v(\{1\}) = v(\{-1\}) \neq 0$$

or there exist $\eta > 0$, n, $a < \infty$, $b \neq 0$ such that in $[1, 1-\eta]$, $\frac{dv(s)}{ds} = f(s)$ exists and satisfies

$$b \leqq \frac{f(s)}{(1-s)^n} \leqq a . \tag{36}$$

Defining the free energy $\psi(\beta)$ as in (3) we have

Theorem 2. *For each H given by* (34) *and each v satisfying* (36), *there exists a complex domain*

$$\mathscr{D} = \{\beta \in \mathbb{C} | \operatorname{Re} \beta \geqq c, \frac{|\operatorname{Im} \beta|}{\operatorname{Re} \beta} \leqq c'\}$$

such that $\psi(\beta)$ is analytic in $\mathscr{D}$.

Moreover, if we assume that $f(s) = \frac{dv}{ds}$ exists in a neighbourhood of 1, and has an asymptotic expansion in powers of $(1-s)$ for which n is the order of the first non-zero coefficient, then

$$\psi(\beta) - \beta \sum_{A \in \mathscr{B}_0} J(A) + (n+1) \log \beta$$

has an asymptotic expansion in powers of T.

Remarks. 1. In the case where $v(\{1\}) = v(\{-1\}) \neq 0$ and is isolated from the rest of the support of v, $\psi(\beta) - \beta \sum_{A \in \mathscr{B}_0} J(A)$ is exponentially small as $\beta \to \infty$, as in the spin $\frac{1}{2}$ Ising model.

2. Theorem 2 combined with the theorem of [4] gives the corollary of [4], which leads to a description of all phases corresponding to H and v, for β large enough.

3. In the case where the system has the decomposition property [12] one may extend the analyticity results to the correlation functions (in some neighbourhood of the real axis for Re β large) and also obtain exponential decay of the correlation functions as in Theorem 1.

4. Theorem 2 can be extended to a somewhat larger class of single-spin distributions requiring only that $\nu([1-\delta, 1]) \geqq \exp\left(-\frac{c}{\delta^\alpha}\right)$, all $\delta < \delta_0$, for some $\alpha < \frac{1}{2}$ and $c < \infty$. The proof uses a contour expansion where $[-1, +1]$ is split into three regions (instead of two) which moreover are chosen in a β dependent way: as β increases, one shrinks the neighbourhoods of the ground states $+1$ and -1 where the spins are outside of the contours and where the high-temperature expansion is made.

Proof of Theorem 2. a) The Contour Expansion. From the proof of Theorem 1, we see that it is enough to be able to write a formula like (11) for the partition function and to have on each term an estimate like Lemma 1, together with a entropy estimate on the number of connected "polymers".

We write $s_i = \sigma_i r_i$, with $\sigma_i = \pm 1$, $r_i \in [0, 1]$ and $s_B = \sigma_{\bar{B}} r_B$ where $\bar{B} = \{i | B(i) \text{ is odd}\}$. Then

$$\int F(\{s_i\}) \prod_{i \in \Lambda} d\nu(s_i) = \sum_{\sigma_i = \pm 1} \int F(\{\sigma_i r_i\}) \prod_{i \in \Lambda} d\tilde{\nu}(r_i), \tag{37}$$

where $\tilde{\nu}(r_i) = \nu(r_i) - \frac{1}{2}\nu(\{0\})$. We apply (37) to the partition function

$$\begin{aligned} Z_\Lambda &= \int \prod_{i \in \Lambda} d\nu_i(s_i) \exp(-\beta H_\Lambda) \\ &= \int \prod_{i \in \Lambda} d\tilde{\nu}_i(r_i) Z_{\Lambda, \mathbf{r}}. \end{aligned} \tag{38}$$

$\mathbf{r} = \{r_i\}_{i \in \Lambda}$; and $Z_{\Lambda, \mathbf{r}}$ is the partition function on an Ising spin $\frac{1}{2}$ model with interactions $J_r(B) = \sum_{\substack{A: \\ \bar{A} = B}} J(A) r_{A \cap \Lambda}$.

We start by expanding $Z_{\Lambda, \mathbf{r}}$ in contours in the usual way [12]: we define the contour of a configuration σ_i as the set of B with $B = \bar{A}$ for some $A \in \operatorname{supp} J$ where $\sigma_B = -1$. Since our lattice is $\mathbb{Z}^d$, the map between configurations and contours is injective [13] and the sum over $\sigma_i = +1$, $i \in \Lambda$, is equivalent to a sum over all possible contours in Λ. We deal first with the case where the Ising system has the decomposition property [12]. One says that a contour is *N-connected* if any two elements can be joined by a sequence of elements each of which is a distance less than N from the next one. The decomposition property means that there exists a N such that the set of contours is in one-to-one correspondence with the set of admissible (i.e. pairwise N-disconnected) families of N-connected contours. Then we may write:

$$Z_\Lambda = \sum_\Gamma Z_\Gamma,$$

where the sum is over all such families and

$$Z_\Gamma=\int\prod_{i\in\Lambda}d\tilde{\nu}(r_i)\prod_{B\in\Gamma}\exp\Big(-\beta\sum_{\bar{A}=B}J(A)r_{A\cap\Lambda}\Big)\prod_{B\neq\Gamma}\exp\Big(\beta\sum_{\bar{A}=B}J(A)r_{A\cap\Lambda}\Big)$$
$$=\exp\Big(\beta\sum_{\substack{A:\\ A\cap\Lambda\neq\emptyset}}J(A)\Big)\int\prod_{i\in\Lambda}d\tilde{\nu}(r_i)\prod_{B\in\Gamma}\exp\Big(-\beta\sum_{\bar{A}=B}J(A)(r_{A\cap\Lambda}+1)\Big)$$
$$\cdot\prod_{B\notin\Gamma}\exp\Big(\beta\sum_{\bar{A}=B}J(A)(r_{A\cap\Lambda}-1)\Big). \tag{39}$$

b) The Mayer Graphs. Now we introduce the variables r_i', $r_i=1-\frac{r_i'}{\operatorname{Re}\beta}$. We define y_A by $r_A-1=y_A-\frac{1}{\operatorname{Re}\beta}\sum_i r_i'A(i)$ and, by $A\notin\Gamma$ we mean $\bar{A}\notin\Gamma$. Then

$$\prod_{B\notin\Gamma}\exp\Big(\beta\sum_{\substack{A:\\ \bar{A}=B}}J(A)(r_{A\cap\Lambda}-1)\Big)$$
$$=\prod_{A\neq\Gamma}\exp(\beta J(A)(r_{A\cap\Lambda}-1))$$
$$=\prod_{A\notin\Gamma}(\exp(\beta J(A)y_{A\cap\Lambda})-1+1)\exp\Big(\frac{-\beta}{\operatorname{Re}\beta}\sum_{i\in\Lambda}r_i'c_i\Big)$$

with

$$c_i=\sum_{A\notin\Gamma}J(A)A(i);\quad \text{if}\quad i\notin\mathbf{\Gamma}=\{i|\exists B\in\Gamma, B\ni i\}$$

then

$$c_i=c_0=\sum_A J(A)A(0).$$

Then we expand the product over $A\notin\Gamma$ into Mayer graphs. This expansion is similar to the one encountered in the proof of Theorem 1, except that the graphs are made with more general bonds than nearest-neighbour ones. We write then

$$Z_\Lambda=\sum_\Gamma Z_\Gamma=\sum_{\Gamma,X}\tilde{\phi}(\Gamma,X),$$

where $\tilde{\phi}(\Gamma,X)$ factorizes over connected pieces of $\mathbf{\Gamma}\cup\mathbf{X}=\{i\in\Lambda|\exists B\in\Gamma\cup X, B\ni i\}$. We define $\phi(\Gamma,X)$ by dividing $\tilde{\phi}(\Gamma,X)$ by the term with $\Gamma=X=\emptyset$ i.e.

$$\prod_{i\in\Lambda}\int_0^{\operatorname{Re}\beta}\exp\Big(-\frac{\beta}{\operatorname{Re}\beta}c_0r_i'\Big)d\nu_\beta(r_i'), \tag{40}$$

where $\nu_\beta(r_i')=\tilde{\nu}\Big(1-\frac{r_i'}{\operatorname{Re}\beta}\Big)$ is the measure $\tilde{\nu}(r_i)$ with r_i expressed as a function of r_i'. This defines $\phi(\Gamma,X)$.

c) Estimates. We have to obtain for $\phi(\Gamma,X)$ estimates similar to Lemma 1. The part coming from the contours is the same because $r_{A\cap\Lambda}+1\geqq 1$. This gives an exponentially small contribution $\exp-\big(\operatorname{Re}\beta\min_A J(A)|\Gamma|\big)$. The denominator (40) is bounded from below by $(\operatorname{Re}\beta)^{-n}$ using (36) as we shall see in the proof of Lemma 4 below. For the Mayer graphs, we have to estimate the ratio

$$\int_0^{\mathrm{Re}\,\beta} \prod_{i\in\mathbf{X}} d\nu_\beta(r_i') \prod_{A\in X} (\exp(\beta J(A) y_{A\cap\Lambda}) - 1) \exp\left(-\frac{\beta}{\mathrm{Re}\,\beta}\sum_{i\in\mathbf{X}} c_i r_i'\right) \tag{41}$$

divided by

$$\left(\int_0^{\mathrm{Re}\,\beta} \exp\left(-\frac{\beta c_0 r_i'}{\mathrm{Re}\,\beta}\right) d\nu_\beta(r_i')\right)^{|X|}. \tag{42}$$

For the numerator we use

$$|\exp(z) - 1| \leqq |z| \exp|z| \tag{43}$$

with $z = \beta J(A) y_{A\cap\Lambda}$. Since $r_i \leqq 1$,

$$\begin{aligned} r_A - 1 &\leqq \frac{1}{|A|}\sum_i (r_i - 1) A(i) \\ &= \frac{1}{\mathrm{Re}\,\beta |A|}\sum_i r_i' A(i). \end{aligned}$$

On the other hand,

$$y_A = \prod_i \left(1 - \frac{r_i'}{\mathrm{Re}\,\beta}\right)^{A(i)} - 1 + \frac{1}{\mathrm{Re}\,\beta}\sum_i A(i)$$

is positive: it is zero if all $r_i' = 0$ and its derivative with respect to any r_i' is positive.
So,

$$|\beta J(A) y_{A\cap\Lambda}| \leqq \frac{|\beta|}{\mathrm{Re}\,\beta} J(A)\left(1 - \frac{1}{|A|}\right)\sum_i r_i' A(i). \tag{44}$$

Choose c' small enough so that

$$\frac{|\beta|}{\mathrm{Re}\,\beta}\left(1 - \frac{1}{|A|}\right) \leqq 1 - k$$

for some $k > 0$ and all $A \in \mathscr{B}_0$. Then (44) gives

$$\exp|\beta|\left(\sum_{A\in X} J(A) y_{A\cap\Lambda}\right) \leqq \exp(1-k)\left(\sum_{i\in A} r_i' d_i\right) \tag{45}$$

with

$$d_i = \sum_{A\in X} J(A) A(i) \leqq c_i. \tag{46}$$

Inserting (43), (45), (46) in (41) gives a bound on (41):

$$\int_0^{\mathrm{Re}\,\beta} \prod_{i\in\mathbf{X}} d\nu_\beta(r_i') \prod_{A\in X} |\beta J(A) y_{A\cap\Lambda}| \exp\left(-k\sum_{i\in\mathbf{X}} d_i r_i'\right).$$

Now, $|\beta J(\mathrm{A}) y_{A\cap\Lambda}|$ is bounded by a sum of terms of the form $\frac{\mathrm{const}}{(\mathrm{Re}\,\beta)^l} r_B'$ with $l = |B| - 1 > 1$.

So each term will give a contribution of the order $(\mathrm{Re}\,\beta)^{-1}$, and the ratio of (41) and (42) will be bounded by $(K\,\mathrm{Re}\,\beta)^{-|X|}$ for some K if we prove the following

Lemma 4. *For β in $\mathscr{D}$*

$$\left|\frac{\int_0^{\operatorname{Re}\beta} d\nu_\beta(r_i')r_i'^{B(i)}\exp(-kr_i')}{\int_0^{\operatorname{Re}\beta} d\nu_\beta(r_i')\exp\left(-\frac{\beta}{\operatorname{Re}\beta}c_0r_i'\right)}\right| \leqq \operatorname{const}(k, c_0, \nu, B(i)).$$

The proof of Lemma 4 is at the end of this section.

Since $\mathscr{B}_0$ is finite, there are only finitely many values of $B(i)$ which may occur and, for a given interaction, our bounds on $|\phi(\Gamma,X)|$ are uniform, for $\beta \in \mathscr{D}$.

We may replace the entropy estimates (22), (23) used in the proof of Lemma 2 and Theorem 1 by more general ones which hold for general ferromagnetic interactions and which are essentially in [12, Proposition 2.5].

Then, to prove analyticity of the free energy we apply the same procedure as in Theorem 1: we define $\phi^T(\Gamma,X)$ and use Lemma 2 whose proof only depends on entropy estimates and the estimates on $|\phi(\Gamma,X)|$ that we just derived. For the asymptotic expansion we proceed as follows: in all the $\phi^T(\Gamma,X)$ we keep only the integration from 0 to $\eta \operatorname{Re}\beta$ and neglect an exponentially small term. Then we expand $f\left(1-\frac{r_i}{\beta}\right)$ as powers of T, which is equivalent to the expansion of $f(s)$ around $s=1$.

d) Finally, let us consider the case where the system does not possess the decomposition property; then the contour expansion is somewhat more difficult: we cannot write directly the sum analogous to (11) as a sum over all admissible families of connected objects and this is an essential step in order to apply the algebraic formalism. Indeed, unless we have a sum over connected objects, we do not have the entropy estimates that are used in the algebraic formalism. We have to use the theory of [12] and in particular Eq. (4.6) of [12], which reduces general interactions to ones with the decomposition property. Let J' be an interaction for a spin $\frac{1}{2}$ Ising system without the decomposition property. Then there exists a set D such that, for all B with $J'(B) \neq 0$, $\sigma_B = \sigma_{DC} = \prod_{i \in D} \prod_{j \in C} \sigma_{i+j}$ for some set C and if we define $J(C) = J'(DC)$ then J has the decomposition property. Moreover the partition functions for the two systems, with free b.c. (denoted by the subscript 0) i.e. restricting the sum in (35) to $A \subseteq \Lambda$, satisfy:

$$Z'_{\Lambda',0} = 2^{|\Lambda'|-|\Lambda|} Z_{\Lambda,0}, \tag{47}$$

where

$$\Lambda' = \bigcup_{i \in D} \Lambda + i. \tag{48}$$

To apply this, we take regions Λ' (instead of Λ) of the form (48) and use (47) in (38)

$$\begin{aligned} Z'_{\Lambda'} &= \int \prod_{i \in \Lambda'} d\tilde{\nu}(r_i) Z'_{\Lambda',\mathbf{r}} \\ &= \int \prod_{i \in \Lambda'} d\tilde{\nu}(r_i) F(\Lambda, \Lambda') Z_{\Lambda,\mathbf{r}}, \end{aligned}$$

where $Z'_{A'}$ corresponds to our original interaction, $Z'_{A,\mathbf{r}}$ is a spin $\frac{1}{2}$ partition function and $Z_{A,\mathbf{r}}$ also, but corresponds to the interaction $J(C)=\sum_{A=CD} J(A)r_A$

$$F(\Lambda,\Lambda')=2^{|\Lambda'|-|\Lambda|}\frac{Z_{\Lambda',\mathbf{r}}}{Z_{\Lambda',0,\mathbf{r}}}\frac{Z_{\Lambda,0,\mathbf{r}}}{Z_{\Lambda,\mathbf{r}}}$$

comes from the change of b.c. from free to $+$. $F(\Lambda,\Lambda')$ is bounded from above and below by $\exp O(|\partial\Lambda|)$ and does not contribute to $\psi(\beta)$ in the thermodynamic limit. So, all we have to do is bound $\frac{1}{|\Lambda'|}\log\int\prod_{i\in\Lambda'} d\tilde{v}(r_i)Z_{\Lambda,\mathbf{r}}$ uniformly in Λ and uniformly on compact subsets of $\mathscr{D}$. But for this we can expand $Z_{\Lambda,\mathbf{r}}$ in contours in Λ (for the reduced system i.e. with the decomposition property) and then do the Mayer expansion in $\Lambda'\supset\Lambda$. The rest of the proof is as before.

Proof of Lemma 4. For the denominator, we write

$$\exp\left(-\frac{\beta c_0 r_i'}{\mathrm{Re}\beta}\right)=\exp(-c_0r_i')\left(1+\exp\left(-i\frac{\mathrm{Im}\beta c_0 r_i'}{\mathrm{Re}\beta}\right)-1\right)$$

and we bound

$$\left|\exp\left(-i\frac{\mathrm{Im}\beta}{\mathrm{Re}\beta}c_0r_i'\right)-1\right|\leqq\frac{|Im\beta|}{Re\beta}(c_0r_i')\exp\left(\frac{|\mathrm{Im}\beta|}{\mathrm{Re}\beta}c_0r_i'\right).$$

We bound the denominator from below by

$$\left(\int_0^{\mathrm{Re}\beta} dv_\beta(r_i')\exp(-c_0r_i')\right)$$
$$\cdot\left(1-\frac{|\mathrm{Im}\beta|}{\mathrm{Re}\beta}c_0\frac{\int_0^{\mathrm{Re}\beta} r_i'\exp\left(\frac{|\mathrm{Im}\beta|}{\mathrm{Re}\beta}c_0r_i'-c_0r_i'\right)dv_\beta(r_i')}{\int_0^{\mathrm{Re}\beta}\exp(-c_0r_i')dv_\beta(r_i')}\right).$$

For $\frac{|\mathrm{Im}\beta|}{\mathrm{Re}\beta}\leqq c'$ sufficiently small the second term will be less than say $\frac{1}{2}$, if we show that the factor multiplying $\frac{|\mathrm{Im}\beta|}{\mathrm{Re}\beta}$ is finite. But this follows from the argument given below which bounds

$$\frac{\int_0^{\mathrm{Re}\beta} dv_\beta(r_i')r_i'^{B(i)}\exp(-kr_i')}{\int_0^{\mathrm{Re}\beta} dv_\beta(r_i')\exp(-c_0r_i')}\tag{49}$$

by a constant.

In order to do this, we restrict the integration in the denominator to $[0,\eta\,\mathrm{Re}\beta]$ where η comes from (36) and use the lower bounds in (36). For the numerator, we split the integration into $[0,\eta\,\mathrm{Re}\beta]$ and $[\eta\,\mathrm{Re}\beta,\,\mathrm{Re}\beta]$ and in the first integral we use the upper bound in (36). We obtain:

$$(49) \leqq \frac{a(\mathrm{Re}\beta)^{-n} \int_0^{\eta \mathrm{Re}\beta} (r_i')^{n+B(i)} \exp(-kr_i') dr_i'}{b(\mathrm{Re}\beta)^{-n} \int_0^{\eta \mathrm{Re}\beta} (r_i')^n \exp(-c_0 r_i') dr_i'} + \frac{\int_{\eta \mathrm{Re}\beta}^{\mathrm{Re}\beta} r_i'^{B(i)} \exp(-kr_i') dr_i'}{b(\mathrm{Re}\beta)^{-n} \int_0^{\eta \mathrm{Re}\beta} (r_i')^n \exp(-c_0 r_i') dr_i'}.$$

The first term is clearly uniformly bounded in $\mathrm{Re}\beta$ $\left[n\right.$ is fixed, depending on ν, $\sup_i B(i)$ depends only on the interaction$\left.\right]$.

For the second term, the numerator can be bounded by $(\mathrm{const})\exp\left(-\frac{k}{2}(\mathrm{Re}\beta)\eta\right)$ and the denominator does not go to zero faster than $(\mathrm{Re}\beta)^{-n}$. This proves the Lemma because in the case where $\nu(\{1\}) \neq 0$, the denominator does not even go to zero as $\mathrm{Re}\beta \to \infty$ and the bound is trivial.

Acknowledgement. It is a pleasure to thank J. Slawny and T. Spencer for useful discussions.

References

1. Minlos, R.A., Sinai, Y.: Trans. Moscow Math. Soc. **17**, 237 (1967); Math. USSR-Sbornik **2**, 355 (1967); Trans. Moscow Math. Soc. **19**, 121 (1968)
2. Glimm, J., Jaffe, A., Spencer, T.: Ann. of Phys. **101**, 610–630, 631–669 (1976)
3. Gallavotti, G., Martin-Löf, A., Miracle-Sole, S.: In: Statistical mechanics and mathematical problems. Battelle Recontres 1971. Lecture Notes in Physics, Vol. 20. Berlin, Heidelberg, New York: Springer 1973
4. Bricmont, J., Pfister, C.E., Lebowitz, J.L.: Periodic Gibbs states of ferromagnetic spin systems J. Stat. Phys. (to appear)
5. Lebowitz, J.L.: Number of phases in one component ferromagnets. In: Mathematical problems in theoretical physics, Proceedings, Rome 1977. Springer Lecture Notes in Physics No. 80
6. Wells, D.: Some moment inequalities and a result on multivariate unimodality. Thesis. Indiana University, 1977
7. Bortz, A., Griffiths, R.B.: Commun. Math. Phys. **26**, 102 (1972)
8. Griffiths, R.B.: Phase transitions. In: Les Houches Lectures 1970. De Witt, C., Stora, R. (eds.). New York: Gordon and Breach 1971
9. Kunz, H., Souillard, B.: Commun. Math. Phys. (to appear)
10. Lebowitz, J.L.: Commun. Math. Phys. **28**, 313 (1972)
11. Schor, R.: Commun. Math. Phys. **53**, 213 (1978)
12. Holsztynski, W., Slawny, J.: Commun. Math. Phys. **66**, 147 (1973)
13. Gruber, C.: General lattice systems. In: Proc. 2nd Int. Coll. on Group Theoretical Methods in Physics, Nijmegen 1973. Janner, A., Janssen, T. (eds.). Nijmegen (1974)

Communicated by A. Jaffe

Received June 3, 1980