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Lattice Systems with a Continuous Symmetry

III. Low Temperature Asymptotic Expansion for the Plane Rotator Model

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Abstract. We prove that the expansion in powers of the temperature T of the correlation functions and the free energy of the plane rotator model on a d-dimensional lattice is asymptotic to all orders in T. The leading term in the expansion is the spin wave approximation and the higher powers are obtained by the usual perturbation series. We also prove the inverse power decay of the pair correlation at low temperatures for $d=3$.

I. Introduction

We investigate the low temperature properties of the classical plane rotator model described by the Hamiltonian:

$$-\beta H=\beta \sum_{\langle ij\rangle} \cos(\phi_i-\phi_j), \qquad \phi_i\in[-\pi,\pi], \tag{1}$$

β is the inverse temperature T and $\langle i,j\rangle$ are nearest neighbor pairs of sites on the d-dimensional simple cubic lattice Z^d.

It has been known for a long time that the SO(2) symmetry of this model is only broken in $d\geqq 3$ where there is a spontaneous magnetization at low temperatures [8, 13]. These results provide a qualitative justification of the spin wave picture. In this paper we prove that in any dimension d, the free energy and the correlations have a low temperature expansion about the spin wave approximation valid to all orders in T. In particular we show how to get higher order correction in T to the spontaneous magnetization ($d\geqq 3$). The zeroth order value for the spontaneous magnetization was obtained in [13].

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The formal expansion in T is obtained [6, 11] by making a change of variables, $\phi_i = \sqrt{T}\phi'_i$, and then expanding the cosine into a power series so that up to a constant,

$$-\beta H = -\frac{1}{2}\sum_{\langle ij\rangle}(\phi'_i-\phi'_j)^2 + \frac{1}{4!}T\sum_{\langle i,j\rangle}(\phi'_i-\phi'_j)^4 + \dots, \qquad \phi'_i \in [-\pi\sqrt{\beta}, \pi\sqrt{\beta}].$$

We see that there are two perturbations of the massless gaussian field (the spin wave approximation): The first one is the power series in T for $-\beta H$ and the other one is the restriction of ϕ' to the interval $|\phi'_i| \leqq \pi\sqrt{\beta}$, i.e. the Gibbs factor $\exp[-\beta H]$ has to be multiplied by a product of the characteristic functions $\chi(|\phi'_i| \leqq \pi\sqrt{\beta})$. The first perturbation, at least when the series is truncated at a given order, can be treated using methods developed in [3] (Part I of this series). The aim of this paper is to get rid of the second one (and of the truncation of the power series). This is done using infrared bounds [13]. We prove that the contribution of this second perturbation is exponentially small in T (when $T \to 0$) and therefore does not appear in the power series expansion. In two dimensions the formal perturbation theory is not defined for functions like the spontaneous magnetization which vanishes. However we can compute asymptotics for all nonvanishing correlations.

As in [3], our method does not give directly any results about the decay of the correlations nor about analyticity in T for $T \neq 0$. However, using ideas from [4] and an improvement of the results in [25] we show, for instance, that the two point function $\langle \sin\phi_0 \sin\phi_x\rangle$ behaves exactly like $|x|^{-1}$ for $d=3$ whenever there is a spontaneous magnetization.

The outline of the paper is as follows: In Sect. II we describe the model and some of its known properties that are used later. In Sect. III we state and prove the main result. In Sect. IV we give an alternative proof of a part of the theorem. In Sect. V we study the decay of the two point correlation function.

II. Definition of the Model

Let H_Λ be the Hamiltonian defined in (1) with periodic boundary conditions on a parallelepiped $\Lambda \subset Z^d$, centered on the origin. We also consider the Hamiltonian:

$$\beta H_{\Lambda,h} = \beta H_\Lambda - h\sum_{i\in\Lambda}\cos\phi_i. \tag{2}$$

A probability measure on $[-\pi,\pi]^{|\Lambda|}$ is defined ($|\Lambda|$ = number of sites in Λ) by:

$$d\mu_{\Lambda,h} = Z_{\Lambda,h}^{-1}\exp\left\{\beta\sum_{\langle i,j\rangle\subset\Lambda}\cos(\phi_i-\phi_j) + h\sum_{i\in\Lambda}\cos\phi_i\right\}. \tag{3}$$

We shall consider the correlations functions:

$$\langle\cos m\phi\rangle_{\Lambda,h} = \int\cos m\phi\, d\mu_{\Lambda,h}(\phi), \tag{4}$$

where $m\phi = \sum_i m(i)\phi_i$, and $m: Z^d \to Z$ is a function of compact support, and the pressure $P(T)$ defined by:

$$P(T) = \lim_{\Lambda\uparrow Z^d}|\Lambda|^{-1}\log Z_{\Lambda,h=0}.$$

It is easy to show that $P(T)$ exists. Using the Lee-Yang theorem [20, 9], we can also show that the thermodynamic limit of the correlation functions exists for $h \neq 0$, and that the state obtained is clustering [12, 18, 23]. The "+" state is defined by:

$$\lim_{h \downarrow 0} \langle \cos m\phi \rangle_h \equiv \langle \cos m\phi \rangle_+ .$$

The limit exists by Ginibre's inequalities. By symmetry, $\langle \sin m\phi \rangle_+ \equiv 0$.

In $d=2$, the uniqueness of the translation invariant equilibrium state [2] implies:

$$\langle \cos m\phi \rangle_+ = \lim_{\Lambda \to Z^d} \langle \cos m\phi \rangle_{\Lambda, h=0} = 0 .$$

In $d=3$, it has been proven [13] that $\langle \cos\phi \rangle_+ \equiv \bar{m}(\beta) > 0$ for β large. This is a consequence of the infrared bounds which we now recall.

Notation. We let

$$\sigma_i = (\sigma_i^0, \sigma_i^1) = (\cos\phi_i, \sin\phi_i) \in R^2 .$$

As in [3], the unit vectors along the coordinate axis are denoted by e_α, $\alpha = 1, \ldots, d$. Given a function $f : Z^d \to R^2$ of compact support, we write

$$\nabla_i^\alpha f = f(i) - f(i + e_\alpha) = (\nabla^\alpha f)(i) ,$$

$$\sum_{e_\alpha, \alpha = 1, \ldots, d} \equiv \sum_\xi , \qquad \sigma(g) = \sum_i g(i) \cdot \sigma_i .$$

Lattice sites will sometimes also be denoted by x, y or z.

Infrared Bounds [13]

IR 1. Let $g^\alpha : Z^d \to R^2$, $\alpha = 1, \ldots, d$ be functions of compact support. Then

$$\left\langle \exp \sigma\left(\sum_{\alpha=1}^{d} \nabla^\alpha g^\alpha \right) \right\rangle_{\Lambda, h} \leqq \exp\left[\sum_{\alpha, i} (g^\alpha(i))^2 \right] \Big/ 2\beta .$$

When $f \in R^2$ is such that $\sum_i f(i) = 0$, then $g^\alpha(i) = -\Delta^{-1} \nabla_i^\alpha f$ is well defined. Applying IR 1 to this particular $g^\alpha(i)$ we get [13]:

$$\langle \exp \sigma(f) \rangle_{\Lambda, h} \leqq \exp[(f, (-\Delta^{-1}) f)/2\beta] .$$

Using the fact that $\langle\ \rangle_h$ is ergodic, IR 1 has the following consequence:

IR 2 [13]. For $d \geqq 3$, let $\hat{\sigma}_i = \sigma_i - \langle \sigma_i \rangle_h$, then

$$\langle \exp \hat{\sigma}(g) \rangle_h \leqq \exp[(g, (-\Delta)^{-1} g)/2\beta]$$

for any g of compact support.

From IR 2, one obtains the bound,

$$[\bar{m}(\beta)]^2 = \langle \cos\phi \rangle_+^2 \geqq 1 - I(d) T , \tag{5}$$

where

$$I(d) \equiv (2\pi)^{-d} \int_{|k_\xi| \leqq \pi} d^d k \left[\sum_\xi (1 - \cos k_\xi) \right]^{-1} .$$

As we shall see later, the r.h.s. of (5) are the first terms in the asymptotic expansion of $\bar{m}^2$.

III. The Main Result

Theorem 1. *For any d, the perturbation expansion in T for the "excess free energy" $Q(T) \equiv P(T) - d\beta - \frac{1}{2}\ln T$ and the correlation functions $\langle \cos m\phi \rangle_+$ is asymptotic to all orders in T.*

1. Strategy of the Proof

The proof will be divided into two parts; we shall first prove the result for $Q(T)$ and for correlation functions of the type

a) $\langle \cos m\phi \rangle_+$ with $\sum_i m(i) \equiv \underline{m} = 0$,

and then for correlation functions of the type

b) $\langle \cos m\phi \rangle_+$ with $\underline{m} \neq 0$.

In some way, Parts a) and b) correspond to the cases $\langle (\nabla_0 \phi)^2 \rangle$ and $\langle \phi_0^2 \rangle$ of [3]. For technical reasons, we are not able to generalize the proof given in [3] for $\langle \phi_0^2 \rangle$ to Case b). Using however the existence of an asymptotic expansion for $\langle \cos(\phi_0 - \phi_x) \rangle_+$ and the decay of the truncated two point function, $\langle \cos\phi_0 \cos\phi_x \rangle_+ - \langle \cos\phi_0 \rangle_0^2 < c|x|^{-1}$, $d \geqq 3$, which we get from IR bounds [4, 25], we are able to generate an asymptotic expansion for $\langle \cos\phi \rangle_+$. Using reflection positivity, we generalize our argument to any $\langle \cos m\phi \rangle_+$ with $\underline{m} \neq 0$. For $d < 3$, Part b) is trivial since by the Dobrushin-Schlosman theorem [8], $\langle \cos m\phi \rangle_+ \equiv 0$ for $\underline{m} \neq 0$.

To keep matters simple, we shall first consider in a) $d \geqq 3$ and then indicate the changes required for $d = 2$. In Sect. 4 we sketch an alternative proof of a) which should also work for non nearest neighbor interactions. This proof uses correlation inequalities.

2. Asymptotic Expansion for $\langle \cos m\phi \rangle$ with $\underline{m} = 0$, $d \geqq 3$

Before giving the proof, we shall derive two technical lemmas from the infrared bound IR 2. Those are the key ingredients which will be used to deal with the second perturbation of the free massless Gaussian field described in the introduction, namely $\chi(|\phi_j| \leqq \pi\sqrt{\beta})$.

We introduce the periodic function $a(\phi) : \mathbb{R} \to [-\pi, \pi]$, s.t. $a(\phi + 2\pi n) = \phi, n \in N$.

Lemma 1. *For $d \geqq 3$ there exists positive constants c_1, c_2 independent of T and h, such that*

$$\langle \exp(\sqrt{\beta}|a(\phi)|/c_1) \rangle_h \leqq c_2 < \infty .$$

Proof. We first prove $\langle \exp(2\sqrt{\beta} a^2(\phi)/\pi^2) \rangle < \infty$. IR 2 with $g = (-\sqrt{\beta}\delta_{i0}, 0)$ implies

$$\langle \exp[-\sqrt{\beta}(\cos\phi - \langle \cos\phi \rangle_h)] \rangle_h \leqq \exp[C_{00}/2] \leqq c_2' \tag{6}$$

with $C_{ij} = (-\Delta)^{-1}(i,j)$ the covariance of the Gaussian measure corresponding to the spin wave approximation. Taking the square root of (5) we obtain,

$$\langle \cos\phi \rangle_h \geqq 1 - \tfrac{1}{2} T I(d) \varepsilon(T) \tag{7}$$

with $\varepsilon(T)\to 1$ when $T\to 0$. The inequalities (6), (7) imply:

$$\langle\exp(\sqrt{\beta}[(1-1/2I(d)T\varepsilon(T)-\cos\phi])\rangle_h \leqq c_2',$$

$1-\cos\phi \geqq 2/\pi^2\phi^2$ for $\phi\in[-\pi,\pi]$, so that

$$\langle\exp(2\sqrt{\beta}a^2(\phi)/\pi^2)\rangle_h \leqq c_2'\exp[1/2I(d)\sqrt{T}\varepsilon(T)] \leqq c_2''. \tag{8}$$

We now use again IR 2 with $\hat{\sigma}(g)=\sqrt{\beta}\sin\phi$. Let us first remark that for any $k<\pi$ $\exists c>0$ such that $|x|\leqq c|\sin x|$ for $|x|\leqq k$. This implies

$$\exp(\sqrt{\beta}|x|/c) \leqq \exp[\sqrt{\beta}\sin x]+\exp[-\sqrt{\beta}\sin x] \quad \text{for} \quad |x|<k. \tag{9}$$

Defining $\mu_h(\phi)$ by $\langle F(\phi)\rangle_h=\int F(\phi)d\mu_h(\phi)$,

$$\int_{-\pi}^{\pi}\exp(\sqrt{\beta}|\phi|/c)d\mu_h(\phi)=\int_{-\pi/2}^{\pi/2}\exp(\sqrt{\beta}|\phi|/c)d\mu_h(\phi) + \int_{|\phi|>\pi/2}\exp(\sqrt{\beta}|\phi|/c)d\mu_h(\phi).$$

To estimate the first integral we use (9):

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\exp(\sqrt{\beta}|\phi|/c)d\mu_h(\phi) &\leqq \int_{-\pi/2}^{\pi/2}(\exp\sqrt{\beta}\sin\phi+\exp-\sqrt{\beta}\sin\phi)d\mu_h(\phi)\\ &\leqq \int_{-\pi}^{\pi}(\exp\sqrt{\beta}\sin\phi+\exp-\sqrt{\beta}\sin\phi)d\mu_h(\phi)\\ &\leqq c_3 \text{ uniformly in } T \text{ by IR 2}.\end{aligned}$$

To estimate the second integral, we use the Chebyshev's inequalities and (8):

$$\begin{aligned}\int_{|\phi|>\pi/2}\exp(\sqrt{\beta}|\phi|/c)d\mu_h(\phi) &\leqq (\exp\sqrt{\beta}\pi/c)\mu_h\{\phi||\phi|\geqq\pi/2\}\\ &\leqq(\exp\sqrt{\beta}\pi/c)\exp(-2\sqrt{\beta}\pi^2/\pi^2 4)\\ &=\exp[-\sqrt{\beta}(1/2-\pi/c)]\leqq c_4\end{aligned}$$

uniformly in T for $c>2\pi$.

Remark. Setting $\sqrt{\beta}\phi=\phi'$ Lemma 1 implies:

$$\int\exp(|\phi'|/c)d\mu'(\phi')\leqq \text{const}$$

when $d\mu'$ is the measure obtained by the change of variables. But,

$$|\phi'|^n/c^n \leqq n!\exp(|\phi'|/c)$$

hence,

$$\int|\phi'|^n d\mu'(\phi')\leqq \text{const}\, n!c^n, \tag{10}$$

where the const is T independent.

Lemma 2. *For $d\geqq 3$, $\langle\delta(\phi\pm\pi)\rangle_+ \leqq c_1\exp[-c_2\beta^{1/2}]$, where c_1 and c_2 are positive constants independent of T and δ is considered as a periodic δ-function.*

Proof. The proof will be based on Lemma 1 and the DLR equations [7, 17]. If μ is the equilibrium measure corresponding to the "+" state, the DLR equations [7, 17] read:

$$\langle \delta(\phi-\pi)\rangle_+ = \int d\mu(\phi_1 \dots \phi_{2d}) F(\phi_1 \dots \phi_{2d}),$$

where

$$F(\phi_1 \dots \phi_{2d}) = \left[\int d\phi \exp\left(\beta \sum_{i=1}^{2d} \cos(\phi-\phi_i)\right)\right]^{-1} \cdot \int \delta(\phi-\pi) \exp\left(\beta \sum_{i=1}^{2d} \cos(\phi-\phi_i)\right) d\phi,$$

and $\{\phi_i\}_{i=1}^{2d}$ are the nearest neighbors of ϕ. Let

$$A = \{\phi_1 \dots \phi_{2d} \mid |\phi_1| < a, \dots, |\phi_{2d}| < a\},$$

($a = \pi/32$ for instance) and let us estimate the integral

$$\int_A F(\phi_1 \dots \phi_{2d}) d\mu(\phi_1 \dots \phi_{2d})$$

by finding a bound for $\sup_A |F(\phi_1 \dots \phi_{2d})|$:

$$\begin{aligned} F(\phi_1 \dots \phi_{2d}) &= \left[\int_{-\pi}^{\pi} \exp\left(\beta \sum_{i=1}^{2d} [\cos(\phi-\phi_i) - \cos(\pi-\phi_i)]\right) d\phi\right]^{-1} \\ &\leqq \left[\int_{-a}^{a} \exp\left(\beta \sum_{i=1}^{2d} [\cos(\phi/2-\phi_i)\cos(\phi/2)]\right) d\phi\right]^{-1} \\ &\leqq [2a \exp 2\beta c']^{-1}, \end{aligned}$$

where $c' = 2d \cos(3a/2)\cos(a/2)$. To prove the result, we still have to estimate

$$\int_{A^c} F(\phi_1 \dots \phi_{2d}) d\mu(\phi_1 \dots \phi_{2d}) \leqq \mu(A^c) \sup_{A^c} F(\phi_1 \dots \phi_{2d}).$$

By Chebyshev and Lemma 1,

$$\mu(A^c) \leqq \text{const} \exp[-c\beta^{1/2}], \quad (c>0).$$

Hence

$$\begin{aligned} \sup_{A^c} F(\phi_1 \dots \phi_{2d}) &\leqq \left[\int_{-\pi}^{\pi} d\phi \exp(-2d\beta|\cos(\phi/2)|)\right]^{-1} \\ &\leqq \left[\int_{-1/\beta}^{0} d\phi \exp(-2d\beta|\sin(\phi/2)|)\right]^{-1} \\ &\leqq \text{const}\, \beta. \end{aligned}$$

Proof of Case a) for $d=3$. We now prove that the expansion for $\langle \cos \nabla_0^e \phi\rangle_+$ is asymptotic up to second order. The proof can be easily generalized to all orders and to all correlation functions with $\underline{m}=0$.

Making the change of variables $\phi_i' = \sqrt{\beta}\phi_i$,

$$\langle \cos V_0^e\phi\rangle_{A,h} = \frac{\int_{-\pi\sqrt{\beta}}^{\pi\sqrt{\beta}} \prod_{i\in A} d\phi_i \cos(\sqrt{T}V_0^e\phi)\exp\left[\beta \sum_{i\in A,\xi}(\cos\sqrt{T}V_i^\xi\phi - 1) + h\sum_{i\in A}\cos\sqrt{T}\phi_i\right]}{\int_{-\pi\sqrt{\beta}}^{\pi\sqrt{\beta}} \prod_{i\in A} d\phi_i \exp\left[\beta \sum_{i\in A,\xi}(\cos\sqrt{T}V_i^\xi\phi - 1) + h\sum_{i\in A}\cos\sqrt{T}\phi_i\right]}.$$

N.B. We shall generally not indicate explicitly whether we are using the original angle variable ϕ or the scaled ϕ' as this should be clear from the context.

We now expand $\cos(\sqrt{T}V_0^e\phi)$ up to second order

$$\cos(\sqrt{T}V_0^e\phi) = 1 - T\frac{(V_0^e\phi)^2}{2} + T^2\frac{(V_0^e\phi)^4}{4!} - T^3\frac{(V_0^e)^6}{6!}\cos\sigma,$$

where σ is fixed by Taylor's theorem. Clearly we have to expand $\langle (V_0^e\phi)^2\rangle$ up to order 1 and $\langle (V_0^e\phi)^4\rangle$ up to order 0. The third term

$$\frac{T^3}{6!}|\langle (V_0^e\phi)^6\cos\sigma\rangle| \leqq \frac{T^3}{6!}\langle (V_0^e\phi)^6\rangle \leqq \text{const}\, T^3$$

because of Lemma 1 (see Remark).

As noted in the introduction there are two perturbations of the massless Gaussian lattice field: a power series in T for

$$\beta\cos(\sqrt{T}V_i^\xi\phi) + \tfrac{1}{2}(V_i^\xi\phi)^2 - 1,$$

and a characteristic function $\chi(|\phi_i| < \pi\sqrt{\beta})$. The perturbation $h\sum_{i\in A}\cos\sqrt{T}\phi$ is irrelevant because h will be set equal to zero.

As in [3] the expansion for $\langle (V_0^e\phi)^2\rangle$ and $\langle (V_0^e\phi)^4\rangle$ will be generated by a regularized form of the integration by parts (I.P.) formula that we now recall: if F is a function of the Gaussian variables ϕ_l, $l = 1, \dots, n$

$$\int \phi_0 F(\{\phi_l\}_{l=1}^n)d\mu_{0A} = \sum_{i\in A} C_{0i}^{A,m}\int \frac{d}{d\phi_i}F(\{\phi_l\}_{l=1}^n)d\mu_{0A} + m^2\sum_{i\in A} C_{0i}^{A,m}\int \phi_i F(\{\phi_l\}_{l=1}^n)d\mu_{0A}, \tag{13}$$

where $d\mu_{0A}$ is the Gaussian measure of the massless field with periodic boundary conditions in a box A, and $C_{0i}^{A,m}$ is the covariance of the massive Gaussian field with periodic boundary conditions in the box A.

As in [3] m will be T-dependent. Let us apply (13) to $\langle (V_0^e\phi)^2\rangle$. After regrouping some terms, see [3, Eq. (9)], we obtain

$$\begin{aligned}\langle (V_0^e\phi)^2\rangle_{A,h} = {} & V_0^eV_0^eC_{00}^{A,m} - \beta\sum_{i,\xi} V_0^eV_i^\xi C_{0i}^{A,m}\langle V_0^e\phi[-\sqrt{T}\sin\sqrt{T}V_i^\xi\phi + T(V_i^\xi\phi)\rangle_{A,h}\\ & + -\sqrt{T}h\sum_{i\in A} V_0^eC_{0i}^{A,m}\langle V_0^e\phi\sin\sqrt{T}\phi_i\rangle_{A,h}\\ & + \sum_i m^2V_0^eC_{0i}^{A,m}\langle V_0^e\phi\phi_i\rangle_{A,h}\\ & + \sum_i V_0^eC_{0i}^{A,m}\langle V_0^e\phi[\delta(\phi + \pi\sqrt{\beta}) + \delta(\phi - \pi\sqrt{\beta})]\rangle_{A,h}.\end{aligned} \tag{14}$$

Taking $\lim_{h\to 0}\lim_{\Lambda\to\infty}$ in (14) we obtain the same equation in the limiting state $\langle\ \rangle$ except that the term proportional to h has disappeared, because $\langle|\nabla_0^e\phi|\rangle_{\Lambda,h}$ is uniformly bounded (see Remark after Lemma 1), thus

$$\begin{aligned}\langle(\nabla_0^e\phi)^2\rangle = \nabla_0^e\nabla_0^e C_{00}^m - \beta\sum_{i,\xi}\nabla_0^e\nabla_i^\xi C_{0i}^m\cdot\langle\nabla_0^e\phi[-\sqrt{T}\sin\sqrt{T}\nabla_i^\xi\phi + T(\nabla_i^\xi\phi)]\rangle\\ +\sum_i m^2\nabla_0^e C_{0i}^m\langle\nabla_0\phi\phi_i\rangle\\ +\sum_i\nabla_0^e C_{0i}^m\langle\nabla_0^e\phi[\delta(\phi+\pi\sqrt{\beta})+\delta(\phi-\pi\sqrt{\beta})]\rangle.\end{aligned}\tag{15}$$

This equation is similar to Eq. (10) of [3] except for the presence of the last term of the r.h.s. This term is the contribution to the expansion of the characteristic functions $\chi(|\phi_i|\leqq\pi\sqrt{\beta})$. We shall use Lemmas 1 and 2 to prove that it is exponentially small in T (as $T\to 0$).

Choosing $m=m(T)=\exp[-(\ln T)^2]$, the first term on the r.s. of (15) gives the 0-th order of the expansion since

$$|\nabla_0^e\nabla_0^e C_{00}^m - \nabla_0^e\nabla_0^e C_{00}^{m=0}|\leqq \mathrm{const}\,m$$

(see [3, Appendix B]). The mass terms

$$m^2\sum_i\nabla_0^e C_{0i}^m\langle\nabla_0^e\phi\phi_i\rangle\leqq\mathrm{const}\,m^2 m^{-1}|\langle\nabla_0^e\phi\phi_i\rangle|$$

because $\sum_i|\nabla_0^e C_{0i}^m|\leqq Cm^{-1}$, see [3, Appendix A].

By Lemma 1 $|\langle\nabla_0^e\phi\phi_i\rangle|\leqq$ const uniformly in T. So the mass term is bounded by const$m$. By our choice of m it is exponentially small with T. The 4-th term of the r.h.s. of (15) is bounded by

$$\mathrm{const}\,m^{-1}|\langle\nabla_0^e\phi\delta(\phi\pm\pi\sqrt{\beta})\rangle|$$

but

$$\langle\nabla_0^e\phi\delta(\phi\pm\pi\sqrt{\beta})\rangle\leqq c_1\exp(-c_2\beta^{1/2})\quad\text{by Lemma 2}.$$

Therefore this term is still exponentially small in T (as $T\to 0$). Applying Taylor's theorem to $\sin(\sqrt{T}(\nabla_i^\xi\phi))$ and using $\sum_i|\nabla_0^e\nabla_i^\xi C_{0i}^m|<\mathrm{const}\ln m$, see [3, Appendix A], the second term (the temperature term) is bounded by $CT\ln m|\langle\nabla_0^e\phi(\nabla_i^\xi\phi)^3\rangle|$ and hence by const$T\ln m$.

In general when we apply the integration by parts formula there appear 4 terms as in (15), the last two are exponentially small and they disappear from the expansion. The second term called, the temperature term or II-term is small compared to the first term called the I-term.

To get the first order of the expansion of $\langle(\nabla_0^e\phi)^2\rangle$, we have to apply I.P. once more to the II-term in (15). This yields after having applied Taylor's theorem to the sin:

$$\begin{aligned}\mathrm{II}=\frac{\beta}{3!}\sum_{i,\xi}\nabla_0^e\nabla_i^\xi C_{0i}^m\langle\nabla_0^e\phi T^2(\nabla_i^\xi\phi)^3\rangle\\ +\frac{\beta}{5!}\sum_{i,\xi}\nabla_0^e\nabla_i^\xi C_{0i}^m\langle\nabla_0^e\phi T^3(\nabla_i^\xi\phi)^5\cos\sigma\rangle.\end{aligned}$$

The second term above is bounded by const $T^3 \ln m$ and is small compared to T.

We now apply I.P. to $\langle \nabla_0^e \phi (\nabla_i^\xi \phi)^3 \rangle$ and then we apply repeatedly I.P. to the I-term produced by the preceding I.P. until we have a purely Gaussian expectation value. By what we have explained before all the remainder terms are small with respect to the purely Gaussian expectation value. We apply the same procedure to $\langle (\nabla_0^e \phi)^4 \rangle$ to get the 0-th order. So

$$\langle \cos \nabla_0^e \phi \rangle_+ = 1 - \frac{T}{2} \langle (\nabla_0^e \phi)^2 \rangle_G^m - \frac{T^2}{4!} \sum_{i,\xi} \langle (\nabla_0^e \psi)^2 (\nabla_i^\xi \phi)^4 \rangle_G^m + \frac{T^2}{4!} \langle (\nabla_0^e \phi)^4 \rangle_G^m + O(T^{2+\varepsilon}),$$

where $\langle \ \rangle_G^m$ is the expectation value in the massive Gaussian field. This yields

$$\langle \cos \nabla_0^e \phi \rangle_+ = 1 - T[C_{00}^m - C_{0e}^m] - \tfrac{3}{2} T^2 [C_{00}^m - C_{0e}^m]^2 + O(T^{2+\varepsilon})$$

As already proved in [3, Appendix B], if

$$\langle \cos \nabla_0^e \phi \rangle = 1 + a_1(m(T))T + a_2(m(T))T^2 + O(T^{2+\varepsilon})$$

then

$$|a_i(m(T)) - a_i(0)| \leqq m(T) = e^{-(\ln T)^2};$$

so

$$\langle \cos \nabla_0^e \phi \rangle_+ = 1 - T[C_{00} - C_{0e}] - \tfrac{3}{2} T^2 [C_{00} - C_{0e}]^2 + O(T^{2+\varepsilon}).$$

But $C_{00} - C_{0e} = \frac{1}{2d}$. So finally,

$$\langle \cos \nabla_0^e \phi \rangle_+ = 1 - T/2d - 3T^2/8d^2 + O(T^{2+\varepsilon}).$$

3. Asymptotic Expansion of $\langle \cos m\phi \rangle$ with $\sum_i m(i) = 0$ in $d=2$

In two dimensions there is no breakdown of the SO(2) symmetry so the measure $d\mu(\phi)$ is not concentrated around $\phi = 0$ and we do not expect Lemmas 1 and 2 to be true. We shall however prove similar results for the difference variables $(\phi_x - \phi_y)$ even when $|x-y| \sim \exp[\beta^{1/4}]$. In $d=2$, $\langle \ \rangle$ will denote $\lim_{\Lambda \to \infty} \langle \ \rangle_{\Lambda, h=0}$.

Lemma 3. *For $d=2$ let $x, y \in Z^2$ and $|x-y| = O(\exp \beta^{1/4})$ then*

$$\langle \exp(\beta^{1/4} |a(\phi_x - \phi_y)|/\pi \rangle \leqq c < \infty,$$

where c is T independent.

Proof. From IR 1, we have,

$$\begin{aligned} \langle \exp[\pm \beta^{1/4}(\cos \phi_x - \cos \phi_y)] \rangle &\leqq \exp[\beta^{-1/2}(C_{00} - C_{yx})] \\ &\leqq \exp[c_1 \beta^{-1/2} \ln |x-y|] \leqq c_2, \end{aligned} \tag{17}$$

where c_1 and c_2 are positive, T-independent constants, and we have used $|x-y| = O(\exp \beta^{1/4})$. Similarly,

$$\langle \exp(\pm \beta^{1/4}(\sin \phi_x - \sin \phi_y)) \rangle \leqq c_3. \tag{18}$$

The bounds (17) and (18) imply

$$\langle \exp\{\pm 2\beta^{1/4}\sin[(\phi_x+\phi_y)/2\rangle \sin[(\phi_x-\phi_y)/2]\} + \exp\{\pm 2\beta^{1/4}\cos[(\phi_x+\phi_y)/2]\sin[(\phi_x-\phi_y)/2]\} \leqq c_4,$$

which clearly gives:

$$\langle \exp[\beta^{1/4}|\sin(\phi_x-\phi_y)/2|]\rangle \leqq c_4. \tag{19}$$

The lemma then follows by noting that $|a(\phi)|/\pi \leqq \sin\phi/2$.

Remark. When $|x-y|=O(1)$, we obviously have

$$\langle \exp\beta^{1/2}|a(\phi_x-\phi_y)|\rangle \leqq c < \infty \tag{20}$$

(c is T independent).

Lemma 4. *For $d=2$, let $|x-x_0|=c\exp[\beta^{1/4}]$. Then*

$$\langle \delta(\phi_{x_0}-\phi_x\pm\pi)\rangle \leqq c_1\exp(-c_2\beta^{1/4}),$$

where c_1 and c_2 are positive, T-independent, constants.

Proof. The proof is similar to the one of Lemma 2 given Lemma 1. By the D.L.R. equations [7, 17], if μ is the equilibrium measure corresponding to $\langle\ \rangle$ then:

$$\langle \delta(\phi_{x_0}-\phi_x-\pi)\rangle = \int d\mu(\phi_1,\dots,\phi_{2d},\psi_1,\dots,\psi_{2d})F(\phi_1,\dots,\phi_{2d},\psi_1,\dots,\psi_{2d}),$$

where

$$F=\left[\int\delta(\phi_{x_0}-\phi_x-\pi)\exp\left(\beta\sum_{i=1}^{2d}[\cos(\phi_{x_0}-\phi_i)+\cos(\phi_x-\psi_i)]\right)d\phi_{x_0}d\phi_x\right] : \left[\int d\phi_{x_0}d\phi_x\exp\left(\beta\sum_{i=1}^{2d}[\cos(\phi_{x_0}-\phi_i)+\cos(\phi_x-\psi_i)]\right)\right]^{-1}, \tag{21}$$

where $\{\phi_i\}_{i=1}^{2d}$ and $\{\psi_i\}_{i=1}^{2d}$, are respectively the nearest neighbor variables to ϕ_{x_0} and ϕ_x.

Let $A=\{\phi_i,\psi_j \,|\, |\phi_i-\psi_i|<a,\ |\psi_j-\psi_{j'}|<a,\ |\phi_i-\phi_{i'}|<a \text{ and } i,j=1,\dots,2d\}$. ($a$ is for instance $\pi/32$.)

Our proof is in 2 steps.

1) Estimate of $\int_A F d\mu$.

$$\int_A d\mu(\phi_1,\dots,\phi_{2d},\psi_1,\dots,\psi_{2d})F \leqq \sup_A F(\phi_1,\dots,\phi_{2d},\psi_1,\dots,\psi_{2d}).$$

Using the double angle formula, the numerator of (21) is bounded by $\exp[2\beta\cos(\pi/2-a/2)]$. The denominator D of (21) obeys:

$$D \geqq \int_{|\phi_{x_0}-\phi_{i_0}|<a} d\phi_{x_0}\exp\left(\beta\sum_i\cos(\phi_0-\phi_i)\right)\cdot\int_{|\phi_x-\psi_{i_0}|<a} d\phi_x\exp\left(\sum_i\cos(\phi_x-\psi_i)\right),$$

where $i_0\in\{1,\dots,2d\}$. $\forall i$, $|\phi_i-\phi_{i_0}|<a$ because $\phi_i,\phi_{i_0}\in A$ and therefore $|\phi_0-\phi_i|<2a$. So

$$D \geqq a^2\exp(\beta 2d 2\cos 2a) \geqq a^2\exp[2d\beta].$$

Finally

$$\sup_A F \leqq a^{-2} \exp(-2\beta(d-\cos(\pi/2-a/2))) \leqq a^{-2} \exp(-c\beta). \tag{22}$$

2) Estimate of $\int_{A^c} F d\mu$.

$$\int_{A^c} F d\mu < |F|_\infty \mu(A^c).$$

By Chebyshev inequality and Lemma 3, $\mu(A^c) \leqq \exp(-\beta^{1/4} a/\pi)$. To estimate $|F|_\infty$ we note that

$$\begin{aligned}|F| = &\left[\int \exp\left(\beta \sum_i [\cos(\phi_{x_0}-\phi_i)+\cos(\phi_{x_0}+\pi-\psi_i)]\right) d\phi_{x_0}\right] \\ &\cdot \left[\int \exp\left(\beta \sum_i [\cos(\phi_{x_0}-\phi_i)+\cos(\phi_x-\psi_i)]\right) d\phi_{x_0} d\phi_x\right]^{-1} \\ \leqq &\sup_{\phi_{x_0}} \Big[\exp(\beta[\cos(\phi_{x_0}-\phi_i)+\cos(\phi_{x_0}+\pi-\psi_i)]) \\ &\cdot \left(\int \exp\left(\beta \sum_i [\cos(\phi_{x_0}-\phi_i)+\cos(\phi_x-\psi_i)]\right) d\phi_x\right)^{-1}\Big] \equiv S.\end{aligned}$$

By compactness, $\exists \tilde{\phi}_{x_0} \in [-\pi, \pi]$ such that

$$\begin{aligned} S = &\exp\left(\beta \sum_i [\cos(\tilde{\phi}_{x_0}-\phi_i)+\cos(\tilde{\phi}_{x_0}+\pi-\psi_i)]\right) \\ &\cdot \left(\int \exp\left(\beta \sum_i [\cos(\tilde{\phi}_{x_0}-\phi_i)+\cos(\phi_x-\psi_i)]\right) d\phi_x\right)^{-1} \\ = &\left\{\int \exp\left(2\beta \sum_i \sin[(\phi_x-\tilde{\phi}_{x_0})/2+\pi/2-\psi_i] \sin[(\phi_x-\tilde{\phi}_{x_0})/2-\pi/2]\right) d\phi_x\right\}^{-1} \\ \leqq &\left[\int_{|\phi_x|<\beta^{-1}} \exp(-2\beta|\phi_x|) d\phi_x\right]^{-1} < \text{const}\,\beta, \end{aligned}$$

where the const is T-independent.

Given Lemmas 4 and 5, we shall perform the expansion in a way which is similar to the case $d=2$ of [3]. We still need an additional lemma:

Lemma 5. *Let $F(\phi_0, \ldots, \phi_N)$ be a periodic function of $\phi_0, \ldots, \phi_N$ with period 2π and such that*

$$F(\phi_0+c, \ldots, \phi_N+c) = F(\phi_0, \ldots, \phi_N). \tag{23}$$

Then

$$\frac{1}{(2\pi)^N} \int F(\phi_0 \ldots \phi_N) d\phi_0 \ldots d\phi_N = \frac{1}{(2\pi)^{N-1}} \int F(0, \phi_1 \ldots \phi_N) d\phi_1 \ldots d\phi_N.$$

Proof.

$$+\frac{1}{2\pi} \int F(\phi_0 \ldots \phi_N) d\phi_0 \ldots d\phi_N = \int d\phi_0 [\int F(\phi_0, \phi_1+\phi_0 \ldots \phi_N+\phi_0) d\phi_1 \ldots d\phi_N]$$

by periodicity. The result then follows from (23). As a direct consequence of Lemma 5 we have:

Corollary 1. *If* $\sum_i m(i)=0$, *then*

$$\langle\cos m\phi\rangle_{\Lambda,h=0}=\langle\cos m\phi\rangle'_{\Lambda,h=0},$$

where the $\langle\ \rangle'$ *state is the state* $\langle\ \rangle$ *with the restriction that* $\phi_{x_0}=0$. *The point* x_0 *can be chosen arbitrarily.*

Proof of Case a) for $d=2$. We expand around a massive Gaussian field with mass $m(T)=e^{-(\ln T)^2}$ in a finite periodic cubical box $\Lambda_0\subset\Lambda$ with sides $R(T)=\exp[\beta^{1/4}]$. In expanding $\langle\cos m\phi\rangle'$ defined in Corollary 1 we shall choose x_0 just outside of Λ_0, but within Λ with $d(x_0,\Lambda_0)=2$. By Lemmas 4 and 5, $\forall x\subset\Lambda_0$, the "spin" at the point x in our system in Λ will be with high probability in the same direction as the one at x_0. Writing $H_\Lambda=H^G_{\Lambda_0}+H'_\Lambda$ with

$$H^G_{\Lambda_0}=\tfrac{1}{2}\sum_{\langle ij\rangle\subset\Lambda_0}(\phi_i-\phi_j)^2+m^2(T)\sum_{i\in\Lambda_0}\phi_i^2+\tfrac{1}{2}\sum_{i\in\partial\Lambda_0}(\phi_i-\phi_{\bar{i}})^2,$$

$$\begin{aligned}H'_\Lambda=&\tfrac{1}{2}\sum_{\substack{\langle i,j\rangle\subset\Lambda\\ i\in\Lambda_0 j\in\Lambda_0}}(\phi_i-\phi_j)^2+\tfrac{1}{2}\sum_{\langle i,j\rangle\subset\Lambda-\Lambda_0}(\phi_i-\phi_j)^2-m^2(T)\sum_{i\in\Lambda_0}\phi_i^2\\ &-\tfrac{1}{2}\sum_{i\in\partial\Lambda_0}(\phi_i-\phi_{\bar{i}})^2+\sum_{\langle i,j\rangle\subset\Lambda}\beta\left(\cos\sqrt{T}(\phi_i-\phi_j)-1+\frac{T}{2}(\phi_i-\phi_j)^2\right),\end{aligned}$$

where $\bar{i}$ and i are nearest neighbor in Λ_0 with periodic boundary conditions.

To expand $\langle\cos V_0^e\phi\rangle=\langle\cos V_0^e\phi\rangle'$, we make as before the change of variable $\phi'=\sqrt{T}\phi$. We have to perform an expansion for $\langle(V_0^e\phi)^2\rangle'$. Let us first consider the zeroth order of $\langle(V_0^e\phi)^2\rangle'$ by doing an integration by parts with respect to $H^G_{\Lambda_0}$. This yields for $\Lambda\uparrow Z^2$,

$$\begin{aligned}\langle(V_0^e\phi)^2\rangle'=&V_0^eV_0^eC_{00}^{\Lambda_0,m} &&\text{(a)}\\ &+\sum_{\substack{i\in\Lambda_0\\ \xi}}V_0^eV_i^\xi C_{0i}^{\Lambda_0,m}\langle V_0^e\phi(-\beta\sqrt{T}\sin\sqrt{T}V_i^\xi\phi+V_i^\xi\phi)\rangle' &&\text{(b)}\\ &+\sum_{\substack{i\in\partial\Lambda_0\\ |i-i'|=1,\, i'\in\Lambda\setminus\Lambda_0}}V_0^eC_{0i}^{\Lambda_0,m}\langle V_0^e\phi(-\beta\sqrt{T}\sin\sqrt{T}(\phi_i-\phi_{i'}))\rangle' &&\text{(c)}\\ &+\sum_{i\in\partial\Lambda_0}V_0^eC_{0i}^{\Lambda_0,m}\langle V_0^e\phi(\phi_i-\phi_{\bar{i}})\rangle' &&\text{(d)}\\ &+m_{0i}^2\sum_{i\in\Lambda_0}V_0C_{0i}^{\Lambda_0,m}\langle V_0^e\phi\phi_i\rangle' &&\text{(e)}\\ &+\sum_{i\in\Lambda_0}V_0C_{0i}^{\Lambda_0,m}\langle V_0^e\phi[\delta(\phi_i+\pi)+\delta(\phi_i-\pi)]\rangle'. &&\text{(f)}\end{aligned}\tag{24}$$

Before considering each term in (24), let us express Lemmas 3 and 4 in the state $\langle\ \rangle'$ using Lemma 5. They are

$$\langle\exp(\beta^{1/4}|a(\phi_i-\phi_{x_0})|)\rangle=\langle\exp(\beta^{1/4}|a(\phi_i)|)\rangle'\leqq c\leqq\infty,\qquad\forall i\in\Lambda_0,\tag{25a}$$

$$\forall i\in\Lambda_0\langle\delta(\phi_i-\phi_{x_0}\pm\pi)\rangle=\langle\delta(\phi_i\pm\pi)\rangle'\leqq c_1\exp(-c_2\beta^{1/4}),\qquad\forall i\in\Lambda_0.\tag{25b}$$

Using now Appendices A and B in [3] we show that the r.s. of (24) has the following properties: (a) is the zeroth order term, (b) is estimated as in the case $d=3$ using (20), (c)–(f) are negligible, with our choice of $m(T)$ and $R(T)$ to all orders in T, e.g.

$$(\mathrm{e}) \leqq m\beta \sum_{i\in\partial\Lambda_0} |\nabla_0^e C_{0i}^{\Lambda_0, m}| \leqq \text{const}\, R(T) \exp(-m(T)R(T)),$$

$$(\mathrm{f}) \leqq \text{const}\, m^{-1} \exp(-c_1 \beta^{1/4}).$$

To finish case a we still have to prove the result for the "free energy" $Q(T) = P(T) - \frac{d}{T} - \frac{1}{2}\ln T$. This follows from the existence of an asymptotic expansion for $\langle \cos \nabla_0^e \phi \rangle$ and the formula

$$Q_\Lambda(T) - Q_\Lambda(a) = \int_a^T -\frac{d}{\mu^2}\left[\langle \cos \nabla\phi \rangle_\Lambda(\mu) - 1 + \frac{\mu}{2d}\right] d\mu$$

when we take in both sides $a \to 0$.

4) Asymptotic Expansion for $\langle \cos m\phi \rangle$ with $\sum_i m(i) \neq 0$, $d > 3$

The method used here is somewhat indirect and consists of two steps:

– find a sequence of functions depending on $\{\nabla_x^e \phi\}$ whose expectations values approximate $\langle \cos m\phi \rangle$, or a product of such expectations; this is basically a clustering property of the state; for example, $\langle \cos(\phi_0 - \phi_x) \rangle$ converges to $\langle \cos \phi_0 \rangle^2$ as $|x| \to \infty$ (see below).

– Define a new function of the temperature, by letting x depend on β above such that the deviation from the desired function, for example $\langle \cos(\phi_0 - \phi_x) \rangle - \langle \cos \phi_0 \rangle^2$, is of order T^n and perform an asymptotic expression for this new function up to order n.

We first carry out this proof for the magnetization, (a) and (b) below, and then show that reflection positivity allows us to make an inductive argument for all other functions $\langle \cos m\phi \rangle$, with $\underline{m} \neq 0$, (c) below.

(a) We write

$$\langle \cos \phi_0 \rangle^2 = \langle \cos(\phi_0 - \phi_x) \rangle - (\langle \cos(\phi_0 - \phi_x) \rangle - \langle \cos \phi_0 \rangle^2). \tag{26}$$

Using reflecting positivity, and the infrared bounds one shows [4] that $0 \leqq \langle \cos(\phi_0 - \phi_x) \rangle - \langle \cos \phi_0 \rangle^2 \leqq |x|^{-1} \ln |x|$. Using correlation inequalities one may improve this to a $|x|^{-1}$ bound [25]. (See also Sect. V for further discussion.) If we want an asymptotic expansion up to order n we choose $|x|$ to be of order β^{n+1} so that the second term in (26) is negligible and we have only to do the expansion of $\langle \cos(\phi_0 - \phi_x) \rangle$ which now depends on β not only because of the measure, but also via x.

(b) As before we expand $\cos \sqrt{T}(\phi_0 - \phi_x)$ into a power series and generate the expansion of the terms in this series, e.g. $\langle (\phi_0 - \phi_x)^{2k} \rangle T^k$, by integration by parts, but with the rule that $\langle (\phi_0 - \phi_x)^{2k} \rangle$ is expanded until order $2k(n + 1/2) + n + 1$. One estimates the remainder as before: when $(\phi_0 - \phi_x)$ is integrated by parts it produces $C_{0i} - C_{0x}$ (or $\nabla_i^\xi C_{0i} - \nabla_i^\xi C_{ix}$) which one estimates by $|x||\nabla_0^e C_{0i}|$ (or $|x||\nabla_0^e \nabla_i^\xi C_{0i}$). So at the end the remainder is multiplied by $|x|^{2k} = \beta^{2k(n+1)}$. The terms which were exponentially small are not affected by this factor. The only terms we have to worry about are the temperature terms. Since $\langle (\phi_0 - \phi_x)^{2k} \rangle$ is expanded until order $2k(n+1/2)+n+1$, these temperature terms have a factor

$$T^{2k(n+1/2)}T^kT^{n+1}\beta^{2k(n+1)}=T^{n+1}$$

and we are in the same situation as before: $T^{n+1}(\ln m)^{2n(n+1/2)}$ is negligible with respect to T^n.

All that is left to consider now are the Gaussian terms produced in the I.P. which depend on β via x. It is easy to see that each of these terms can be written as a sum of Gaussian expectation involving only ϕ_0 or only ϕ_x (and the latter do not depend on x by translation invariance) and of terms mixing ϕ_0 and ϕ_x as for example

$$\sum_{\substack{x_1\dots x_l\\ \xi_1\dots\xi_l}} V_{x_1}^{\xi_1}C_{0x_1}\prod_{(ij)}V_{x_i}^{\xi_i}V_{x_j}^{\xi_j}C_{x_ix_j}V_{x_l}^{\xi}C_{x_l\cdot x} \tag{27}$$

or products of such terms.

For the terms involving only ϕ_0 or ϕ_x, we use the estimates of Appendix B in [3] to show that the difference between these terms and those with a massless covariance is of order $\exp(-(\ln T)^2)$. We now show that terms like (27) are small compared to T^n:

Call the term (27) $F(x)$ and $G(x)=xF(x)$. Then

$$\sup_x |x||F(x)|\leqq \int_{-\pi}^{\pi}|\tilde{G}(p)|d^dp=\int_{-\pi}^{\pi}\sum_{i=1}^{d}\left|\frac{d}{dp_i}\tilde{F}(p)\right|d^dp\,. \tag{28}$$

Now, by explicit computation $\tilde{F}(p)$ is of the form

$$\prod_e(\exp \mathrm{i}\,p_e-1)^{n_e}\Big(\sum_\xi(1-\cos p_\xi)+m^2(T)\Big)^{-(l+1)}, \quad \text{with} \quad \sum n_e=2l\,.$$

(28) is therefore bounded by $\ln m(T)\sim(\ln T)^2$ which implies that

$$|F(x)|\leqq|x|^{-1}(\ln T)^2=T^{n+1}(\ln T)^2$$

which is small compared to T^n.

This finishes the proof for the spontaneous magnetization. Its expansion (or rather the square of it) will be given in terms of these graphs mentioned above which involve only ϕ_0 or only ϕ_x (with the massless covariance).

(c) Now we give the general inductive argument which allows us to prove the asymptotic expansion for all correlation functions $\langle\cos m\phi\rangle$, $\underline{m}\equiv\sum_i m(i)\neq 0$. This uses heavily the known decay in $|x|^{-1}$ of $\langle\cos\phi_0\cos\phi_x\rangle-\langle\cos\phi_0\rangle^2$ and of $\langle\sin\phi_0\sin\phi_0\rangle$ [4, 25] and also Theorem 3 of [4] which gives a kind of "domination by the two point function" based on reflection positivity.

1. By symmetry we have only to consider $\underline{m}\geqq 0$. We start with the case $\underline{m}=1$, and write

$$\begin{aligned}\langle\cos m\phi\rangle=[&\langle\cos(m\phi-\phi_x)\rangle-(\langle\cos(m\phi-\phi_x)\rangle\\ &-\langle\cos m\phi\rangle\langle\cos\phi_0\rangle)]/\langle\cos\phi_0\rangle\end{aligned}$$

noting that by translation invariance $\langle\cos\phi_0\rangle=\langle\cos\phi_x\rangle$.

We already have the asymptotic expansion for $\langle\cos\phi_0\rangle$ [it starts with $(1+O(T)]$ and can therefore be inverted); $\langle\cos(m\phi-\phi_x)\rangle$ is a function of the differences and can be treated as $\langle\cos(\phi_0-\phi_x)\rangle$ above. So, all we have to show is that $\cos(m\phi-\phi_x)-\langle\cos m\phi\rangle\langle\cos\phi_0\rangle$ is of order T^n for $|x|$ increasing like some power of β (actually here we take $|x|=\beta^{2n}$). We write $\cos(m\phi-\phi_x)=\cos m\phi\cos\phi_x+\sin m\phi\sin\phi_x$; and suppose for simplicity that x is along the e_1 axis, that $i_1<0$ whenever $m(i)\neq 0$. Using reflection positivity with respect to the plane perpendicular to the e_1 axis at zero, $x\to\bar{x}$, then gives

$$|\langle\sin m\phi\sin\phi_x\rangle|\leqq c_m\langle\sin\phi_x\sin\phi_{\bar{x}}\rangle^{1/2},$$

where c_m depends only on m. (This is similar to Theorem 3 in [4].) Since the r.s. decays like $|x|^{-1}$ [25], we get, with $|x|\sim\beta^{2n}$,

$$|\langle\sin m\phi\sin\phi_x\rangle|\leqq c_m T^n.$$

Similarly we write,

$$\begin{aligned}\langle\cos m\phi\cos\phi_x\rangle-\langle\cos m\phi\rangle\langle\cos\phi_0\rangle&=\langle\cos m\phi(\cos\phi_x-\langle\cos\phi_0\rangle)\rangle\\&\leqq c_m\langle(\cos\phi_x-\langle\cos\phi_0\rangle)(\cos\phi_{\bar{x}}-\langle\cos\phi_0\rangle)\rangle^{1/2}\\&=c_m(\langle\cos\phi_x\cos\phi_{\bar{x}}\rangle-\langle\cos\phi_0\rangle^2)^{1/2}.\end{aligned}$$

2. For general m, let us assume that we have done the proof of the expansion for $\underline{m}\leqq k$ and let $\underline{m}=k+1$ and $\underline{n}=k$. Let n_x denote n translated by an amount x in the positive e_1 direction, and y a point in the positive e_2 direction. m is assumed to be such that $i_1<0$, $i_2<0$ if $m(i)\neq 0$. We shall prove that $\langle\cos(m\phi-n_x\phi-\phi_y)\rangle$ is close to $\langle\cos m\phi\rangle\langle\cos n\phi\rangle\langle\cos\phi_0\rangle$ with an error of order T^n if $|x|$ and $|y|$ are of order β^{2n}. Then, since we know the expansion of $\langle\cos n\phi\rangle$, $\langle\cos\phi_0\rangle$ by our recursive hypothesis, and can invert them, we obtain the expansion of $\langle\cos m\phi\rangle$ by the same arguments as above.

To prove the cluster property, we write

$$\cos(m\phi-n_x\phi-\phi_y)=\cos(m\phi-\phi_y)\cos n_x\phi+\sin(m\phi-\phi_y)\sin n_x\phi.$$

Now, by reflection positivity,

$$|\langle\sin(m\phi-\phi_y)\sin n_x\phi\rangle|\leqq c_m(\langle\sin n_x\phi\sin n_{\bar{x}}\phi\rangle)^{1/2}.$$

To bound $\langle\sin n_x\phi\sin n_{\bar{x}}\phi\rangle$ we use the fact that, by reflection positivity,

$$\langle\cos n_x\phi\cos n_x\phi\rangle-\langle\cos n\phi\rangle^2=1/2\langle(\cos n_x\phi-\langle\cos n_x\phi\rangle)(\cos n_{\bar{x}}\phi-\langle\cos n_x\phi\rangle)\rangle$$

is positive and therefore

$$\langle\sin n_x\phi\sin n_{\bar{x}}\phi\rangle\leqq\langle\cos(n_x\phi-n_{\bar{x}}\phi)\rangle-\langle\cos n\phi\rangle^2.$$

In the right hand side of this inequality, the first term can be expanded in powers of T because it is a function of the differences and the second by assumption because $\underline{n}=k$. But the coefficients in this expansion cancel up to order n except for an error of order $T^{2n}\ln T$ if $|x|\sim\beta^{2n}$.

We apply the same argument to control the difference between $\langle\cos(m\phi-\phi_y)\cos n_x\phi\rangle$ and $\langle\cos(m\phi-\phi_y)\rangle\langle\cos n_x\phi\rangle$ and then between $\langle\cos(m\phi-\phi_y)\rangle$ and $\langle\cos m\phi\rangle\langle\cos\phi_0\rangle$.

IV. An Alternative Proof of Part a) of Theorem 1

In this section, we carry out a proof of Part a) of the Theorem which does not use the infrared bounds. It proceeds in three steps:

1. It follows from Ginibre's inequalities [14] that for any

$$\Lambda, \Lambda' \subset Z^d, \quad \langle \cos m\phi \rangle^0_{\Lambda'} \leqq \langle \cos m\phi \rangle \leqq \langle \cos m\phi \rangle^+_\Lambda, \tag{30}$$

where $\langle\ \rangle^0_{\Lambda'}$ is the Gibbs measure with free b.c. (i.e. no coupling with Λ'^c) on Λ', and $\langle\ \rangle^+_\Lambda$ is the Gibbs measure with 0 boundary condition ($\phi_i = 0 \forall i \in \Lambda^c$). We carry out the expansion of the right and left side of (30), where we let Λ and Λ' depend on the temperature in such a way, [e.g. $|\Lambda|, |\Lambda'| \sim \exp(\sqrt{\beta})$] that the coefficients of the asymptotic expansion of both sides of (30) are the same, giving the expansion of $\langle \cos m\phi \rangle$.

2. We show (Proposition 2) that, if $|\Lambda|, |\Lambda'|$ grow like $\exp(\sqrt{\beta})$ then the left and right hand side of (30) are exponentially close as $\beta \to \infty$ to the corresponding expectation values conditioned on the event: $|\phi_i - \phi_j| \leqq \eta$, $\eta > 0$ for all nearest neighbor pairs $\langle ij \rangle$, with $\langle ij \rangle \cap \Lambda \neq \emptyset$ or $\langle ij \rangle \subset \Lambda'$.

3. We show that, for $\eta < \pi/2$, these conditioned expectation values are equal to the ones where we let all ϕ_i run from $-\infty$ to $+\infty$ (Proposition 2 and Corollary 2). Once this is done, we integrate by parts with respect to a massive Gaussian as before.

We define $\langle\ \rangle^+_{\Lambda,\eta}$ and $\langle\ \rangle^0_{\Lambda,\eta}$ as the expectation value $\langle\ \rangle^+_\Lambda$ or $\langle\ \rangle^0_\Lambda$ conditioned on the event that for all n.n. pairs, $|\phi_i - \phi_j| \leqq \eta \bmod 2\pi$, that is

$$d\mu^+_{\Lambda,\eta} = (Z_{\Lambda,\eta})^{-1} \exp(-\beta H_{\Lambda,+}) \prod_{\langle i,j \rangle \cap \Lambda \neq \emptyset} \chi(|\phi_i - \phi_j| \leqq \eta \bmod 2\pi) \cdot \prod_{i \in \Lambda} d\phi_i$$

with $\phi_j = 0$ if $j \notin \Lambda$ and similarly for $d\mu^0_{\Lambda,\eta}$.

Then we have:

Proposition 1. *There exists constants c_1, c_2 such that, for all β, Λ, η and for $\gamma = +$ or 0*

$$|\langle \cos m\phi \rangle^\gamma_\Lambda - \langle \cos m\phi \rangle^\gamma_{\Lambda,\eta}| \leqq c_2 [|\Lambda| \exp(-c_1 \beta \eta^2)] \exp(c_2 |\Lambda| \exp(-c_1 \beta \eta^2)).$$

Proof. The proof is the same for $+$ or 0; we write

$$\begin{aligned}\langle \cos m\phi \rangle^+_\Lambda &= \sum_X \langle \cos m\phi \rangle^+_{\Lambda,X} P(X) \\ &= \langle \cos m\phi \rangle^+_{\Lambda,X=\emptyset} P(X=\emptyset) + \sum_{X \neq \emptyset} \langle \cos m\phi \rangle^+_{\Lambda,X} P(X) \\ &\langle \cos m\phi \rangle^+_{\Lambda,\eta} + \sum_{X \neq \emptyset} (\langle \cos m\phi \rangle^+_X - \langle \cos m\phi \rangle^+_{\Lambda,\eta}) P(X),\end{aligned}$$

where the sum over X runs over all subsets of $\{\langle ij \rangle \cap \Lambda \neq \emptyset\}$ (or $\{\langle ij \rangle \subset \Lambda\}$ for free b.c.). $\langle\ \rangle_{\Lambda,X}$ is the expectation value conditioned on the event that $|\phi_i - \phi_j| \geqq \eta \bmod 2\pi$ for $\langle ij \rangle \in X$ and $|\phi_i - \phi_j| \leqq \eta \bmod 2\pi$ for $\langle ij \rangle \notin X$; $P(X)$ is the probability of that event. Since $\chi \leqq 1$ we estimate

$$\begin{aligned}P(X) &\leqq \left\langle \prod_{\langle ij \rangle \in X} \chi(|\phi_i - \phi_j| < \eta \bmod 2\pi) \right\rangle \\ &\leqq \exp(-c_1 \beta \eta^2 |X|)\end{aligned}$$

by the contour estimates of [1]. Therefore

$$\begin{aligned}|\langle\cos m\phi\rangle_{\Lambda}^{+}-\langle\cos m\phi\rangle_{\Lambda,\eta}^{+}| &\leqq 2\sum_{X\neq\emptyset}\exp(-c_1\beta\eta^2|X|)\\ &\leqq 2[(1+\exp(-c_1\beta\eta^2))^{2d|\Lambda|}-1]\\ &\leqq c_2(|\Lambda|\exp(-c_1\beta\eta^2))\exp(c_2|\Lambda|\exp(-c_1\beta\eta^2)).\end{aligned}$$

In the second step we use the fact that the sum runs over all subsets of $\{\langle ij\rangle\cap\Lambda\neq\emptyset\}$ whose number is less than $2d|\Lambda|$. In the last step we use

$$\begin{aligned}(1+x)^n-1&=\exp(n\log(1+x))-1\\ &\leqq\exp(nx)-1\leqq nx\exp(nx), \qquad (x>0).\end{aligned}$$

Proposition 2. *Let Λ be a hypercube in Z^d and let $k\in Z^d$ be, either in Λ, or nearest neighbor to some point in Λ. $B(\{\phi_i\},\phi_k)$, where $\{\phi_i\}$ is the set $\{\phi_i\}_{i\in\Lambda'}$ is any periodic function (in $L^1([-\pi,\pi]^{|\Lambda|})$) of period 2π in each of its variables. Then for any $\eta<\pi/2$ and any $\phi_k\in[-\pi,\pi]$,*

$$\begin{aligned}L&\equiv\int_{-\pi}^{\pi}B(\{\phi_i\},\phi_k)\prod_{\langle ij\rangle\subset\Lambda\cup\{k\}}\chi(|\phi_i-\phi_j|\leqq\eta \bmod 2\pi)\prod_{\substack{i\in\Lambda\\ i\neq k}}d\phi_i\\ &=\int_{-\infty}^{+\infty}B(\{\phi_i\},\phi_k)\prod_{\langle ij\rangle\subset\Lambda\cup\{k\}}\chi(|\phi_i-\phi_j|\leqq\eta)\prod_{\substack{i\in\Lambda\\ i\neq k}}d\phi_i\equiv R.\end{aligned}\tag{31}$$

Proof. In L we write

$$\chi(|\phi_i-\phi_j|\leqq\eta \bmod 2\pi)=\sum_{n_{ij}\in Z}\chi(|\phi_i-\phi_j+2\pi n_{ij}|\leqq\eta)$$

and expand the product over $\langle ij\rangle$. We always write the subscripts ij in lexicagraphic order corresponding to a directed bond. This gives

$$L=\sum_{\substack{\{n_{ij}\}\\ n_{ij}\in Z}}\int_{-\pi}^{+\pi}B(\{\phi_i\},\phi_k)\prod_{\langle ij\rangle\subset\Lambda\cup\{k\}}\chi(|\phi_i-\phi_j+2\pi n_{ij}|\leqq\eta)\prod_{i\in\Lambda}d\phi_i.\tag{32}$$

In R we write for each integral

$$\int_{-\infty}^{+\infty}=\ldots=\sum_{m\in Z}\int_{2\pi m-\pi}^{2\pi m+\pi}\ldots.$$

After changing variables $\phi_i\to\phi_i-2\pi m_i$ (which leaves B unchanged because of periodicity) we get:

$$R=\sum_{\substack{\{m_i\}\\ m_i\in Z}}\int_{-\pi}^{+\pi}B(\{\phi_i\},\phi_k)\cdot\prod_{\langle ij\rangle\subset\Lambda\cup\{k\}}\chi(|\phi_i-\phi_j+2\pi(m_i-m_j)|\leqq\eta)\prod_{i\in\Lambda}d\phi_i\tag{33}$$

($m_k=0$ in the above expression). All we have to show in order to prove the equality of (32) and (33) is that each set $\{n_{ij}\}$ for which the integral does not vanish is such that n_{ij} can be written as $n_{ij}=m_i-m_j$ for all oriented pairs $\langle ij\rangle$. Given such a set

$\{n_{ij}\}$ we define for all $m, n, i, j, n_{ji}=n_{ij}$. Since we may take $m_k=0$ and since for any $j\in\Lambda$ there exists always a path $(ki_1),\ldots,(i_n,j)$ of n.n. connecting k and j we may define $-m_j=n_{ki_1}+n_{i_1i_2}+\ldots+n_{i_n,j}$. Then, of course $n_{ij}=m_i-m_j$ but the trouble might be that two different paths may lead to two different definitions of m_j. This does not occur however if for each closed loop on the lattice $\sum_{\text{loop}} n_{ij}=0$.

For this to hold it is enough that $\sum_{\text{Sq.}} n_{ij}=0$ for each elementary square with 4 n.n. pairs in the lattice. But $\sum_{\text{Sq.}}(\phi_i-\phi_j)=0$ and we may write

$$2\pi\left|\sum_{\text{Sq.}} n_{ij}\right| = \left|\sum_{\text{Sq.}}(2\pi n_{ij}+\phi_i-\phi_j)\right|$$

$$\leqq \sum_{\text{Sq.}}|2\pi n_{ij}+\phi_i-\phi_j|\leqq 4\eta<2\pi$$

by our condition on η and the fact that the integral in (32) vanishes unless $|2\pi n_{ij}+\phi_i-\phi_j|\leqq\eta$. Since $|\sum n_{ij}|$ is a positive integer strictly less than one, it is zero.

We apply now Proposition 2 to the measures $\mu^{\gamma}_{\Lambda,\eta}$. Let Λ be a cube in Z^d: $\Lambda=[-L,L]^d$, $l=(L,0,\ldots,0)$, $k=(L+1,0,\ldots,0)$, $\Lambda'=\Lambda\cup\{k\}$. We define two measures on $R^{||\Lambda|}$:

$$d\mu^{(1)}_\Lambda=(Z^{(1)}_\Lambda)^{-1}\prod_{\langle ij\rangle\subset\Lambda}\exp(\beta\cos(\phi_i-\phi_j))\chi(|\phi_i-\phi_j|\leqq\eta)$$

$$\cdot\exp\beta\cos\phi_k)\chi(|\phi_k|\leqq\eta)\prod_{i\in\Lambda}\mathrm{d}\phi_i, \tag{34}$$

$$d\mu^{(2)}_\Lambda=(Z^{(2)})^{-1}\prod_{\langle ij\rangle\subset\Lambda}\exp(\beta\cos(\phi_i-\phi_j))$$

$$\cdot\chi(|\phi_i-\phi_j|\leqq\eta)\prod_{i\in\partial\Lambda}\exp(\beta\cos\phi_i)\chi(|\phi_i|\leqq\eta)\prod_{i\in\Lambda}d\phi_i, \tag{35}$$

where in $Z^{(1)}_\Lambda, Z^{(2)}_\Lambda$ the integration runs from $-\infty$ to $+\infty$.

Corollary 2. *For $\eta<\pi/2$ and any m with* $(\operatorname{supp} m)\subset\Lambda$ *and with* $\underline{m}=\sum m(i)=0$,

$$\langle\cos m\phi\rangle^0_{\Lambda',\eta}=\langle\cos m\phi\rangle^{(1)}_\Lambda, \tag{36}$$

$$\langle\cos m\phi\rangle^+_{\Lambda,\eta}=\langle\cos m\phi\rangle^{(2)}_\Lambda. \tag{37}$$

Proof. For the free b.c. we first remark that, since we integrate in the numerator and the denominator of the l.h.s. of (36) functions which are invariant under rotations of all the spins in Λ', we may as well set $\phi_k=0$ since that integration is redundant. Then we apply Proposition 2 to the N. and D. with $\phi_k=0$ being the spin fixed outside Λ and

$$B=\exp\left(\beta\left(\sum_{\langle ij\rangle\subset\Lambda}\cos(\phi_i-\phi_j)\right)+\beta\cos\phi_k\right)$$

in the D. and the same thing multiplied by $\cos m\phi$ in the N.

For the $+$b.c. we apply also the identity (31) to the N. and D. including in B a factor

$$\prod_{\substack{j\notin\Lambda\\ j\neq k}}\ \prod_{\substack{i\in\Lambda\\ |i-j|=1}} \exp(\beta\cos\phi_i)\chi(|\phi_i|\leqq\eta \bmod 2\pi) \tag{38}$$

(which is periodic). We have to do this because Proposition 2 only allows one spins fixed outside of Λ (at site k). But now we have a periodic characteristic function which is left. In order to get (37) (no periodic χ in $\mu_\Lambda^{(2)}$) we write

$$\chi(|\phi_i|\leqq\eta \bmod 2\pi)=\sum_{n_i\in Z}\chi(|\phi_i-2\pi n_i|\leqq\eta)$$

and expand the product over i in (36) both in the N. and D. We shall see that only the term where all $n_i=0$ contributes and this will prove (37). Let us go around $\partial\Lambda$ starting from l and let j be the first point where $n_j\neq 0$. Then there is a $j'\in\partial\Lambda$ (just before j) with $|j-j'|=1$, $n_{j'}=0$. Since $\langle jj'\rangle\subset\Lambda$, $|\phi_j-\phi_{j'}|\leqq\eta$ and since $n_{j'}=0$, $|\phi_{j'}|\leqq\eta$ (for the integrand not be zero). So $|\phi_j|\leqq 2\eta$ and $2\pi|n_j|\leqq|2\pi n_j-\phi_j|+|\phi_j|\leqq 3\eta<2\pi$ and so $n_j=0$.

Now it is easy to check that the measures (34) and (35) are log concave perturbation of Gaussian ones: for $|\phi_i-\phi_j|<\pi/2$ one can find a $\tau>0$ such that $\cos(\phi_i-\phi_j)+\tau(\phi_i-\phi_j)^2$ is concave [5].

Now we generate the asymptotic expansion of $\langle\cos m\phi\rangle_\Lambda^{(1)}$, $\langle\cos m\phi\rangle_\Lambda^{(2)}$ by integrating by parts as before. We let the radius of the box $\Lambda, R(T)=\exp(\sqrt{\beta})$, which grows sufficiently fast to show that the boundary terms are negligible $[m(T)R(T)\sim\exp-(\log T)^2\exp(\sqrt{\beta})\to\infty$ as $T\to 0]$. Since our measures are log concave perturbations of Gaussian ones, we can control the $(\nabla\phi')^{2n}$, $\phi'=\sqrt{\beta}\phi$ and the δ functions $\delta(|\nabla\phi|=\eta)$ by the Brascamp-Lieb inequalities [5]. We also control the expectation values appearing in the mass term with these inequalities, because $\langle\phi^2\rangle_\Lambda^{(i)}$, $i=1,2$ is bounded by the corresponding Gaussian expectation which is finite for $d\geqq 3$ and diverge like $\log\Lambda\sim\sqrt{\beta}$ for $d=2$ [for $d=1$ one would have to choose $R(T)=\exp(-3/2(\ln T)^2)$] (see [3]).

Remark. 1. Since we do not use reflection positivity, the above proof works, in principle, for ferromagnetic finite range interactions instead of nearest-neighbor ones. One would have to check the Gaussian estimates of Appendices A and B of [3] for non-nearest-neighbor Gaussian measures. The proof however only works for $\langle\cos m\phi\rangle$ with $\underline{m}=0$.

2. One could, with this method, perform directly the expansion for $\langle\cos m\phi\rangle$ with $\sum m_i\neq 0$ and, in particular show that $\langle\cos\phi_0\rangle\neq 0$ for T small enough if one could prove one of the following statements, which are presumably true:

1. The difference $|\langle\cos m\phi\rangle_\Lambda-\langle\cos m\phi\rangle_{\Lambda,\eta}|$ between the correlation functions and the one conditioned on $|\phi_i-\phi_j|\leqq\eta$ for all pairs $\langle i,j\rangle$ is exponentially small as $\beta\to\infty$ uniformly in Λ.

2. The expectation value $\langle\cos m\phi\rangle_\Lambda^+$ converge sufficiently fast to their infinite volume limit. A convergence of order $(\log|\Lambda|)^{-1}$ would suffice to prove the phase transition.

3.The limit of $\langle\cos m\phi\rangle_h$ as $h\downarrow 0$ is fast enough; again $(\log h)^{-1}$ would be enough to prove the phase transition.

Any of these statements would provide a proof of phase transition for three or more dimensional rotator models without using reflection positivity.

V. On the Decay of the Two-Point Function

We show here that whenever there is a spontaneous magnetization, the transverse two-point function of the plane rotator, $\langle \sin\phi_0 \sin\phi_x \rangle$, behaves exactly like $\beta^{-1}|x|^{-(d-2)}$ for large $|x|$. This is what the Gaussian (spin-wave) approximation would predict. The upper bound was proven in [25]. We prove here the lower bound, which improves Goldstone theorem type of result (e.g. [19]) showing that $\sum_x \langle \sin\phi_0 \sin\phi_x \rangle^2$ diverges. The proof relies on the infrared bound, the Mermin-Wagner argument and the correlation inequalities of [22, 24, 15].

Theorem. *For any $d \geqq 3$, there exists constants c_1, c_2 such that for all $x \in Z^d$*

a) $$\frac{c_2 \bar{m}^2}{\beta|x|^{d-2}} \leqq \langle \sin\phi_0 \sin\phi_x \rangle \leqq \frac{c_1}{\beta|x|^{d-2}},$$

b) $$\frac{c_2 \bar{m}^2}{\beta|x|^{2(d-2)}} \leqq \langle \cos\phi_0 \cos\phi_x \rangle - \bar{m}^2 \leqq \frac{c_1}{\beta|x|^{d-2}},$$

where $\bar{m} = \langle \cos\phi_0 \rangle$ is the spontaneous magnetization and $|x| = \sum_{e=1}^{d} |x_e|$.

Proof. The upper bounds are proven in [25]. b) Follows from a), the Dunlop-Newman inequality [9] and the inequality $\langle \cos\phi_0 \cos\phi_x \rangle - \bar{m}^2 \leqq \langle \sin\phi_0 \sin\phi_x \rangle$ proven in [10, 16]. Let us prove a) for $d=3$ (the general case is similar). By the infrared bounds and the Mermin-Wagner argument, (see e.g. [21]) $S(p)$, the Fourier transform of $\langle \sin\phi_0 \sin\phi_x \rangle$ satisfies the bounds,

$$\frac{c_2' \bar{m}^2}{\beta|p|^2} \leqq S(p) \leqq \frac{c_i'}{\beta|p|^2}.$$

Therefore, if $f_L(x)$ is the characteristic function of a cube $\Lambda \leqq Z^3$ centered at the origin of volume L^3:

$$c_2'' \beta^{-1} L^5 \leqq \sum_{x,y} \langle \sin\phi_x \sin\phi_y \rangle f_L(x) f_L(y) \leqq c_1'' \beta^{-2} L^5$$

$\left(\text{one has only to show that } \int_{-\pi}^{\pi} p^{-2} |\hat{f}_L^2(p)|^2 d^3p \sim L^5, \text{ see e.g. [4]}\right)$.

By positivity of $\langle \sin\phi_x \sin\phi_y \rangle$ and translation invariance, we have:

$$\left(\frac{L}{2}\right)^3 \sum_{x \in 1/2\Lambda} \langle \sin\phi_0 \sin\phi_x \rangle \leqq \sum_{x,y} \langle \sin\phi_x \sin\phi_y \rangle f_L(x) f_L(y)$$
$$\leqq c_1'' \beta^{-1} L^5 \tag{40}$$

and

$$(L)^3 \sum_{x \in \Lambda} \langle \sin\phi_0 \sin\phi_x \rangle \geqq \sum_{x,y} \langle \sin\phi_x \sin\phi_y \rangle f_L(x) f_L y)$$
$$\geqq c_2'' \beta^{-1} L^5, \tag{41}$$

where $k\Lambda$ is a cube of size $(kL)^3$ centered at the origin.

Now we use the fact that, by correlation inequalities [22, 24, 15] $\langle \sin\phi_0 \sin\phi_x \rangle$ reaches its maximum for x inside Λ at the corners of Λ and its maximum outside Λ for x along the axis $|x| = \left[\frac{L}{2}\right] + 1$. (The minimum property is used in [25] for the upper bound.) To get the lower bound, let

$$x = \left(\left[\frac{L}{2}\right] + 1, 0, \ldots, 0\right).$$

Then

$$\begin{aligned}
&\langle \sin\phi_0 \sin\phi_x \rangle \\
&\geqq ((kL^3) - (L)^3)^{-1} \sum_{\substack{x \in k\Lambda \\ x \notin \Lambda}} \langle \sin\phi_0 \sin\phi_x \rangle \\
&= ((kL)^3 - (L)^3)^{-1} \left(\sum_{x \in k\Lambda} \langle \sin\phi_0 \sin\phi_x \rangle - \sum_{x \in \Lambda} \langle \sin\phi_0 \sin\phi_x \rangle \right) \\
&\geqq ((kL)^3 - (L)^3)^{-1} \frac{c_2''(kL)^2}{\beta} - \frac{2^5 c_1'' L^2}{\beta}
\end{aligned}$$

using inequalities (40) and (41). This proves the lower bound when x is along a coordinate axis, if we choose k such that $c_2''k^2 \geqq 2^5 c_1'' + 1$ and $c_2 = 2(k^3 - 1)^{-1}$.

When x is not along a coordinate axis we use the result of [15] which says that for given $|x| = \sum_{e=1}^{d} |x_e|$, $\langle \sin\phi_0 \sin\phi_x \rangle$ reaches its maximum along the coordinate axis. This is proven in [15] for Ising models, but the same inequalities hold for the plane rotator model: one has to use the fact that the state $\langle \ \rangle$ can be obtained as a limit of Gibbs states in Λ_n for any increasing sequence $\Lambda_n \uparrow Z^d$, [2, 23], so that we can use all the inequalities of [22, 24, 15] at once.

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