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Lattice Systems with a Continuous Symmetry

I. Perturbation Theory for Unbounded Spins

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Abstract. We investigate a continuous Ising system on a lattice, equivalently an anharmonic crystal, with interactions:

$$\sum_{\langle x,y\rangle} (\phi_x-\phi_y)^2+\lambda(\phi_x-\phi_y)^4, \qquad \phi_x\in\mathbb{R}, \qquad x\in\mathbb{Z}^d .$$

We prove that the perturbation expansion for the free energy and for the correlation functions is asymptotic about $\lambda=0$, despite the fact that the reference system ($\lambda=0$) does not cluster exponentially. The results can be extended to more general systems of this type, e.g. an even polynomial semibounded from below instead of a quartic interaction. By a suitable scaling, λ corresponds to the temperature.

I. Introduction

In recent years there have been several works giving a mathematical justification to the high and low temperature (H.T., L.T.) perturbation theory frequently used by physicists in statistical mechanics and quantum field theory, see for example [4, 6–9, 11, 13].

In all these cases the perturbation is made around an unperturbed system which is explicitly known, e.g. a Gaussian field ($\lambda P(\Phi)_2$ theories) or a product of uncoupled systems (statistical mechanics in H.T. regime), and which is massive in the sense that its correlation functions are exponentially clustering. No similar justification has been given, however, for the case where the reference system is not exponentially clustering. This occurs for example in the anharmonic crystal and the n component Heisenberg (classical or quantum) spin system. In both cases the lack of exponential clustering appears related to the invariance of the Hamiltonian

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under a continuous group. It is the purpose of this paper, the first of a series to investigate such systems, to prove the existence of an asymptotic expansion for such a case: the reference system is a massless Gaussian field on a lattice, for which the covariance is not even summable.

Specifically, we consider a lattice $\mathbb{Z}^d$ and the following Hamiltonian

$$\beta H=\beta\left[\sum_{\langle x,y\rangle}(\phi_x-\phi_y)^2+\sum_{\langle x,y\rangle}(\phi_x-\phi_y)^4\right], \tag{I}$$

where the sums are over the nearest neighbour pairs, and ϕ_x is a real random variable uniformly distributed on $\mathbb{R}$, $\beta=\frac{1}{T}$ the reciprocal temperature. Making a change of variable $\phi_x\to\sqrt{\beta}\,\phi_x$, this is equivalent to

$$\beta H=\sum_{\langle x,y\rangle}(\phi_x-\phi_y)^2+T\sum_{\langle x,y\rangle}(\phi_x-\phi_y)^4. \tag{II}$$

We shall prove that expectations values of the form

$$\left\langle\prod_x\phi_x^{n_x}\prod_{\langle x,y\rangle}(\phi_x-\phi_y)^{n_{xy}}\right\rangle,\qquad (n_x=0\ \text{if}\ d<3)$$

defined by adding a mass term to (II) and letting it go to zero in the thermodynamic limit, have an asymptotic expansion in powers of T.

This can be viewed as a low-temperature expansion for the model (I) or an expansion in the coupling constant for the perturbed harmonic crystal (II). It can also be related to a low temperature expansion for the isotropic rotator model. [One would expand $\cos(\phi_x-\phi_y)-1$, keep the first two terms, let ϕ_x run from $-\infty$ to ∞ and change the sign of the quartic term to insure stability.]

The basic tool for performing this low temperature expansion is, as in the massive case, the integration by parts formula for Gaussian measures (see [5, 11, 13]). This has the advantage of expressing the remainder of the expansion up to a given order in a form which is easier to estimate than the remainder in a Taylor series. Because our covariance is not summable, we cannot proceed as directly as in the massive case. What we do is to add and substract a temperature dependent mass term to the Hamiltonian and perform the integration by parts with respect to the massive Gaussian measure. By a suitable choice of the dependence of the mass on the temperature, we can show that to each order n, the remainder is indeed small compared to T^n.

The same method works for more general interactions of finite range of the type (II) and also for any even polynomial semi-bounded from below instead of a quartic.

Our method gives only an asymptotic, not analytic, expansion around $T=0$. This is as expected since for $T<0$ the system is not stable. Indeed, for the one dimensional case the equilibrium measure corresponding to (II) factorises into a product of measures of the form $\exp[-\theta^2-T\theta^4]d\theta$ with $\theta=\phi_x-\phi_y$, where it is known that the integral and moments are not analytic around $T=0$.

One would expect however, in analogy with the one dimensional case, that the correlation functions are analytic in T for $\operatorname{Re}T>0$, since this system is not expected to have any phase transition for any T. Our method does not give such a result.

The outline of the paper is as follows: in Sect. 2 we describe the model. Section 3 contains the statement and proof of the main theorems. Section 4 is devoted to various remarks and simple extensions. We gather in the appendix the estimates on the Gaussian lattice field that are used in the text.

Decay properties of the correlation functions for systems described by (II) which are valid for all $T>0$ are given in Part II of this series [3].

Finally, a modification of these methods proves asymptoticity of the low temperature expansion for the classical rotator model. This will be described in Part III by the present authors and E. Lieb.

II. The Model

At each point x of a lattice $\mathbb{Z}^d$, there is a real random variable ϕ_x and we consider the following Hamiltonian H_Λ, with periodic boundary conditions on Λ, Λ being a parallelipiped in $\mathbb{Z}^d$ centered at the origin:

$$H_\Lambda = \sum_{\langle x,y\rangle \subset \Lambda} (\phi_x - \phi_y)^2 + T \sum_{x,y\in\Lambda} J(x-y)(\phi_x-\phi_y)^4 . \tag{1}$$

$\langle x, y\rangle$ means that x and y are nearest neighbour, or that they are at opposite ends of Λ, and T stands for the temperature. (β has been absorbed in H.) $J(x-y) \geqq 0$, has finite range D. If $J(x-y)=1$ when $|x-y|=1$ and 0 otherwise, we call it a *nearest-neighbour interaction.*

We also consider the Hamiltonian

$$H_{\Lambda,m} = H_\Lambda + m^2 \sum_{x\in\Lambda} \phi_x^2 . \tag{2}$$

For $m \neq 0$ we define the expectation of functions of the type $\prod_x \phi_x^{n_x}$, $n_x \in \mathbb{N}$, via the formula:

$$\langle f\rangle_{\Lambda,m} = Z_{\Lambda,m}^{-1} \int_{\mathbb{R}^{|\Lambda|}} f \exp(-H_{\Lambda,m}) \prod_{x\in\Lambda} d\phi_x$$

$$Z_{\Lambda,m} = \int_{\mathbb{R}^{|\Lambda|}} \exp(-H_{\Lambda,m}) \prod_{x\in\Lambda} d\phi_x .$$

Notation

$\{e_\alpha\}$ $\alpha = 1 \dots d$ is a basis of $\mathbb{Z}^d$ given by:

$$e_\alpha = (0 \dots 0, 1, 0 \dots 0) \quad (1 \text{ in the } \alpha\text{th column}).$$

The difference variables $\nabla_x^{e_\alpha}\phi = \phi_x - \phi_{x+e_\alpha}$ will be called gradients, e_α will sometimes be replaced by e or, when used in a summation by ξ, i.e. $\sum_{e_\alpha, \alpha=1}^{d}$ will be written $\sum_\xi$.

If a is a finite subset of $(\mathbb{Z}^d, \{e_\alpha\})$ (with "repetition" to avoid the use of exponents), we introduce products of gradients:

$$\prod_{(x,\xi)\in A} \nabla_x^\xi \phi . \tag{3}$$

Given a function $f: \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, we define

$$\nabla_x^e \nabla_y^{e'} f(x,y) = [f(x,y) - f(x+e,y)] - [f(x,y+e') - f(x+e,y+e')]. \tag{4}$$

We use below the letter C to denote a constant which may be different from one formula to the other.

States

We now define the infinite volume states that we shall consider. The construction differs slightly when $d=1,2$, and when $d \geqq 3$.

When $T=0$ in (1), we have a Gaussian model and, for $d \geqq 3$,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \left\langle \prod_x \phi_x^{n_x} \right\rangle_{\Lambda,m}$$

exists and is uniformly bounded in m. Since the measure $\langle\ \rangle_{\Lambda,m}$ is a log concave perturbation of a Gaussian, it follows from the Brascamp and Lieb inequalities [1, 2] that the same is true for $T \neq 0$ (modulo the fact that the limit $\Lambda \uparrow \mathbb{Z}^d$ may have to be taken via a subsequence). We therefore let for $d \geqq 3$ $\langle\ \rangle$ denote any expectation value defined on functions of the form $\prod \phi_x^{n_x}$ obtained as a limit $\Lambda \uparrow \mathbb{Z}^d$ and then $m \downarrow 0$ (possibly via a subsequence).

The Brascamp-Lieb inequalities also tell us that, for $d \geqq 3$, the Fourier transform of $\langle \phi_0 \phi_x \rangle$ is bounded by $\mathrm{const}|p|^{-2}$ (p Fourier variable).

Using these inequalities one easily shows that the *free energy* $P(T)$ exists (for all d):

$$P(T) = \lim_{m \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-1} \log Z_{\Lambda,m}. \tag{5}$$

(See Sect. IV for more details.)

When $d=1$ or 2, we construct the infinite volume expectation values in the following way:

Let f denote any function of the form (3). Define

$$\langle f \rangle_{\Lambda,m_0} = Z_{\Lambda,m_0}^{-1} \int_{\mathbb{R}^{|\Lambda|}} f \exp(-H_\Lambda - m_0^2 \phi_{x_0}^2) \prod_{x \in \Lambda} d\phi_x. \tag{6a}$$

That is, we put a mass term only at x_0. We notice that $\langle f \rangle_{\Lambda,m_0}$ is independent of x_0 and m_0, because, if we integrate all ϕ_x in (6) for $x \neq x_0$, the result is independent of ϕ_{x_0}. This can be checked by changing variables $\phi_x \to \phi_x - \phi_{x_0}\ \forall x \neq x_0$. In what follows we let $m_0 \to \infty$ in (6); that is, we fix ϕ_{x_0} to be equal to zero. Again, by the Brascamp-Lieb inequalities, for any f of the form (3), $|\langle f \rangle_{\Lambda,m_0=\infty}|$ is uniformly bounded in Λ and we can take the limit $\Lambda \to \mathbb{Z}^d$ via a subsequence. For $d=1,2$, $\langle\ \rangle$ will denote any limit obtained in that way. We define the corresponding free energy for $d=1,2$ as that obtained with $x_0=0$,

$$P(T) = \lim_{\Lambda \to \infty} |\Lambda|^{-1} \log Z_{\Lambda,m_0=\infty}. \tag{6b}$$

Actually, this definition coincides with the previous one (5), in particular $P(T)$ is independent of m_0:.

III. The Results

Theorem 1. *The free energy $P(T)$ and the correlation functions $\left\langle \prod_{(x,\xi)\in A} (\nabla_x^\xi \phi_x)\right\rangle$ have an asymptotic expansion to all orders in T whose coefficients are given by the usual perturbation theory.*

Theorem 2. *For $d \geqq 3$ the correlation functions $\left\langle \prod_{x\in D} \phi_x^{n_x}\right\rangle$ (D, finite subset of $\mathbb{Z}^d$) have an asymptotic expansion to all orders in T whose coefficients are given by the usual perturbation theory.*

Outline of Proof. Our expansion is based on the integration by parts formula (I.P.) [6, 9, 11, 13]:

If $\{\phi_z\}_{z=1}^n$ is a set of Gaussian variables and μ is the corresponding Gaussian measure with covariance C_{xy}, then I.P. gives

$$\int \phi_y F(\{\phi_z\}_{z=1}^n) d\mu = \sum_{x=1}^n C_{xy} \int \frac{d}{d\phi_x} F(\{\phi_z\}_{z=1}^n) d\mu \tag{7}$$

for F a differentiable function of the $\{\phi_z\}_{z=1}^n$ and $\phi_y F \in L^1(d\mu)$.

In order to give an idea of the proof, we restrict ourselves to nearest-neighbour interactions and consider the zeroth order of $\langle \nabla_0^e \phi \nabla_0^{e'} \phi\rangle$ in $d \geqq 3$.

We first consider finite volume expectation values and absorb the interaction into the function F of formula (7). This yields

$$\langle \nabla_0^e \phi \nabla_0^{e'} \phi\rangle_{\Lambda,m} = \nabla_0^e \nabla_0^{e'} C_{00}^{\Lambda,m} - 4T \sum_{x\in\Lambda} \sum_{|y-x|=1} \nabla_0^e C_{0x}^{\Lambda,m} \langle \nabla_0^{e'} \phi (\phi_x - \phi_y)^3\rangle_{\Lambda,m}. \tag{8}$$

Grouping together the terms

$$\nabla_0^e C_{0x}^{\Lambda,m} \langle \nabla_0^{e'} \phi (\phi_x - \phi_{x+e_\alpha})^3\rangle_{\Lambda,m} + \nabla_0^e C_{0,x+e_\alpha}^{\Lambda,m} \langle \nabla_0^e \phi (\phi_{x+e_\alpha} - \phi_x)^3\rangle_{\Lambda,m}$$

and taking $\Lambda \uparrow \infty$, (8) becomes

$$\langle \nabla_0^e \phi \nabla_0^{e'} \phi\rangle_m = \nabla_0^e \nabla_0^{e'} C_{00}^m - 4T \sum_{x\in\mathbb{Z}^d} \sum_{\xi} \nabla_0^e \nabla_x^\xi C_{0x}^m \langle \nabla_0^{e'} \phi (\nabla_x^\xi \phi)^3\rangle_m. \tag{9}$$

If we are interested in the zeroth order, we have to prove that the second term in the r.h.s. of (9) goes to zero with T (uniformly in m).

Since we have uniform bounds on $|\langle \nabla_0^{e'} \phi (\nabla^\xi \phi)^3\rangle_m|$ by the Brascamp-Lieb inequalities, it might be thought possible to bound $\sum_{x\in\mathbb{Z}^d} \sum_{\xi} |\langle \nabla_0^e \nabla_x^\xi C_{0x}^m\rangle|$ uniformly in m. This is however impossible because after summing over x, this expression diverges as $\ln m$ when $m \downarrow 0$ [see Appendix, Proposition A1 d].

The idea is then to add and substract in the Hamiltonian a mass term which is temperature dependent. We write

$$H_{\Lambda,m'} = \sum_{\langle x,y\rangle\subset\Lambda} (\phi_x - \phi_y)^2 + m^2 \sum_{x\in\Lambda} \phi_x^2 + m^2(T) \sum_{x\in\Lambda} \phi_x^2 - m^2(T) \sum_{x\in\Lambda} \phi_x^2 + T \sum_{\langle x,y\rangle\subset\Lambda} (\phi_x - \phi_y)^4.$$

We expand the new interaction by I.P. with respect to the Gaussian theory with mass $m' = m + m(T)$. After letting $\Lambda \to \mathbb{Z}^d$, $m \to 0$, this gives:

$$\langle \nabla_0^e \phi \nabla_0^{e'} \phi \rangle - \nabla_0^e \nabla_0^{e'} C_{00}^{m(T)} = R(T)$$
$$-4T \sum_{x \in \mathbb{Z}^d} \sum_{\xi} \nabla_0^e \nabla_x^{\xi} C_{0x}^{m(T)} \langle \nabla_0^{e'} \phi (\nabla_x^{\xi} \phi)^3 \rangle$$
$$+ 2m^2(T) \sum_{x \in \mathbb{Z}^d} \nabla_0^e C_{0x}^{m(T)} \langle \nabla_0^{e'} \phi \phi_x \rangle . \tag{10}$$

We can now by choosing $m(T)$ very small as $T \to 0$, for instance $m(T) = \exp(-(\ln T)^2)$, indeed prove that $\nabla_0^e \nabla_0^{e'} C_{00}$ is the zeroth order term in the asymptotic expansion.

The argument is in two steps:

i) we show that the Gaussian expectation with mass $m(T)$ are close to the massless one (Appendix B).

$$|\nabla_0^e \nabla_0^{e'} C_{00}^{m(T)} - \nabla_0^e \nabla_0^{e'} C_{00}| \leqq m^2(T) .$$

ii) We estimate $R(T)$ using the Brascamp-Lieb bounds:

$$|\langle \nabla_0^{e'} \phi (\nabla_x^{\xi} \phi)^3 \rangle| \leqq C$$
$$|\langle \nabla_0^{e'} \phi \phi_x \rangle| \leqq C \quad \text{(Sect. II)} .$$

Gaussian estimates then give (Appendix A)

$$\sum_{x \in \mathbb{Z}^d} \sum_{\xi} |\nabla_0^e \nabla_x^{\xi} C_{0x}^{m(T)}| \leqq C \ln m(T)$$
$$\sum_{x \in \mathbb{Z}^d} |\nabla_0^e C_{0x}^{m(T)}| \leqq C m^{-1}(T)$$

and $R(T)$ is thus bounded by

$$C T \ln m(T) + C m(T) .$$

Clearly, with our choice of $m(T)$, $R(T) \to 0$ as $T \to 0$.

Proof of Theorem 1. 1. The result for $P(T)$ follows from the one for the correlation functions and the formula:

$$P(T) = P(0) + 2 \int_0^T \sum_{\xi} \langle (\nabla_0^{\xi} \phi)^4 \rangle (T') dT' .$$

(See Sect. IV for a simpler argument.)

2. For the correlation functions, we start with the case of nearest neighbour interactions and $d \geqq 3$. Take out a factor from $\prod_{(x,\xi) \in A} \nabla_x^{\xi} \phi$ (which, for simplicity, we denote $\nabla_0^e \phi$) and call f the product of the other factors. We then integrate by parts in $\langle (\nabla_0^e \phi) f \rangle$ and obtain, in analogy with (10),

$$\langle (\nabla_0^e \phi) f \rangle = \sum_x \nabla_0^e C_{0x}^{m(T)} \left\langle \frac{d}{d\phi_x} f \right\rangle , \tag{I}$$
$$-4T \sum_{x \in \mathbb{Z}^d} \sum_{\xi} \nabla_0^e \nabla_x^{\xi} C_{0x}^{m(T)} \langle f (\nabla_x^{\xi} \phi)^3 \rangle , \tag{II}$$
$$+ 2m^2(T) \sum_{x \in \mathbb{Z}^d} \nabla_0^e C_{0x}^{m(T)} \langle f \phi_x \rangle . \tag{III}$$

We see that I.P. produces three kinds of terms. The first term (I) comes from a contraction with f and contains only a finite sum. The second one (II), the temperature term, comes from the differentiation of the interaction $T(\nabla\phi)^4$ and is bounded by $CT\ln m(T)$ (Proposition A1 c and d). The mass term (III) is bounded by $Cm(T)$, (Proposition A1 b). Choosing $m(T)=\exp(-(\ln T)^2)$ (III) is smaller than any power of T when $T\to 0$, and can therefore be neglected in the expansion.

To get the zeroth order, we apply I.P. to $\langle f\nabla_0^e\phi\rangle$ and then repeatedly to the I-term produced until we have a fully Gaussian expectation value. (This is called Step 1.) To obtain the coefficient of order 1, we apply the same procedure to the (non Gaussian) expectation value of the terms produced by Step 1 which have a factor T, producing then also terms with a factor T^2. If we continue the procedure until order n, the remainder $R(T)$ (again neglecting mass terms) will be bounded by a sum of terms of the form:

$$CT^{n+1}\prod_{(x,\xi)\in A}F(x,\xi),$$

where

$$F(x,\xi)=\sum_{\substack{y_1,\dots,y_{p_x}\\ \xi_1,\dots,\xi_{p_x}}}|\nabla_x^{\xi}\nabla_{y_1}^{\xi_1}C_{xy_1}|\left|\prod_{(ij)}\nabla_{y_i}^{\xi_i}\nabla_{y_j}^{\xi_j}C_{y_iy_j}\right| \tag{11}$$

with $\sum_{x\in A}p_x=n+1$.

We associate a graph to each $F(x,\xi)$, given by the tree attached to each point (x,ξ). This tree is made of vertices (x,ξ), $(y_1,\xi_1)\dots(y_{p_x},\xi_{p_x})$ and of edges $((y_i\xi_i),(y_j\xi_j))$ for (ij) in the product in (11). (See Appendix B for a more detailed construction.)

We have bounded $|\nabla_x^{\xi}\nabla_y^{\xi'}C_{xy}|\leqq C$ to suppress the loops of the graph. For each power of T, there has to be one new summation over a variable x_s and a factor $\nabla_{x_{s-1}}^{\xi_{s-1}}\nabla_{x_s}^{\xi_s}C_{x_{s-1}x_s}$ coming from the I.P. Only these factors are kept in (11).

The non Gaussian expectation value are also bounded by a constant (Sect. II). Using Gaussian estimates (Proposition A1 c and d):

$$F(x,\xi)\leqq(\ln m)^{p_x}\leqq(\ln T)^{2p_x}$$

and

$$R(T)\leqq CT^{n+1}(\ln T)^{2(n+1)}.$$

We obtain then

$$\langle(\nabla_0^e\phi)f\rangle=\sum_{i=1}^{n}a_i(T)T^i+\text{error of higher order in } T,$$

where $a_i(T)$ are the coefficient computed in a Gaussian theory of mass $m(T)$. By Appendix B, Proposition A6,

$$|a_i(T)-a_i(0)|\leqq Cm^2(T)=C\exp(-2(\ln T)^2)$$

and $a_i(0)$ are computed with a massless Gaussian measure. This finishes the proof for $d\geqq 3$ and nearest-neighbour interactions.

3. Now we turn to the case $d\leqq 2$ but still with nearest-neighbour interactions. Since for $d=1$ and nearest-neighbour interactions, the measure factorizes into a product of measures over the gradient variables, we shall now consider only $d=2$.

In this case we integrate by parts with respect to a massive Gaussian measure with mass $m(T)=\exp(-(\ln T)^2)$ restricted to a box Λ_0 whose radius $r(T)=\exp(T^{-1})$ grows as $T\to 0$. We put periodic b.c. on the Gaussian measure. We write

$$\begin{aligned}
H_\Lambda &= H^G_{\Lambda 0} + H'_\Lambda ,\\
H^G_{\Lambda_0} &= \sum_{\langle xy\rangle \subset \Lambda_0} (\phi_x-\phi_y)^2 + m^2(T) \sum_{x\in\Lambda_0} \phi_x^2 ,\\
H_{\Lambda'} &= \sum_{\substack{\langle xy\rangle \\ x\in\Lambda_0,\, y\notin\Lambda_0}} (\phi_x-\phi_y)^2 + \sum_{\langle xy\rangle\subset \Lambda\setminus\Lambda_0} (\phi_x-\phi_y)^2 ,\\
&\quad - \sum_{x\in\partial\Lambda_0} (\phi_x-\phi_{\bar{x}})^2 + T \sum_{\langle xy\rangle\subset\Lambda} (\phi_x-\phi_y)^4 ,\\
&\quad - m^2(T) \sum_{x\in\Lambda_0} \phi_x^2 ,
\end{aligned}$$

where x and $\bar{x}$ are at opposite sides of Λ_0.

We let x_0 in (6) be outside Λ_0 and such that $\operatorname{dist}(x_0, \Lambda_0)=2$. Let $C^{\Lambda_0,m}_{x,y}$ be the covariance of the Gaussian measure given by $H^G_{\Lambda_0}$.

Then, I.P. with respect to that measure gives:

$$\begin{aligned}
\langle(\nabla^e_0\phi)^2\rangle &= \nabla^e_0\nabla^e_0 C^{\Lambda_0,m}_{00}\\
&\quad -4T\sum_{x\in\Lambda_0}\sum_{\xi} \nabla^e_0\nabla^\xi_x C^{\Lambda_0,m}_{0x}\langle(\nabla^e_0\phi)(\nabla^\xi_x\phi)^3\rangle\\
&\quad -\sum_{\substack{x\in\partial\Lambda\\ y\notin\Lambda_0}} \nabla_0 C^{\Lambda_0,m}_{0,x}\langle(\nabla^e_0\phi)(\phi_x-\phi_y)\rangle\\
&\quad +\sum_{x\in\partial\Lambda_0} \nabla^e_0 C^{\Lambda_0,m}_{0,x}\langle(\nabla^e_0\phi)(\phi_x-\phi_{\bar{x}})\rangle\\
&\quad -4T\sum_{\substack{x\in\partial\Lambda_0\\ y\notin\Lambda_0}} \nabla^e_0 C^{\Lambda_0,m}_{0,x}\langle(\nabla^e_0\phi)(\phi_x-\phi_y)^3\rangle\\
&\quad +m^2\sum_{x\in\Lambda_0} \nabla^e_0 C^{\Lambda_0,m}_{0,x}\langle\nabla^e_0\phi\phi_x\rangle .
\end{aligned} \tag{12}$$

The second term can be estimated as before by $CT\log m(T)$. The third, fourth and fifth terms can be shown to be negligible to any order in T because all the covariances entering the sums are of order $\exp(-mr)$, provided we suitably bound the corresponding expectation values. The sixth term is bounded by

$$m \sup_{x\in\Lambda_0} |\langle\nabla^e_0\phi\phi_x\rangle| \tag{13}$$

(by Appendix A1b).

The expectation values can be bounded because

$$\langle(\phi_x-\phi_y)^2\rangle \leqq \text{const}\ln|x-y| \tag{14}$$

as one can see, using the Brascamp-Lieb inequalities and the explicit computation for the Gaussian measure. If we take $y=x_0$ in (14), since $\phi_{x_0}=0$ we have that, for all $x\in\Lambda_0\langle\phi_x^2\rangle^{1/2}\leqq\text{const}(\ln|x|)^{1/2}=\text{const}|T^{-1}|^{1/2}$. Using this and the Schwartz inequality one proves that (13) and the third, fourth and fifth terms in (12) are negligible to all orders in T.

We repeat the procedure for all higher orders and the only estimate we have to check is that the Gaussian coefficients [in Λ_0 with mass $m(T)$] are close to the massless infinite volume ones. This is done in Proposition A7.

4. In the general case where we have interactions of range D we do not get gradients of the covariance as in (12) but differences $C_{0x}-C_{0y}$ where x and y are at most at a distance D. These differences can be written as a finite sum of gradients and the estimates on the way the sum over the lattice diverges with m remain unchanged except for factors of D.

For $d=1$, we use in this case the same method as for $d=2$ above. But we take $r(T)=\exp(\frac{3}{2}(\ln T)^2)$ so that $m(T)r(T)\to\infty$ and $m(T)r(T)1/2\to 0$ as $T\to 0$. Indeed, instead of (14) we have a bound $\langle(\phi_x-\phi_y)^2\rangle \leqq \mathrm{const}|x-y|$ and $\langle\phi_x^2\rangle^{1/2} \leqq |x|^{1/2} \leqq r(T)^{1/2}$.

Proof of Theorem 2. We first consider the simpler case of nearest-neighbour interactions in (1) below and then explain how to extend the proof to general interactions (2).

1) Let us start with $\langle\phi_0^2\rangle$ and do more general expectation values later. As in the proof of Theorem 1, we add and subtract a temperature dependent term, $m^2(T)\sum_{i\in\Lambda}\phi_i^2$, and perform the integration by parts with respect to the massive Gaussian. The integration by parts reads:

$$\langle\phi_0^2\rangle = C_{00}^{m(T)} - 4T\sum_{x\in\mathbb{Z}^d,\xi} \nabla_x^\xi C_{0x}^{m(T)} \langle\phi_0(\nabla_x^\xi\phi)^3\rangle - 2m^2(T)\sum_{x\in\mathbb{Z}^d} C_{0x}^{m(T)}\langle\phi_0\phi_x\rangle\,. \tag{15}$$

To obtain the zeroth order of our expansion, namely $\lim_{T\to 0}\langle\phi_0^2\rangle = C_{00}$, we use the fact that

$$\sum_{x\in\mathbb{Z}^d,\xi} |\nabla_x^\xi C_{0x}^{m(T)}| \leqq \frac{c}{m} \quad\text{and}\quad \sum_{x\in\mathbb{Z}^d} C_{0x}^{m(T)} = \frac{1}{m^2}$$

(Proposition A1 a and b).

The second term in (15) is thus of order T/m. To bound the third term we can use the fact that $|\langle\phi_0\phi_x\rangle| < \frac{c}{|x|^k}$ for some $k>0$ (actually, for any $k<1$, see Theorem 2 in [3]). As we show in 2), we can avoid the use of this theorem, but it simplifies the proof.

By Proposition A1e, the third term is of order m^k. Now, we can let $m(T)\to 0$ in such a way that $\frac{T}{m}\to 0$. We know that $C_{00}^{m(T)}\to C_{00}$ as $m\to 0$ with an error of order m (Proposition A2). This finishes the proof for the zeroth order. Let us remark that in this case we have a constraint in the choice of $m(T)$ namely $Tm^{-1}(T)\to 0$, so we cannot take $m(T)=e^{-(\ln T)^2}$ as in the first theorem.

For the higher orders, let us call as before the first term in (15) the *I-term*, the second one the *temperature term* and the third the *mass term*. If we want the asymptotic expansion up to order n, we start with (15) and leave the first and third term as they are, and integrate by parts one $\nabla_x^\xi \phi$ in the second term. This gives

$$\begin{aligned}\langle\phi_0(\nabla_x^\xi\phi)^3\rangle = \nabla_x^\xi C_{0x}\langle(\nabla_x^\xi\phi)^2\rangle + 2\nabla_x^\xi\nabla_x^\xi C_{xx}\langle\phi_0\nabla_x^\xi\phi\rangle \\ -4T\sum_{y\in\mathbb{Z}^d,\xi'}\nabla_x^\xi\nabla_y^{\xi'}C_{xy}\langle\phi_0(\nabla_x^\xi\phi)^2(\nabla_x^{\xi'}\phi)^3\rangle \\ -2m^2\sum_{y\in\mathbb{Z}^d}\nabla_x^\xi C_{xy}\langle\phi_0(\nabla_x^\xi\phi)^2\phi_y\rangle\,,\end{aligned} \tag{16}$$

where the first two terms are I-terms.

We insert (16) back in (15) and then we apply the following rule: repeat the integration by parts of all factors $\nabla_x^\xi\phi$ that occur in the expectation value of (16) until all the terms are *either*

- pure Gaussian [e.g. C_{00} in (15)].
- multiplied by a factor T^l, $l=\left[\frac{n+1}{k}\right]+n+1$, for some $k<1$
- multiplied by a factor m^4,
- do not contain any more gradients $\nabla_x^\xi\phi$ in the expectation value [e.g. the third term in (15)].

To obtain an asymptotic expansion up to order n we isolate all pure Gaussian terms which have coefficients that are powers of T up to order n. These, as we know (Proposition A6), converge to the same expression with $m=0$, and the error is of order m. We shall choose $m(T)=T^{(n+1)/k}$, so that the error is small compared to T^n (k is less than one). So we have only to show that all the other terms are small compared to T^n. We treat them one by one:

a) The remaining Gaussian terms are trivial to estimate: the series are uniformly bounded in m and they come with a coefficient which is a power of T larger than n.

b) For the terms with a factor T^l, each of them is bounded by a series of the form:

$$C\sum_{\substack{x_1,\dots,x_l\in\mathbb{Z}^d\\ \xi_1,\dots,\xi_l}}|\nabla_{x_1}^{\xi_1}C_{0x_1}|\left|\prod_{(ij)}\nabla_{x_i}^{\xi_i}\nabla_{x_j}^{\xi_j}C_{x_ix_j}\right|, \tag{17}$$

where the graph associated with the vertices $\{1,\dots,l\}$ and the edges (ij) in the product form a connected tree [as in (11)]. We use as before $(\nabla_x^\xi\nabla_y^{\xi'}C_{xy})\leqq C$ to avoid loops in the graph and we disregard the possible mass factor, $m^2\sum_{y\in\mathbb{Z}^d}\nabla_{x_i}^{\xi_i}C_{x_iy}$ which is of order m (there is at most one such factor because otherwise we would have a factor m^4 and this case is treated below). Now from the estimates in the Appendix Proposition A1c and d, it follows that (17) is of the order of $\frac{1}{m}(\log m)^{l-1}$. With our choice of l, and m, this multiplied by T^l is small compared to T^n.

c) For the terms with a factor m^4 we have a bound of the form (17), with possibly a smaller power T^v, $v\leqq l$, (and only v summation variables) but with a factor

$$m^4\sum_{y_1,y_2\in\mathbb{Z}^d}|\nabla_{x_i}^{\xi_i}C_{x_iy_1}||\nabla_{x_j}^{\xi_j}C_{x_jy_2}|$$

(i and j may coincide).

This sum (times m^4) is of order m^2 (Proposition A1b). Multiplied by the factor $\frac{1}{m}$ coming from $\sum_{x_1\in\mathbb{Z}^d}|\nabla_{x_1}^{\xi_1}C_{0x_1}|$, it still gives a contribution of order m, which is small compared to T^n.

d) Now we consider the case where there is a term with no more $\nabla_x^{\xi}\phi$ to integrate by parts. This is bounded by a series of the form

$$CT^{\nu}m^2\sum_{x_1,\ldots,x_\nu,y\in\mathbb{Z}^d}|\nabla_{x_1}^{\xi_1}C_{0x_1}|\prod_{(ij)}|\nabla_{x_i}^{\xi_i}\nabla_{x_j}^{\xi_j}C_{x_ix_j}|\,|\nabla_{x_\nu}^{\xi_\nu}C_{x_\nu y}|\,|\langle\phi_0\phi_y\rangle|, \tag{18}$$

where the product over (ij) is as in (17). Indeed, we know that we have exactly one mass term, because, if there was none, it is easy to see that either the term would be purely Gaussian or there would be gradients left in the expectation value and if there was more than one, we would be in the case of the factor m^4 considered above.

By Proposition A1e and Theorem 2 in [3]

$$m\sum_{y\in\mathbb{Z}^d}|\nabla_{x_\nu}^{\xi_\nu}C_{x_\nu y}|\,|\langle\phi_0\phi_y\rangle|\leqq cm^k. \tag{19}$$

On the other hand the sum over $x_2,\ldots,x_\nu$, involving only factors with two gradients diverges at most like $(\log m)^{\nu-1}$ and the sum over x_1 multiplied by m is bounded. So, with our choice of $m(T)$, this term is also small compared to T^n, which finishes the proof for $\langle\phi_0^2\rangle$.

Now in the general case, $\left\langle\prod_{x\in D}\phi_x\right\rangle$, the procedure is exactly the same. We start by integrating by parts one ϕ_x; we keep on integrating ϕ_x's in the I-terms produced by that integration until either we have a Gaussian expectation or a temperature or mass term. Then, after that, we integrate only gradients as before. This leads to the same four kinds of terms. The only differences with $\langle\phi_0^2\rangle$ is in Case d: in (18) $\langle\phi_0\phi_y\rangle$ is replaced by $\left\langle\prod_{x\in D'}\phi_x\phi_y\right\rangle$ for some $D'\subseteqq D$. Using Theorems 2 and 3 (or 4) in [3], $\left|\left\langle\prod_{x\in D'}\phi_x\phi_y\right\rangle\right|\leqq\text{const}\frac{1}{|y|^{k/2}}$. Then the same arguments as before hold with k changed into $k/2$ in the choice of m and l.

2) For non nearest-neighbour interactions, we do not know whether Theorem 2 of [3] holds, but we have the following Lemma, which we prove after the end of this proof:

Lemma. *Let $|x-y|=T^{n+1}$. Then the function $\langle\phi_x\phi_y\rangle$ (depending on T via $\langle\ \rangle$ and via $|x-y|$) is of order T^n.*

We use the Lemma to bound (18). [(15) is similar.] As a matter of fact, the proof of the Lemma establishes the asymptotic expansion for any two point function, so we consider (18) directly for general functions with $\prod_{x\in D'}\phi_x$ instead of ϕ_0.

First we restrict the sums in (18) to $|x_i|\leqq m^{(-1-\varepsilon)}$ $i=1\ldots\nu$ and $|y|\leqq m^{-1-\varepsilon}$ for some fixed $\varepsilon>0$. The error made on (18) by such a restriction is of order $\exp(-m^{-\varepsilon})$ which, since m is chosen to be a power of T turns out to be negligible in the expansion. This is easily seen because all the covariances are of order $\exp(-m|x|)$.

Now if we look at (19) with the restrictions on x_v, y, and $\prod_{x\in D'} \phi_x$ instead of ϕ_0, we split the sum over y into

$$\sum_{|y|\leqq m^{-1-\varepsilon}} = \sum_{|y|\leqq m^{-\frac{1}{2d}}} + \sum_{m^{-\frac{1}{2d}}\leqq y\leqq m^{-1-\varepsilon}},$$

where d = dimension of the lattice; the first sum contains $m^{-1/2}$ terms; so multiplied by m, this term is of order $m^{1/2}$. For the second sum we use F.K.G. inequalities [3] and the Lemma to show that $\left\langle \prod_{x\in D} \phi_x\phi_y \right\rangle$ is of order $(\log m)m^{1/2d}$ since $|y|\geqq m^{-1/2d}$ and m is a certain power of T. Since the sum $\sum_y |\nabla_{x_v}^{\xi_v} C_{x_v y}| \leqq m^{-1}$, (19) holds in this case for any $k<\frac{1}{2d}$. It is easy to see from the proof of Part 1) that, if we choose $m(T)=T^{2(n+1)/k}$ with $k<\frac{1}{2d}$, this is sufficient.

Proof of the Lemma. We integrate by parts ϕ_y and the gradients as we would do for a fixed y in the nearest-neighbour case. The main point is that in the expression corresponding to (18) we have

$$T^v m^2 \sum_{\dots} \sum_{y'\in\mathbb{Z}^d} \nabla_{x_1}^{\xi_1} C_{yx_i} \prod_{(i,j)} \nabla_{x_i}^{\xi_i} \nabla_{x_j}^{\xi_j} C_{x_i x_j} \nabla_{x_v}^{\xi_v'} C_{x_v y'} \langle \phi_0 \phi_{y'} \rangle .$$

We may directly estimate

$$\sum_{y'} \nabla_{x_v}^{\xi_v'} C_{x_v y'} \langle \phi_0 \phi_{y'} \rangle$$

uniformly in x by going to Fourier transforms and using the p^{-2} bound given by the Brascamp-Lieb inequalities [1]. The result is that it diverges at most like $\log m$. We estimate the other sums as before and we get a result of order m times some power of $\log m$. The mass term in (15) can be handled in a similar way. It is easy to check that the estimates on the temperature terms are uniform in y and also on the m^4 term. For the Gaussian terms, one easily checks, by using Fourier transforms that

$$\sup_y |yG_m(y)| \leqq \int \left|\frac{d}{dp} G_m(p)\right| d_p^d ,$$
$$\leqq \log m ,$$

where $G_m(y)$ is any massive Gaussian term coming in the asymptotic expansion of $\langle \phi_0 \phi_y \rangle$. So, all the Gaussian terms are bounded by $|y|^{-1} \log m$.

IV. Remarks

1. If we restrict ourselves to the free energy [defined in (5)] then our method works very easily in any dimension.

Suppose we simply add to (1) (and do not substract) a term $m^2(T)\sum_x \phi_x^2$ with $m(T)=\exp(-(\log T)^2)$. Then we carry out the integration by parts with respect to the massive Gaussian and we obtain the correct asymptotic series (we do not have any mass terms to worry about because we did not substract the mass). All we have to show is that the free energies at $T \neq 0$ with the mass and without the mass are close with an error of order $\exp(-(\log T)^2)$. This is easy because (with periodic b.c.)

$$\frac{1}{|\Lambda|}(\log Z_{\Lambda,m} - \log Z_{\Lambda,m'}) = 2\int_{m'}^{m} \tilde{m} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \phi_x^2 \rangle_{\Lambda,\tilde{m}} d\tilde{m}$$
$$= 2\int_{m'}^{m} \tilde{m} \langle \phi_0^2 \rangle_{\Lambda,\tilde{m}} d\tilde{m}.$$

In the thermodynamic limit, using the Brascamp-Lieb inequalities [1, 2].

$$\langle \phi_0^2 \rangle_m \leqq c, \quad d \geqq 3,$$
$$\langle \phi_0^2 \rangle_m \leqq c \log m \, d = 2,$$
$$\leqq m^{-1}, \quad d = 1.$$

So for all d,

$$\lim_{m' \downarrow 0} \lim_{\Lambda \uparrow \infty} \frac{1}{|\Lambda|} |\log Z_{\Lambda,m} - \log Z_{\Lambda,m'}| \leqq m.$$

2. We can consider, for $d \geqq 3$, the theory with $m = 0$ in (2) and Dirichlet boundary conditions (i.e. $\phi_x = 0$ outside Λ). When performing the integration by parts in a finite box Λ, we would have a boundary term but it would disappear in the thermodynamic limit because the covariance is massive and we would get the same formulas as in the proof of Theorem 1. We can also handle other "reasonable" boundary conditions (e.g. $|\phi_x| \leqq \text{const}\, x \notin \Lambda$) for functions of the gradients.

3. One generalization of Theorems 1 and 2 (for nearest-neighbour interactions) is to replace in (1) and (2) $(\phi_x - \phi_y)^4$ by an arbitrary even polynomial $Q(\phi_x - \phi_y)$ whose coefficient of highest degree is positive. The first remark is that, even if the polynomial is not convex, we can still use the Brascamp-Lieb inequalities [1]: let us write

$$(\phi_x - \phi_y)^2 + TQ(\phi_x - \phi_y) = \tfrac{1}{2}(\phi_x - \phi_y)^2 + \tfrac{1}{2}(\phi_x - \phi_y)^2 + TQ(\phi_x - \phi_y).$$

The matrix of the second derivatives of

$$\tfrac{1}{2}(\phi_x - \phi_y)^2 + TQ(\phi_x - \phi_y)$$

is

$$a_{xy} = (-1)^{|x-y|}(1 + TQ''(\phi_x - \phi_y));$$

Q'' is positive for large values of the argument (because the coefficient of highest degree is positive) and, for T small enough, $1 + TQ''$ is everywhere positive, which implies the Brascamp-Lieb inequalities.

Using such a Q we obtain the same formulas as in (9) and (10) with $4(\nabla_x^\xi \phi)^3$ replaced by $Q'(\nabla_x^\xi \phi)$.

Notice that F.K.G. inequalities also hold in this case (see [3]).

On the other hand, we do not know how to deal with non-even polynomials because then some terms on the perturbation series are only conditionally convergent.

Appendix: Gaussian Estimates

A. Sum of Covariances and Their Derivatives

We collect here some formulas which are used in the proof of Theorems 1 and 2. They concern sums of expectation values of the (infinite volume) Gaussian lattice field on $\mathbb{Z}^d$ given by the covariance:

$$\langle\phi_0\phi_x\rangle=\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\frac{e^{ikx}}{\sum_{i=1}^{d}(1-\cos k_i)+m^2}\prod_{i=1}^{d}dk_i\,. \tag{A1}$$

Proposition A1

a) $$\sum_{x\in\mathbb{Z}^d}|\langle\phi_0\phi_x\rangle|=\sum_{x\in\mathbb{Z}^d}\langle\phi_0\phi_x\rangle=\frac{1}{m^2}$$

b) $$\sum_{x\in\mathbb{Z}^d}|\langle\phi_0(\phi_x-\phi_{x+e})\rangle|=2\sum_{\substack{x\in\mathbb{Z}^d\\ x_e\geqq 0}}\langle\phi_0(\phi_x-\phi_{x+e})\rangle$$
$$=2\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi}\frac{dk}{1-\cos k+m^2}=\frac{2}{\sinh\tilde{m}}\leqq cm^{-1}\,,$$

where $\tilde{m}$ *is defined by* $\cosh\tilde{m}-1=m^2$.

c) $$\sum_{x\in\mathbb{Z}^d}|\langle(\phi_0-\phi_e)(\phi_x-\phi_{x+e})\rangle|$$
$$=\sum_{x\in\mathbb{Z}^d,x_e=0}\langle(\phi_0-\phi_e)(\phi_x-\phi_{x+e})\rangle$$
$$-2\sum_{x\in\mathbb{Z}^d,x_e\geqq 1}\langle(\phi_0-\phi_e)(\phi_x-\phi_{x+e})\rangle$$
$$=4\left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi}\frac{(1-\cos k)}{1-\cos k+m^2}dk=\frac{4(1-e^{-\tilde{m}})}{\sinh\tilde{m}}\leqq 4$$

with $\tilde{m}$ *as in* b).

d) $$\sum_{x\in\mathbb{Z}^d}|\langle(\phi_0-\phi_e)(\phi_x-\phi_{x+e'})\rangle|$$
$$=4\sum_{\substack{x_e\leqq 0\\ x_{e'}\geqq 0}}\langle(\phi_0-\phi_e)(\phi_x-\phi_{x+e'})\rangle$$
$$=4\left(\frac{1}{2\pi}\right)^2\int_{-\pi}^{\pi}\frac{dk_e dk_{e'}}{1-\cos k_e+1-\cos k_{e'}+m^2}\leqq c|\log m|\,.$$

e) For $0<k<1$,

$$\sum_{x\in\mathbb{Z}^d}\langle\phi_0\phi_x\rangle\frac{1}{(1+|x|)^k}\leqq\frac{C_1}{m^{2-k}}\,,$$
$$\sum_{x\in\mathbb{Z}^d}|\langle\phi_0(\phi_x-\phi_{x+e})\rangle|\frac{1}{(1+|x|)^k}\leqq\frac{C_2}{m^{1-k}}\,,$$

where C_1, C_2 *are constants independent of* m. *Similar estimates hold if we replace* $\frac{1}{(1+|x|)^k}$ *by* $\frac{1}{(1+|x-y|)^k}$ *for any* $y\in\mathbb{Z}^d$ (C_1, C_2 *independent of* y).

Proof. The proof of a) is immediate, since $\langle\phi_0\phi_x\rangle \geqq 0$ and summing over x_e amounts to putting $k_e=0$ in the integral (A1). For b) we notice that $\langle\phi_0(\phi_x-\phi_{x+e})\rangle \geqq 0$ (resp. $\leqq 0$) for $x_e \geqq 0$ (resp. <0). This follows from correlation inequalities (Theorem 1 and Corollary 1 in [10]). We write explicitly

$$\langle\phi_0(\phi_x-\phi_{x+e})\rangle = \int_{-\pi}^{+\pi} \frac{e^{ikx}(1-e^{ik_e})}{\sum_{i=1}^{d}(1-\cos k_i)+m^2} \prod_{i=1}^{d} dk_i .$$

The sum over $x_{e'}$, $e' \neq e$ puts $k_{e'}=0$ in the integral and the sum over x_e cancels exactly the factor $(1-e^{ik_e})$, using the Riemann-Lebesgue lemma. The integral can be computed by contour integration, introducing the variable $z=e^{ik}$.

For c) and d) the proof is the same; one uses the method of [10] to determine the signs of the terms. The formula for $\langle(\phi_0-\phi_e)(\phi_x-\phi_{x+e'})\rangle$ is

$$\int_{-\pi}^{\pi} \frac{(1-e^{ik_{e'}})(1-e^{-ik_e})\,e^{ikx}}{\sum_{i=1}^{d}(1-\cos k_i)+m^2} \prod_{i=1}^{d} dk_i .$$

For e) we use the explicit computation

$$\sum_{\substack{x:\\ x_1=a}} \langle\phi_0\phi_x\rangle = \frac{1}{(2\pi)} \int_{-\pi}^{\pi} \frac{e^{ik_1a}}{1-\cos k_1+m^2}\,dk_1$$

$$= \frac{e^{-\tilde{m}a}}{\sinh\tilde{m}} \quad \text{together with} \quad |x| \geqq |x_1| . \tag{A2}$$

So we have to estimate

$$\sum_{a=1}^{\infty} \frac{e^{-\tilde{m}a}}{a^k} \quad \text{which is of order } m^{-1+k} .$$

For the next estimate we use the same method, knowing the sign of $\langle\phi_0(\phi_x-\phi_{x+e})\rangle$. Obviously, sums like $\sum_{a=1}^{\infty} \frac{e^{-\tilde{m}a}}{(1+|a-b|)^k}$ satisfy similar estimates as with $b=0$.

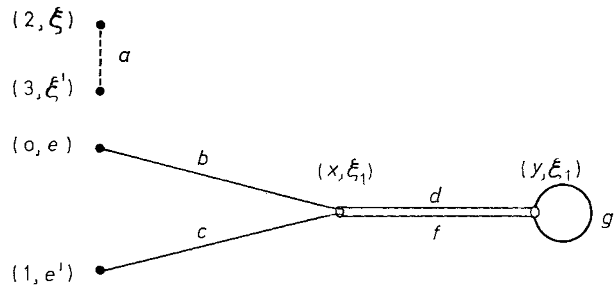
B. *Rate of Convergence of the Coefficients in the Asymptotic Expansion when* $m \to 0$

With each coefficient produced in the expansion, we shall associate a set of graphs. In this section, $\langle\nabla_x^\xi\phi\nabla_y^{\xi'}\phi\rangle$ or $\langle\phi_x\nabla_y^\xi\phi\rangle$ denote Gaussian expectation values. Whether $\langle\ \rangle$ is massive or massless is indicated by the context.

To see how we assign graphs, let us first consider an example: one second-order term in the expansion of $\langle\nabla_0^e\phi\nabla_1^{e'}\phi\nabla_2^\xi\phi\nabla_3^{\xi'}\phi\rangle$ will be

$$\begin{aligned}&\langle\nabla_2^\xi\phi\nabla_3^{\xi'}\phi\rangle \sum_{\xi_1\xi_2} \sum_{y,x\in\mathbb{Z}^d} \langle\nabla_0^e\phi\nabla_x^{\xi_1}\phi\rangle\langle\nabla_1^{e'}\phi\nabla_x^{\xi_1}\phi\rangle\\ &\cdot\langle\nabla_x^{\xi_1}\phi\nabla_y^{\xi_2}\phi\rangle^2\langle(\nabla_y^{\xi_2}\phi)^2\rangle .\end{aligned} \tag{A3}$$

We want to represent (A3) by



(A4)

(0, e), (1, e') (2, ξ), (3, ξ') will be called *external points* because they do not enter into a summation over the lattice. (x, ξ_1), (y, ξ_2) will be called *internal points* or *vertices.*

The line a which connects two external points will be called an *external contraction.* The lines b, c which connect an external point to an internal one are *external lines.* The lines d and f which connect two different internal points are *internal lines.*

The line g connecting an internal point to itself is a *tadpole.*

The graph (A4) which represent a second order term has two vertices because of the two lattice sums.

In general, we are interested in an expansion for

$$\langle(\nabla_a^{e_a}\phi)\dots(\nabla_n^{e_n}\phi)\rangle \qquad (n \text{ external points}), \tag{A5}$$

or for

$$\langle\phi_a\text{----}\phi_n\rangle. \tag{A6}$$

[In this case, the external points are not pairs of symbol (x, e_x).]

If we want to exhibit a term of order N in the expansion, we first draw a graph with n external points, and N vertices. As it is well known, the graph must be an union of connected subgraphs, each of which is connected to a different subset of external points, and each vertex is the intersection of four lines (a tadpole represents two lines).

To compute the graph we apply the usual rule: [We first give the rules for case (A5)].

0) To each external contraction $((a, e_a), (b, e_b))$ we associate a number $\langle\nabla_a^{e_a}\phi\nabla_b^{e_b}\phi\rangle$.

1) To each tadpole $((x, \xi), (x, \xi))$ we associate a number $\langle(\nabla_x^{\xi}\phi)^2\rangle$. We then take it out of the graph.

2) To each external line $((a, e_a), (x, \xi_x))=l$ we associate a variable k_l and a function $f_l(k_l)$ which is the Fourier transform of $\langle\nabla_a^{e_a}\phi\nabla_x^{\xi_x}\phi\rangle$.

3) To each internal line $((x, \xi), (y, \xi'))=n$ we associate a variable k_n and a function $f_n(k_n)$ which is the Fourier transform of $\langle\nabla_x^{\xi}\phi\nabla_y^{\xi'}\phi\rangle$.

4) To each vertex x, we associate a δ-function $\delta\left(\sum_{l\ni x} k_l\right)$ where $l\ni x$ means $l=((x, \xi), (y, \xi'))$ or $l=((y, \xi'), (x, \xi))$.

5) We do the product of all the functions and δ-functions and integrate over all the variables [from $(-\pi$ to $\pi)$]. We then multiply the result by the product of numbers associated to the tadpoles and to the external contractions.

For expectation values of (A6) the only changes are rule (0) and (2) which become:

(0′) At each external contraction (a, b) we associate a number $\langle\phi_a\phi_b\rangle$

(2′) At each external line $(a,(x,\xi))=l$, we associate a variable k_l and a function $f'_l(k_l)$ which is the Fourier transform of $\langle\phi_a V^\xi_x\varphi\rangle$.
Explicitly if

$$l=((x,e)(y,e')),\, f_l(k_l)=\frac{(e^{ik_e}-1)(e^{ik_{e'}}-1)}{\sum\limits_\xi(1-\cos k_\xi)+m^2}$$

if

$$l=(a,(x,e')),\, f'_l(k_l)=\frac{(e^{ik_{e'}}-1)}{\sum\limits_\xi(1-\cos k_\xi)+m^2},$$

(m may be 0 or not; To precise this fact we shall sometimes use $f_{l,m}$ or $f_{l,0}$).

Remark. In the following we shall use the following bounds:

$$\begin{aligned} 1-\cos y &\geqq \frac{2}{\pi^2}y^2,\; y\in[-\pi,\pi],\\ 1-\cos y &\leqq \tfrac{1}{2}y^2,\; y\in[-\pi,\pi],\\ |e^{iy}-1| &\leqq (1+\pi)|y|,\; y\in[-\pi,\pi]. \end{aligned} \tag{A7}$$

Notation. Let $G_m(l_1\ldots l_p)$ be a graph, computed in a Gaussian theory of mass m, appearing in the expansion of (A5) $l_1\ldots l_p$ are the lines of the graph. $G'_m(l_1\ldots l_p)$ will denote a graph (computed in a Gaussian theory of mass m) appearing in the expansion of (A6). When a statement is valid for both G_m and G'_m we shall express it using $G^\#_m$. Also $f^\#_{l,m}(k_e)$ will represent $f_{l,m}(k)$ or $f'_{l,m}(k_l)$.

In this Sect. we want to estimate $\Delta G^\#_m=G^\#_m-G^\#_0$. Since disconnected components of $G^\#$ factorize, we shall restrict ourselves to one connected component.

By using a triangular inequality, $\Delta G^\#_m(l_1\ldots l_p)$ may be estimated by sums of Δ_{l_i} $G^\#(l_1\ldots l_p)$ where $l_j(j\neq i)$ are lines with associated function $f_{l_j,\alpha}$ if they are internal lines and $f^\#_{l_j,\alpha}$ if they are external lines. ($\alpha=m$ or $\alpha=0$.) The line l_i has an associated function $f_{l_i,m}-f_{l_i,0}$ if it is internal and $f^\#_{l_i,m}-f^\#_{l_i,0}$ if it is external.

Proposition A2

Suppose a connected graph has n external lines labeled by $1,\ldots n$ with associated variables $k_1,\ldots,k_n$. Then it may be computed as

$$G=\int\limits_{-\pi}^{\pi} dk_1\ldots dk_n f^\#_{1,m}(k_1),\ldots,f^\#_{n,m}(k_n)\,\delta(\textstyle\sum a_ik_i)\,F(k_1\ldots k_n)$$

where $a_i\in\mathbb{R}$ and F is a bounded function for all $m\geqq 0$.

Proof. We may consider that G is constructed by the following procedure: to each external point is associated a tree. In this case the number of lines equals the number of vertices. Since the graph is connected, there exists at least $n-1$ lines connecting the different trees. So there exists at least $n-1$ independent variables after having integrated out the δ-function of the graph. Clearly these variables can be attached to the external lines. Because the interaction we consider is an even polynomial, each time a vertex appears, a new line must also appear. So there exists no line where the momentum k is constrained to be zero. (No zero momentum line.)

$F(k_1 \ldots k_n)$ is then obtained by doing integrals over bounded regions of bounded functions and so is bounded.

Proposition A3

$$|\Delta G_m(l_1 \ldots l_p)| \leqq \text{const}\, m^2 .$$

Proof. As remarked above it is sufficient to estimate

$$\Delta_{l_i} G(l_1 \ldots l_p) .$$

Suppose e and e' are the two directions fixed by the line l_i

$$(f_{l_i,m} - f_{l_i,0})(k) = \frac{(e^{ik_e}-1)(e^{ik_{e'}}-1)m^2}{\left[\sum_\xi (1-\cos k_\xi) + m^2\right]\left[\sum_\xi (1-\cos k_\xi)\right]} . \tag{A8}$$

Since there is no line of zero momentum (see proof of Proposition A2),

$$\Delta_{l_i} G(l_1 \ldots l_p) = \int (f_{l_i,m} - f_{l_i,0})(k) F^*(k)\, dk , \tag{A9}$$

where F^* is a bounded function (see proof of Proposition A2).

Introducing (A8) in (A9), and using $|F^*|_\infty < \infty$,

$$|\Delta_{l_i} G(l_1 \ldots l_p)| < \sup_k m^2 \frac{|(e^{ik_e}-1)(e^{ik_e}-1)|}{[\sum (1-\cos k_\xi) + m^2]} \int \frac{|F^*(k)|}{\sum (1-\cos k_\xi)} d^3k .$$

Using the bounds (A7) we have,

$$|\Delta_{l_i} G(l_1 \ldots l_p)| \leqq \text{const}\, m^2 .$$

Proposition A4

$$|\Delta G'_m(l_1 \ldots l_p)| \leqq m\, \text{const} .$$

Proof. In this case, one has to be more careful, and distinguish between internal and external lines. As before let us examine $\Delta_{l_i} G'(l_1 \ldots l_p)$.

1) l_i is an external line.

Let $f'_1(k_1)$ the function associated to l_i.

Define $g_1(k_1) = f'_{1,m}(k_1) - f'_{1,0}(k_1)$.

By Proposition A2,

$$\Delta_{l_i} G = \int_{-\pi}^{\pi} dk_1 \ldots dk_{n-1} g_1(k_1) f'_2(k_2) \ldots$$

$$f'_n \left(-\sum_{i=1}^{n-1} \frac{a_i k_i}{a_n} \right) F(k_1 \ldots k_{n-1}) . \tag{A10}$$

$\int_{-\pi}^{\pi} f'_2(k_2) f'_n \left(-\sum_{i=1}^{n-1} \frac{a_i k_i}{a_n} \right) dk_2$ is bounded in $k_1 \quad k_3 \ldots k_{n-1}$

uniformly in m because of the Schwartz inequality, f_2' and $f_n' \in L_2[-\pi,\pi]$. Then

$$\begin{aligned}|\Delta_{l_i} G| &\leqq \text{const} \int dk_1 dk_3 \dots dk_{n-1} (g_1(k_1)) f_3'(k_3) \dots f_{n-1}'(k_{n-1}) \\ &\leqq \text{const} \int dk_1 |g_1(k_1)| \\ &\quad (\text{because each } f_i' \in L_1[-\pi\pi]).\end{aligned}$$

Let e the direction fixed by l_i, then

$$g_1(k) = \frac{(e^{ik_e}-1)m^2}{\left[\sum_{\xi}(1-\cos k_\xi)+m^2\right][\sum(1-\cos k_\xi)]}.$$

Using the estimate (A7),

$$|\Delta_{l_i} G| \leqq \text{const} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d^3k \frac{|k_e| m^2}{k^2(k^2+m^2)}.$$

By scaling $k' = mk$, one finds

$$|\Delta_{l_i} G| \leqq \text{const}\, m. \tag{A11}$$

[Remark: If $n=2$, we directly do the scaling into (A10) and get (A11).]

2) l_i is an internal line.

By Proposition A1,

$$\begin{aligned}G(l_1 \dots l_p) &= \int dk_1 \dots dk_{n-1} f_1'(k_1) \cdot \dots \\ &\cdot \dots \cdot f_n'\left(-\sum_{i=1}^{n-1} \frac{a_i k_i}{a_n}\right) F(k_1 \dots k_{n-1}),\end{aligned} \tag{A12}$$

l_i being an internal line, enters in the computation of $F(k_1 \dots k_{n-1})$. Suppose that after having integrated out the δ-functions in G, l_i is represented by a function of r variables

$$k_1 \dots k_{n-1} k_n \dots k_r : f = f\left(\sum_{i=1}^{n-1} b_i k_i + \sum_{i=n}^{r} c_i k_i\right) \qquad b_i, c_i \in \mathbb{R}.$$

Expliciting this fact in (A12), we have:

$$G = \int dk_1 \dots dk_r f_1'(k_1) \dots f_n'\left(-\sum_{i=1}^{n-1} \frac{a_i k_i}{a_n}\right) f\left(\sum_{i=1}^{n-1} b_i k_i + \sum_{i=n}^{r} c_i k_i\right) \cdot F'(k_1 \dots k_r)$$

with F' being bounded.

Let $g(k) = f_m(k) - f_0(k)$

$$\begin{aligned}\Delta_{l_i} G(l_1 \dots l_p) = \int dk_1 \dots dk_r f_1'(k_1) \dots f_n'\left(-\sum_{i=1}^{n-1} \frac{a_i k_i}{a_n}\right) \\ \cdot g\left(\sum_{i=1}^{n-1} b_i k_i + \sum_{i=n}^{r} c_i k_i\right) F'(k_1 \dots k_r).\end{aligned}$$

For $z = (z^{e_1} \dots z^{e_d})$, $z' = (z'^{e_1} \dots z'^{e_d})$, z^{e_i} and $z'^{e_i} \in [-\pi,\pi]$. Define

$$H(z,z') = \int\int\int_{-\pi}^{\pi} f_1'(k_1) f_n'(k_1+z) g(k_1+z') d^3k_1. \tag{A13}$$

We want to prove the bound uniform in z and z'

$$|H(z,z')| \leqq \text{const}\, m\,. \tag{A14}$$

The integral (A13) contains three singularities at $k_1=0$, $k_1=-z$, $k_1=-z'$. Clearly the worse situation is when these singularities coincide.

Let us consider a case when this may happen, the other cases being easier: Assume

$$\begin{array}{l}|z^{e_i}| \leqq \frac{\pi}{2}\\ |z'^{e_i}| \leqq \frac{\pi}{2}\end{array} \quad \text{and estimate:}$$

$$\int_{-\pi/2}^{\pi/2} f_1'(k) f_n'(k+z) g(k+z') d^3k\,.$$

Suppose that f_1' represents a line $l_1=(a,(x,\xi))$, f_n' represents a line $l_n=(b,(x',\xi_2))$, f represents a line $l_i=((y,\xi_3),(y',\xi_4))$.

Using (A7) we have,

$$\begin{aligned}&\int_{-\pi/2}^{\pi/2} f_1'(k) f_n'(k+z) g(k+z') d^3k\\ &\leqq \int_{-\pi}^{\pi} \frac{|k^{\xi_1}|\,|k^{\xi_2}+z^{\xi_2}|\,|k^{\xi_3}+z'^{\xi_3}|\,|k^{\xi_4}+z'^{\xi_4}|m^2 d^3k}{k^2(k+z)^2((k+z')^2+m^2)(k+z')^2}\\ &\leqq \int_{-\pi}^{\pi} m^2 \frac{|k^{\xi_1}|\,|k^{\xi_2}+z^{\xi_2}|}{k^2(k+z)^2((k+z')^2+m^2)}\, d^3k\end{aligned}$$

by scaling

$$\begin{aligned}&k=k'm\\ &=m\int_{-\pi/m}^{\pi/m} \frac{|k^{\xi_1}|\,|k^{\xi_2}+z^{\xi_2}/m|}{k^2(k+z/m)^2((k+z'/m)^2+1}\, d^3k\\ &\leqq m\,\text{const}\,.\end{aligned}$$

We then have

$$\begin{aligned}|\Delta_{l_i}G| \leqq \int dk_1 dk_{n-1} |f'(k_e)\dots f_{n-1}'(k_{n-1})| \left| H\left(\sum_{i=2}^{n-1} a_i k_i, \sum_{i=2}^{n-1} b_i k_i + \sum_{i=n}^{r} c_i k_i\right)\right.\\ |F'(k_1\dots k_r)|dk_n\dots dk_r\,.\end{aligned} \tag{A15}$$

Using (A14), (A15) is bounded by:

$$|\Delta_{l_i}G| \leqq \text{const}\, m\,.$$

In order to be complete we have also to control the speed of convergence of the tadpoles and of the external contractions. For that it is sufficient to estimate $C_{0x}^m - C_{0x}^{m=0}$ where C_{0x}^m is the covariance of the Gaussian field of mass m.

Proposition A5

$$|C_{0x}^m - C_{0x}^{m=0}| \leqq m\,\text{const}\,.$$

Proof.

$$|C_{0x}^{m}-C_{0x}^{m=0}|=\left|\int_0^m \frac{d}{dm'}C_{0x}^{m'}dm'\right|$$
$$=\left|\int_0^m\int_{-\pi}^{\pi} e^{ikx}\frac{d^3k(-2m')}{(\sum(1-\cos k_\xi)+m'^2)^2}dk\,dm'\right|$$
$$\leqq C2m\sup_{m'}\int_{-\pi}^{\pi}\frac{d^3km'}{(k^2+m'^2)^2}.$$

By (A7) by scaling

$$k=k'm'$$
$$=c2m\sup_{m'}\int_{-\pi/m'}^{\pi/m'}\frac{d^3k'}{(k'^2+1)^2}$$
$$\leqq \text{const}\,m.$$

In Sect. III we proved that

$$\langle(\nabla_a^{e_a}\phi)\dots(\nabla_r^{e_r}\phi)\rangle=a_0(m)+a_1(m)T+\dots+a_k(m)T^k+o(T^k)$$

and

$$\langle\phi_a\dots\phi_r\rangle=b_0(m)+b_1(m)T+\dots+b_k(m)T^k+o(T^k),$$

where $a_i(m)$ and $b_i(m)$ are the usual coefficients given by perturbation theory but computed in a Gaussian theory of mass m.

Using Proposition A5 one proves:

Proposition A6

$$|a_i(m)-a_i(0)|\leqq\text{const}\,m^2,$$
$$|b_i(m)-b_i(0)|\leqq\text{const}\,m.$$

Proposition A7

$$|a_i(m,r)-a_i(0,\infty)|\leqq\text{const}(m^2+m^{-1}r^{-1}),$$

where $a_i(m,r)$ are the Gaussian coefficients in Λ_0 with mass m introduced in Part 3 of the proof of Theorem 1.

Proof. By Proposition A6 and the triangle inequality, we have only to estimate $|a_i(m(T),r(T))-a_i(m(T),\infty)|$. Using the graph representation, we have to consider

$$|G_m(l_1,\dots,l_p,r)-G_m(l_1,\dots,l_p,\infty)|.$$

Λ_0 is a square of size $(2r(T)+1)^d$ and the dual Λ_0^* is the set of k such that

$$k_e=\frac{2\pi n_e}{(2r+1)}\qquad n_e=0,1\dots 2r,\qquad e=1,\dots,d.$$

By Proposition A2,

$$G_m(l_1, \ldots, l_p, r) = |\Lambda_0|^{-p} \sum_{\substack{k_1, \ldots, k_p \\ k_i \in \Lambda_0^*}} G^*(k_1, \ldots, k_p),$$

where G^* is bounded uniformly in m.

$$G_m(l_1, \ldots, l_p, \infty) = (2\pi)^{-d} \int_{-\pi}^{\pi} dk_1 \ldots dk_p G^*(k_1, \ldots, k_p).$$

The difference between the Riemann sum and the integral is bounded by:

$$\sup (2r+1)^{-1} \sum_{\substack{j=1,\ldots,p \\ e=1,\ldots,d}} \left| \frac{d}{dk_{e,j}} G(k_1, \ldots, k_p) \right|.$$

By inspection one sees that

$$\left| \frac{d}{dk_{e,j}} G(k_1 \ldots k_p) \right| \leqq \text{const}\, m^{-1}$$

which finishes the proof.

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