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Statistical Mechanics of Systems of Unbounded Spins

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Abstract. We develop the statistical mechanics of unbounded n-component spin systems on the lattice Z^ν interacting via potentials which are superstable and strongly tempered. We prove the existence and uniqueness of the infinite volume free energy density for a wide class of boundary conditions. The uniqueness of the equilibrium state (whose existence is established in general) is then proven for one component ferromagnetic spins whose free energy is differentiable with respect to the magnetic field.

1. Introduction

The study of continuous unbounded spin systems on a lattice has received great impetus in recent years from its close connection with Euclidian quantum field theory [1, 2]. While the applications to field theory require the passage to the limit of zero lattice spacing which poses great difficulties (yet to be overcome for the physically interesting situations) the lattice results are of interest in their own right and many of them carry over, more or less directly, to field theory once the existence of the latter is proven. Indeed certain lattice results are very helpful in proving the existence of the corresponding field theory.

In this paper we develop the general statistical mechanical theory of such systems: making use of what was done for continuum particle systems and Ising systems in the last decade [3, 4]. For this reason we consider interaction potentials of fairly long range and not necessarily of ferromagnetic type, although these are not currently of interest in field theory. We do however restrict ourselves essentially to pair-wise interactions; all higher spin interactions would have to be bounded. More general many spin interactions can be dealt with by an extension of those methods [5] but they will not be considered here.

Our main results are: a) the existence and uniqueness (independence of boundary conditions) of the infinite volume free energy density F, b) the existence of infinite volume "regular" equilibrium states as limits of finite volume Gibbs

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states and as solutions of the *DLR* equations and c) the existence of a unique regular equilibrium state for one component ferromagnetic systems at all values of the magnetic field h at which $\partial F/\partial h$ exists. Part c) is an extension of the result of Lebowitz and Martin-Löf [6] for spin-$\frac{1}{2}$ Ising systems and implies in particular that for ferromagnetic systems satisfying the Lee-Yang theorem there is a unique regular equilibrium state whenever $h \neq 0$. No comparable results have yet been derived for field theory: only equivalence of a few boundary conditions has been shown so far [1, 2, 7]. It seems reasonable to expect however that our stronger results also hold in field theory. We might even hope that the connection between properties of the free energy and equilibrium states extends to more general systems [8].

Notation and Assumptions. We consider the lattice $\mathbb{Z}^\nu$ at each site of which there is a vector spin variable S_x, $x \in \mathbb{Z}^\nu$, $S_x \in \mathbb{R}^d$. Each S_x has associated with it an intrinsic or free, positive measure $\mu(dS)$, the same for all sites, and we will consider both interactions between the spins as well as self interactions. We use the following notation, essentially the same as in [5].

Definition 1.1. A configuration of spins in $\mathbb{Z}^\nu$ is a function $\underline{S}: \mathbb{Z}^\nu \to \mathbb{R}^d$ with values S_x at $x \in \mathbb{Z}^\nu$. S_Λ, $\Lambda \subset \mathbb{Z}^\nu$, is the restriction of $\underline{S}$ to Λ and denotes a configuration of spins in Λ. We denote by $X(\Lambda)$ the configuration for the region Λ. Both X and $X(\Lambda)$ are topological spaces for the product topology of $\mathbb{R}^d$.

Definition 1.2. For $x = (x^1, \ldots, x^\nu)$, $S = (S^1, \ldots, S^d)$

$$|x| = \max_{1 \leq i \leq \nu} |x^i| \qquad |S| = \left[\sum_{i=1}^{d} (S^i)^2\right]^{\frac{1}{2}}.$$

Λ and Δ will hereafter denote bounded sets in $\mathbb{Z}^\nu$.

Definition 1.3. We assume μ to be a Borel measure on $\mathbb{R}^d$ and

$$\int \mu(dS) e^{-\alpha s^2} \quad \text{for every} \quad \alpha > 0. \tag{1.1}$$

Definition 1.4. The energy is a real function U on the configurations of all Λ satisfying the following conditions:

(a^1) $U(S_\Lambda)$ is a Borel function on $(\mathbb{R}^d)^\Lambda$.

(a^2) U is invariant under translation of $\mathbb{Z}^\nu$.

(b) Superstability. There are $A > 0$ and $c \in \mathbb{R}$ such that for every S_Λ

$$U(S_\Lambda) \geq \sum_{x \in \Lambda} [A S_x^2 - c].$$

(c) Regularity. Let Λ_1 and Λ_2 be disjoint sets in $\mathbb{Z}^\nu$ with $\Lambda = \Lambda_1 \cup \Lambda_2$ and S_Λ a spin configuration in Λ with restrictions S_{Λ_1} and S_{Λ_2} in Λ_1 and Λ_2 respectively. The interaction between the spins in Λ_1 and Λ_2 is defined as

$$W(S_{\Lambda_1} | S_{\Lambda_2}) = U(S_\Lambda) - U(S_{\Lambda_1}) - U(S_{\Lambda_2}). \tag{1.2}$$

The following regularity condition is assumed for the interaction. There is a decreasing positive function on the natural integers such that

$$|W(S_{\Lambda_1} | S_{\Lambda_2})| \leq \tfrac{1}{2} \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} \Psi(|x-y|)(S_x^2 + S_y^2) \tag{1.3}$$

with

$$\Psi(r) \leqq K r^{-\nu-\varepsilon} \tag{1.4}$$

for some K and $\varepsilon > 0$.

Remark. Many results in this paper hold also with a regularity condition weaker than Definition 1.4 (c), namely

$$\sum_{x \in \mathbb{Z}^\nu} \Psi(|x|) < +\infty . \tag{1.5}$$

In particular the basic estimates in [4, 5] are proven under the assumption of Equation (1.5). We actually need Equation (1.4) only when we fix the external spins in Sections 3, 4, 5, as $S_x^2 \leqq a \log|x|$. Equation (1.4) implies then that the interaction of any spin with all the others is finite.

It is convenient to state explicitely the following consequence of Definitions 1.4 and 1.3 (see [5]).

Definition 1.4 (d). There is a bounded Borel set Σ in $\mathbb{R}^d$ and $b > 0$ such that

$$\int_{\Sigma^\Lambda} \prod_{x \in \Lambda} \mu(dS_x) \exp[-U(S_\Lambda)] \geqq b^{|\Lambda|} .$$

Definition 1.5. The finite region partition function with zero boundary conditions, $Z(\Lambda)$, is defined as

$$Z(\Lambda) = \int \mu(dS_\Lambda) \exp[-U(S_\Lambda)] = \exp[|\Lambda| F(\Lambda)]$$

where $F(\Lambda)$ is the free energy per site and

$$\mu(dS_\Lambda) = \prod_{x \in \Lambda} \mu(dS_x) .$$

The finite volume equilibrium measure $\nu_\Lambda(dS_\Lambda)$ is given by

$$\nu_\Lambda(dS_\Lambda) = Z(\Lambda)^{-1} \mu(dS_\Lambda) \exp[-U(S_\Lambda)] .$$

The dependence on temperature and magnetic field is included in $U(S_\Lambda)$: it will be made explicit when necessary. (Later we will also consider partition functions and finite volume equilibrium measures with non-zero boundary conditions.)

Definition 1.6. For Λ and $\Delta \subset \Lambda$ we denote by $\varrho_\Lambda(S_\Delta)\mu(dS_\Delta)$ the restriction of ν_Λ to $X(\Delta)$, that is

$$\varrho_\Lambda(S_\Delta) = \exp[-U(S_\Delta)] Z(\Lambda)^{-1} \int \mu(dS_{\Lambda \setminus \Delta}) \exp[-U(S_{\Lambda \setminus \Delta}) - W(S_\Delta | S_{\Lambda \setminus \Delta})] .$$

We also introduce the set of bad configurations

$$B_\Lambda(N^2, \Delta) = \left\{ S_\Lambda \,\middle|\, \sum_{x \in \Delta} S_x^2 \geqq N^2 |\Delta| \right\} .$$

The main tool in our analysis will be the following theorem due to Ruelle [5].

Theorem 1.1. *Let the conditions Definitions 1.3, 1.4(a¹), (b), (c) be satisfied. Then there are* γ, δ, $\bar{\gamma}$, $\bar{\delta}$ *such that for every* $\Delta \subseteqq \Lambda$

$$\varrho_\Lambda(S_\Delta) \leqq \exp\left[-\sum_{x\in\Delta}(\bar{\gamma}S_x^2-\bar{\delta})\right], \bar{\gamma}>0\,, \tag{1.6}$$

$$\nu_\Lambda[B_\Lambda(N^2|\Delta)] \leqq \exp[-(\gamma N^2-\delta)|\Delta|], \gamma>0\,. \tag{1.7}$$

Proof. Equation (1.6) is proven in [5]. Equation (1.7) is easily obtained from Equation (1.6) with $\gamma<\bar{\gamma}$ and

$$e^{\delta}=e^{\bar{\delta}}\int\mu(dS)\exp[(\gamma-\bar{\gamma})S^2]\,,$$

where the integral is finite because of Definition 1.3.

2. Thermodynamic Limit of the Free Energy with Zero Boundary Conditions

Using the estimates given in Theorem 1.1 we essentially reduce the problem to a "bounded spin" lattice system; the strategy of the proof is then the usual one [3, 4]. We first prove that the free energy is uniformly bounded. Then by compactness there exists an increasing sequence of cubes $\Gamma_n \to \mathbb{Z}^\nu$ for which the free energy $F(\Gamma_n)$ has a limit, F. We then consider a general sequence of domains $\Lambda_n \to \mathbb{Z}^\nu$, in the Van Hove sense (see Def. 2.3). For n sufficiently large we will be able to decompose (almost completely) Λ_n in cubes Γ_m and find upper and lower bounds for the interactions between the spins in different cubes. At this point use of Theorem 1.1 is crucial since the interactions are not uniformly bounded: what we actually prove is that they can be bounded in a set of configurations of sufficiently large measure. As a result we obtain that the free energy $F(\Lambda_n)$ differs from $F(\Gamma_m)$ by a quantity for which we have explicit bounds. As n increases Γ_m can be made larger and the error smaller, so that in the thermodynamic limit $F(\Lambda_n) \to F$.

We carry out the above program through a series of partial results stated as lemmas, some of whose proofs will be given in an appendix.

The first point is to obtain a uniform bound on the free energy of a finite region Λ. The upper bound is a direct consequence of the superstability assumption (Def. 1.4b) and of the finiteness of the measure μ, Equation (1.1). The lower bound is obtained by restricting the spins in each site to belong to the bounded set Σ (see Def. 1.4). We therefore obtain the following lemma.

Lemma 2.1. *Let the assumption of Section 1 hold, then there exists a* λ *such that for every* Λ, $|F(\Lambda)| \leqq \lambda$.

It now follows by compactness that

Corollary 2.2. *There is an increasing sequence of cubes* $\Gamma_m \to \mathbb{Z}^\nu$ *such that*

$$\lim_{m\to\infty} F(\Gamma_m) = F\,.$$

Our next step will be to consider general domains Λ and to fill them up almost completely with some cube Γ_m of the sequence used in Corollary 2.2. For this purpose we restrict the sequences of domains $\{\Lambda_n\}$ as follows [3].

Definition 2.3. We say that $\{\Lambda_n\}$ tends to $\mathbb{Z}^\nu$ in the sense of Van Hove if: a) $\Lambda_{n+1}\supset\Lambda_n$, b) $\Lambda_n\supset\Delta\,\forall\Delta$ and some n, c) given any parallelepiped Γ and the partition $\Pi(\Gamma)$ of $\mathbb{Z}^\nu$ generated by translation of Γ

$$\lim N_\Gamma^-(\Lambda_n)=\infty \qquad \lim N_\Gamma^-(\Lambda_n)/N_\Gamma^+(\Lambda_n)=1$$

where $N_\Gamma^-(\Lambda_n)$ is the number of sets of $\Pi(\Gamma)$ contained in Λ_n, $N_\Gamma^+(\Lambda_n)$ the number of sets with non-void intersection with Λ_n.

The main point in the whole procedure is to control the interaction between two disjoint regions, Δ_1 and Δ_2, contained in some region Λ. We proceed as in the proof of Lemma 3.1 of [4]. Assumption Definition 1.4c gives the following bound

$$\begin{aligned} W(S_{\Delta_1}|S_{\Delta_2}) &\le \tfrac{1}{2}\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)(S_x^2+S_y^2)\\ &=\tfrac{1}{2}\sum_{x\in\Delta_1}S_x^2\Big(\sum_{y\in\Delta_2}\Psi(|x-y|)\Big)+\tfrac{1}{2}\sum_{y\in\Delta_2}S_y^2\Big(\sum_{x\in\Delta_1}\Psi(|x-y|)\Big)\,. \end{aligned} \tag{2.1}$$

We observe that the first sum in the r.h.s. of Equation (2.1) depends only on the sum of the S_x^2 for x varying in the equipotential surfaces of the potential $\sum_{y\in\Delta_2}\Psi(|x-y|)$. This enables us to bound W on a set of configurations which have a non-vanishing equilibrium measure. Let $V_i(\Delta_1,\Delta_2)$, $i=1,2,\ldots$, be the values taken by the function $\sum_{y\in\Delta_2}\Psi(|x-y|)$ when x is in Δ_1, ordered in decreasing order. Let

$$g_i^\Lambda(\Delta_1,\Delta_2)=\Big\{S_\Lambda\Big|\sum_{x\in\bar v_i(\Delta_1,\Delta_2)}S_x^2\le N^2|\bar v_i(\Delta_1,\Delta_2)|\Big\} \tag{2.2}$$

where

$$\bar v_i(\Delta_1,\Delta_2)=\Big\{x\in\Delta_1\Big|\sum_{y\in\Delta_2}\Psi(|x-y|)\ge V_i(\Delta_1,\Delta_2)\Big\}$$

and let

$$g^\Lambda(\Delta_1,\Delta_2)=\bigcap_I^\infty g_i^\Lambda(\Delta_1,\Delta_2)\,. \tag{2.3}$$

A direct consequence of the definition of g and of Equation (2.1) is the following

Lemma 2.4. *Let Δ_1 and Δ_2 be contained in Λ and be disjoint. Let the interaction satisfy D.1.4c, then with the notation of Equations (2.2) and (2.3)*

$$|W(S_{\Delta_1}|S_{\Delta_2})|\le N^2\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|) \tag{2.4}$$

for $S_\Lambda\in g^\Lambda(\Delta_1,\Delta_2)\cap g^\Lambda(\Delta_2,\Delta_1)$.

Lemma 2.4 allows us to bound the interactions in regions of the configuration space whose equilibrium measure can be easily controlled by the superstability estimates given in Theorem 1.1. In the Appendix we prove

Lemma 2.5. *Let $\Delta_1\cap\Delta_2=\emptyset$ and $\Delta_1\subset\Lambda$. Let the assumptions Definition 1.4 hold then there is N independent of Λ, Δ_1, Δ_2 such that*

$$\nu_\Lambda(g^c)=Z^{-1}(\Lambda)\int_{g^c}\mu(dS_\Lambda)\exp(-U(S_\Lambda))\le\tfrac{1}{4} \tag{2.5}$$

where g^c = complement of $g^\Lambda(\Delta_1,\Delta_2)$.

From now on N will be considered fixed as in Lemma 2.5. By Lemma 2.4 and 2.5 we can obtain estimates on the additivity of the free energy. We prove in the Appendix the following

Lemma 2.6. *Let the assumptions of Section 1 hold and let*

$$\Lambda=\bigcup_i^n \Delta_i,\ \Delta_i\cap\Delta_j=\emptyset \quad \textit{for} \quad i\neq j.$$

Then

$$\prod_i^n \left[\tfrac{1}{2}Z(\Delta_i)\exp\left(-N^2/2\sum_{x\in\Delta_i}\sum_{y\notin\Delta_i}\Psi(|x-y|)\right)\right]\leqq Z(\Lambda)\leqq\prod_i^n\left[2Z(\Delta_i)\exp\left(N^2/2\sum_{x\in\Delta_i}\sum_{y\notin\Delta_i}\Psi(|x-y|)\right)\right].$$

It is now straightforward to prove the following theorem

Theorem 2.7. *Let the assumptions of Section 1 hold and let $\Lambda_n\to\mathbb{Z}^\nu$ as in Definition 2.3; then the thermodynamic limit for the free energy exists and is independent of the sequence Λ_n.*

Proof. Let F be defined as in Lemma 2.2 then we will prove that

$$\limsup F(\Lambda_n)\leqq F\leqq\liminf F(\Lambda_n).$$

We only give the proof for the lower bound, the upper one can be proven analogously.

Let $\varepsilon>0$, we then fix n_0 so that

(i) $|\Gamma_{n_0}|^{-1}\ln Z(\Gamma_{n_0})-F<\varepsilon/4,\ |\Gamma_{n_0}|^{-1}\ln 2<\varepsilon/4$,

(ii) $|\Gamma_{n_0}|^{-1}N^2/2\sum_{x\in\Gamma_{n_0}}\sum_{y\notin\Gamma_{n_0}}\Psi(|x-y|)<\varepsilon/4$

we then fix m_0 so large that for $n\geqq m_0$ condition (iii) on Λ_n, which will be stated later, holds.

For $n\geqq m_0$ we decompose Λ_n into cubes of the decomposition $\Pi(\Gamma_{n_0})$, see Definition 2.3, which are contained in Λ_n, their number is $N^-_{\Gamma_{n_0}}(\Lambda_n)$ and the individual remaining points in Λ_n, their number is $|\Lambda_n|-N^-_{\Gamma_{n_0}}(\Lambda_n)|\Gamma_{n_0}|$. We then use for this decomposition of Λ_n the upper bound of Lemma 2.6 and we have

$$\begin{aligned}\log Z(\Lambda_n)/|\Lambda_n| &\leqq N^-_{\Gamma_{n_0}}/|\Lambda_n|\left[\log\left\{2Z(\Gamma_{n_0})\cdot\exp\left(N^2/2\sum_{x\in\Gamma_{n_0}}\sum_{y\notin\Gamma_{n_0}}\Psi(|x-y|)\right)\right\}\right]+|\Lambda_n|^{-1}(|\Lambda_n|-N^-_{\Gamma_{n_0}}(\Lambda_n)|\Gamma_{n_0}|)\log\{2Z(\{0\})\\ &\quad\exp\left[N^2/2\sum_{y\neq 0}\Psi(|y|)\right]\right\}\\ &\leqq N^-_{\Gamma_{n_0}}(\Lambda_n)[N^-_{\Gamma_{n_0}}(\Lambda_n)|\Gamma_{n_0}|]^{-1}\log(2Z(\Gamma_{n_0}))\\ &\quad+(|\Gamma_{n_0}|)^{-1}N^2/2\sum_{x\in\Gamma_{n_0}}\sum_{y\notin\Gamma_{n_0}}\Psi(|x-y|)\\ &\quad+\left(1-\frac{N^-}{N^+}\right)\log\{2Z(\{0\})\left[\exp[N^2/2\sum_{y\neq 0}\Psi(|y|)\right]\right\}.\end{aligned}$$

Therefore the theorem is proven if m_0 is so chosen that for $n>m_0$

$$\text{(iii)}\ \frac{N^-_{\Gamma_{n_0}}(\Lambda_n)}{N^+_{\Gamma_{n_0}}(\Lambda_n)} \geqq 1-\varepsilon\left[\log\left\{2Z(\{0\})\exp\left(N^2/2\sum_{y\neq 0}\Psi(|y|)\right)\right\}\right]^{-1}.$$

This is possible because of Definition 2.3 and since by assumption Definition 1.4(c) and (d) the denominator in the *rhs* of (iii) is bounded.

3. Non-Zero Boundary Conditions: Free Energy

We first establish the notation used for the free energy and Gibbs measure with non-zero boundary conditions (b.c.). We then prove that the thermodynamic limit of the free energy exists and is the same for "all" b.c. In the next section we discuss the thermodynamic limit of the Gibbs measures.

We will consider pure b.c., general b.c., and periodic b.c.

Definition 3.1. Pure b.c.: For $a>0$ let

$$X(a)=\{\underline{S}\,|\,S_x^2\leqq a\log|x|,\ |x|\geqq 1\}. \tag{3.1}$$

For $\hat{\underline{S}}\in X(a)$ we define the partition function, free energy and Gibbs measure as

$$Z(\Lambda|\hat{\underline{S}})=\int\mu(dS_\Lambda)\exp[-U(S_\Lambda)-W(S_\Lambda|\hat{S}_{\Lambda^c})] \tag{3.2}$$

$$F(\Lambda|\hat{\underline{S}})=|\Lambda|^{-1}\log Z(\Lambda|\hat{\underline{S}}) \tag{3.3}$$

$$\nu_\Lambda(dS_\Lambda|\hat{\underline{S}})=Z(\Lambda|\hat{\underline{S}})^{-1}\mu(dS_\Lambda)\exp[-U(S_\Lambda)-W(S_\Lambda|\hat{S}_{\Lambda^c})]=\varrho(S_\Lambda|\hat{\underline{S}})\mu(dS_\Lambda). \tag{3.4}$$

For $\Delta\subset\Lambda$ let

$$\nu_\Delta(dS_\Delta|\hat{\underline{S}})\ \text{be the projection of}\ \nu_\Lambda(dS_\Lambda|\hat{\underline{S}})\ \text{on}\ X(\Delta), \tag{3.5}$$

$$\nu_\Delta(dS_\Delta|\hat{\underline{S}})=\varrho_\Delta(S_\Delta|\hat{\underline{S}})\mu(dS_\Delta)=\mu(dS_\Delta)\int\mu(dS_{\Lambda\setminus\Delta})\varrho(S_\Lambda|\hat{\underline{S}}). \tag{3.6}$$

Remark. The choice of $X(a)$ in Definition 3.1 is quite arbitrary. The criterion was to include sufficiently many configurations in the allowed external conditions. We assumed Equation (3.1) to hold because when a is sufficiently large any spin configuration is asymptotically in $X(a)$ with probability one for the (physically relevant) infinite equilibrium measures ν, see Section 4: namely for sufficiently large a the set

$$\bar{X}(a)=\{\underline{S}\,|\,\exists\Lambda(\underline{S}):S_x^2\leqq a\log|x|\quad\text{for}\quad x\notin\Lambda(\underline{S})\}, \tag{3.7}$$

has

$$\nu[\bar{X}(a)]=1. \tag{3.8}$$

To introduce general b.c. we first give the definition of regular measures on X.

Definition 3.2. A regular measure λ is a Borel probability measure on X whose projection $\lambda(dS_\Lambda)$ on any $X(\Lambda)$ satisfies (1.6), i.e. $\exists$

$$g(S_\Lambda|\lambda)\leqq\exp\left[-\sum_{x\in\Lambda}(\bar{\gamma}S_x^2-\bar{\delta})\right],\ \bar{\gamma}>0 \tag{3.9}$$

such that

$$\lambda(dS_\Lambda)=\mu(dS_\Lambda)g(S_\Lambda|\lambda). \tag{3.10}$$

As in Theorem 1.1 we can deduce from Equation (3.9) that there are $\gamma > 0$, δ such that for every Δ

$$\lambda[B(N^2|\Delta)] \leqq \exp[-|\Delta|(\gamma N^2 - \delta)] .$$

The following property holds for regular measures:

Lemma 3.1. *Let λ be regular with coefficients $\bar{\gamma}$ and $\bar{\delta}$. Then there is an a sufficiently large so that $\lambda[\bar{X}(a)] = 1$.*

Proof (c.f. [9]). We have for every Λ

$$\mathrm{compl}\,\bar{X}(a) \subset \bigcup_{x \notin \Lambda} X_x(a)$$

where

$$X_x(a) = \{\underline{S} | S_x^2 > a \log|x|\} .$$

Hence

$$\lambda[\mathrm{compl}\,\bar{X}(a)] \leqq \lim_{\Lambda} \sum_{x \notin \Lambda} \lambda[X_x(a)] .$$

If Λ is sufficiently large

$$\begin{aligned} \lambda[X_x(a)] &= \int \mu(dS_x) \varrho(S_x|\lambda) \chi(S_x^2 > a \log|x|) \\ &\leqq \mathrm{const} \exp(-\bar{\gamma} a \log|x|) \end{aligned}$$

and therefore the result is obtained if $\bar{\gamma} a > v$; v the dimension of the space $\mathbb{Z}^v$.

Definition 3.3. General boundary conditions: For a regular measure λ we define the free energy as

$$F(\Lambda|\lambda) = \int \lambda(d\hat{S}) F(\Lambda|\hat{\underline{S}}) . \tag{3.11}$$

The Gibbs measure $v_\Lambda(dS_\Lambda|\lambda)$ is defined via

$$\int v_\Lambda(dS_\Lambda|\lambda) f(S_\Lambda) = \int \lambda(d\hat{\underline{S}}) \int v_\Lambda(dS_\Lambda|\hat{\underline{S}}) f(S_\Lambda) \tag{3.12}$$

where $v_\Lambda(dS_\Lambda|\hat{\underline{S}})$ is defined in Equation (3.4). We finally define for $\Delta \subset \Lambda$

$$v_\Delta(dS_\Delta|\lambda) \text{ as the projection on } X(\Delta) \text{ of } v_\Lambda(dS_\Lambda|\lambda) \tag{3.13}$$

$$v_\Delta(dS_\Delta|\lambda) = \varrho_\Delta(S_\Delta|\lambda)\mu(dS_\Delta) = \mu(dS_\Delta) \int \mu(dS_{\Lambda\setminus\Delta}) \lambda(d\hat{S}_{\Lambda^c}) \varrho(S_\Lambda|\hat{\underline{S}}) . \tag{3.14}$$

Remark. In Definition 3.3 we defined the free energy as the λ-average of the free energies with pure b.c. We could have given a different definition by taking the λ-average of the pure b.c. partition functions and take the logarithm to define the free energy. By our choice the thermodynamic potentials are λ-average of the pure b.c. ones. This makes them generating functionals for the correlation functions with λ b.c.

Definition 3.4. Periodic B.C. Let Γ be a parallelepiped in $\mathbb{Z}^v$, then periodic b.c. amount to considering a new energy defined as

$$U(S_\Gamma) \to U(S_\Gamma) + W(S_\Gamma|S_{\Gamma^c}) \equiv U_p(S_\Gamma) \tag{3.15}$$

where S_{Γ^c} is obtained by translations of S_Γ by Γ all over the lattice $\mathbb{Z}^\nu$. To treat periodic b.c. we strengthen the superstability condition Definition 1.4(b) by asking that

$$A - \sum_{|x|>0} \Psi(|x|) = A_p > 0\,. \tag{3.16}$$

We define the partition function and the free energy as

$$Z_p(\Gamma) = \int \mu(dS_\Gamma) \exp[-U_p(S_\Gamma)]\,, \tag{3.17}$$

$$F_p(\Gamma) = |\Gamma|^{-1} \log Z_p(\Gamma)\,. \tag{3.18}$$

The Gibbs measure is naturally obtained from the zero b.c. by the substitution given in Equation (3.15).

In the Appendix we prove the following (see Note added in proof)

Theorem 3.1. *Let the assumptions of Section 1 hold and let $\Lambda_n \to \mathbb{Z}^\nu$ as in Definition 2.3. Then $F(\Lambda_n|\hat{S})$ and $F(\Lambda|\lambda)$ have the thermodynamic limit F, the same as for zero b.c. in Theorem 2.7. If Γ_n is an increasing sequence of parallelepipeds then $F_p(\Gamma_n)$ also has F as its thermodynamic limit.*

4. Equilibrium States

In this section we study infinite volume equilibrium measures. We use compactness arguments to prove the existence of limits of finite volume Gibbs measures with zero and non-zero b.c. The main technical tool is again the superstability estimates of Theorem 1.1, and their "extensions" to the non-zero b.c. case, Theorem 4.1 and Corollary 4.2 below. In the remainder of the section we introduce a DLR equation, [10], whose solutions are all the limiting measures of the finite Gibbs measures.

First of all we observe that, because of the assumption (3.16), the periodic b.c. can be treated as the zero one, just as in Section 3 for the free energy. For the sake of brevity we therefore do not mention it explicitely.

As we said before the main point is to extend the estimates of Theorem 1.1 to non zero b.c. In the pure b.c. case the external spins can be very large and can "drive" the spins of sites close to the boundaries to large values also. The idea therefore will be to look well inside the region, and then arguments similar to those of [4, 5] can be reproduced.

In the Appendix we prove the following

Theorem 4.1. *Let the hypotheses of Section 1 hold. Then there exists $\gamma > 0$ and δ such that the following holds: For every Δ there is $\Lambda(\Delta)$ such that for $\hat{\underline{S}} \in X(a)$*

$$\varrho_\Lambda(S_\Delta|\hat{\underline{S}}) \leqq \exp\left[-\sum_{x\in\Delta} (\gamma S_x^2 - \delta)\right] \quad \textit{for} \quad \Lambda \supseteqq \Lambda(\Delta)$$

where ϱ_Λ is defined in Equation (3.6).

The extension of Theorem 1.1 to general b.c. can be related to Theorem 4.1 as follows:

Corollary 4.2. *Let the assumptions of Section 1 hold, then there are $\gamma>0$ and δ such that the following hold: For every $\varepsilon>0$ and Δ there is $\Lambda(\varepsilon,\Delta)$ and a decomposition of $\varrho_\Lambda(\hat{S}_\Delta|\lambda)$, see Equation (3.14),*

$$\varrho_\Lambda(S_\Delta|\lambda)=\varrho'_\Lambda(S_\Delta|\lambda)+\varrho''_\Lambda(S_\Delta|\lambda),\ \Lambda\supset\Lambda(\varepsilon,\Delta)$$

such that

$$0\leqq\varrho'_\Lambda(S_\Delta|\lambda)\leqq\exp\left[-\sum_{x\in\Delta}(\gamma S_x^2-\delta)\right],\ \Lambda\supset\Lambda(\varepsilon,\Delta),$$

$$0\leqq\varrho''_\Lambda(S_\Delta|\lambda),\ \int\mu(dS_\Delta)\varrho''_\Lambda(S_\Delta|\lambda)<\varepsilon,\ \Lambda\supset\Lambda(\varepsilon,\Delta).$$

Proof. Since λ is regular there is an a such that $\lambda[\bar{X}(a)]=1$ see Lemma 3.1. Then for $f(\underline{S})$ cylindrical and bounded in $X(\Delta)$ define ν'_Λ by

$$\int\nu'_\Lambda(dS_\Delta)f(S_\Delta)=\int\lambda(d\hat{\underline{S}})\chi\{S_x^2\leqq a\log|x|\quad\text{where}\quad x\notin\Lambda\}$$
$$Z^{-1}(\Lambda|\hat{S}_{\Lambda^c})\int\mu(dS_\Lambda)\exp\{-U(S_\Lambda)-W(S_\Lambda|\hat{S}_{\Lambda^c})\}f(S_\Delta).$$

For this measure we have the estimates of Theorem 4.1. The proof is then completed by noting that for Λ sufficiently large

$$\int\lambda(d\underline{S})\chi\{S_x^2\leqq a\log|x|\quad\text{when}\quad x\notin\Lambda\}\geqq 1-\varepsilon.$$

We restrict our attention to the following class M of measures on X.

Definition 4.1. A measure $\nu\in M$ is

(i) ν is a Borel probability measure on X,

(ii) ν is tempered [4]: it is carried by the union over N of the sets

$$R_N=\left\{\underline{S}|\forall j\in\mathbb{Z}^+,\ \sum_{|x|\leqq j}S_x^2\leqq N^2(2j+1)^\nu\right\}.$$

(iii) $\nu(dS_\Lambda)$ is absolutely continuous w.r.t. $\mu(dS_\Lambda)$ and its density is denoted by $\varrho_\Lambda(S_\Lambda|\nu)$.

We will say that $\nu_\alpha\in M$ converge to $\nu\in M$ if for every Borel cylindrical set f,

$$\lim\nu_\alpha(f)=\nu(f). \tag{4.1}$$

We have the following

Theorem 4.3. *Given any sequence Λ_m increasing to $\mathbb{Z}^\nu$ and the corresponding zero (pure, general) b.c. Gibbs measures, there is a subsequence $\Lambda_{m'}$ and a measure $\nu\in M$ such that*

$$\lim\nu_{\Lambda_{m'}}=\nu$$

in the sense of Equation (4.1).

Proof. To fix our ideas let us consider the measures ν_{Λ_m} as general b.c. Gibbs measures w.r.t. a regular measure λ [the other cases can be treated analogously]. We fix Λ and consider the sequence of measures $\nu_{\Lambda_m}(dS_\Lambda|\lambda)$ on $X(\Lambda)$, $\Lambda\subset\Lambda_m$. As in Corollary 4.2

$$\nu_{\Lambda_m}(dS_\Lambda|\lambda)=\nu'_{\Lambda_m}(dS_\Lambda|\lambda)+\nu''_{\Lambda_m}(dS_\Lambda|\lambda)$$

and $\nu''_{\Lambda_m}(dS_\Lambda|\lambda)$ approaches zero in the sense of Equation (4.1) for $f=f(S_\Lambda)$ in $X(\Lambda)$. Again by use of Corollary 4.2 and then by compactness arguments, a subsequence can be found for which $\nu'_{\Lambda_m}(dS_\Lambda)$ and therefore $\nu_{\Lambda_m}(dS_\Lambda)$ have a limit. By a diagonalization procedure and the Kolmogorov theorem we therefore prove that the subsequence $\{\Lambda_{m'}\}$ and the measure ν exist. Still by Theorem 4.2 it is finally proven that ν is tempered and therefore in M; the proof is then completed.

Theorem 4.3 tells us that the limits of finite volume Gibbs states exist. To characterize them we use equilibrium equations. For any Δ we introduce the operators τ_Δ on the Borel cylindrical sets as

$$(\tau_\Delta f)(\underline{S})=\int \nu_\Delta(dS_\Delta|\underline{S})f(\underline{S})\,. \tag{4.2}$$

Definition 4.2. We say that a measure ν satisfies the equilibrium equations if $\nu\in M$ and for any Borel cylindrical set f

$$\nu(\tau_\Delta f)=\nu(f)\,. \tag{4.3}$$

The same arguments as used in [4] allow us to prove the following

Theorem 4.4. *Let the hypotheses of Section 1 hold. Then there are $\bar{\gamma}$, $\bar{\delta}$ (the same as in Theorem 1.1) such that if ν satisfies the equilibrium equations and $\varrho(S_\Delta)\mu(dS_\Delta)$ is its relativization to $X(\Delta)$, then*

$$\varrho(S_\Delta)\leqq\exp\left\{-\sum_{x\in\Delta}(\bar{\gamma}S_x^2-\bar{\delta})\right\}. \tag{4.4}$$

As a consequence ν is a finite volume Gibbs measure with general b.c. determined by a regular measure λ which is just ν. Therefore ν is the (trivial) limit of general b.c. Gibbs measures.

With the next theorem we prove a converse of the last statement in Theorem 4.4 and we show that there are solutions to the equilibrium equations which characterize the limiting Gibbs measures.

Theorem 4.5. *Let the hypotheses of Section 1 hold. Let ν_n be a sequence of Gibbs measures for the regions $\Lambda_n\to\mathbb{Z}^\nu$. Let ν be the limiting measure, then ν satisfies the equilibrium equations.*

The proof is given in the appendix.

5. Uniqueness of Equilibrium States for Ferromagnetic Systems

Definition 5.1. The one component spin system, $S_x\in R$, will be called ferromagnetic [1, 11] if the energy has the form

$$U(S_\Lambda)=-\sum_{x\neq y\in\Lambda}J(x,y)S_xS_y-\sum_{x\in\Lambda}\Psi(S_x)-\sum_{x\in\Lambda}h(x)S_x,\quad\text{with}\quad J(x,y)\geqq 0\,. \tag{5.1}$$

A ferromagnetic system satisfies the FKG [1, 11] inequalities: Let $\nu_\Lambda(f|\hat{\underline{S}})$ be the expectation value of $f(\underline{S})$ with respect to the Gibbs measure in Λ with b.c. $\hat{\underline{S}}$,

$$\nu_\Lambda(f|\hat{\underline{S}})=\int f(S_\Lambda,\hat{S}_{\Lambda^c})\varrho(S_\Lambda|\hat{\underline{S}})\mu(dS_\Lambda)\,, \tag{5.2}$$

where $\varrho(S_\Lambda|\hat{\underline{S}})$ is defined in Equation (3.4).

Define a partial ordering on the configurations X; $\underset{\sim}{S} > \underset{\sim}{S}'$, if and only if $S_x \geqq S'_x$. We call f an increasing function if $f(\underset{\sim}{S}) \geqq f(\underset{\sim}{S}')$ whenever $\underset{\sim}{S} > \underset{\sim}{S}'$.

Lemma 5.2. (FKG). *Let f and g be increasing functions then, for a ferromagnetic system,*

$$\nu_A(fg|\underset{\sim}{S}) \geqq \nu_A(f|\underset{\sim}{S})\nu_A(g|\underset{\sim}{S}). \tag{5.3}$$

The inequality clearly remains valid in the limit $\Lambda \to \infty$.

It follows directly from the FKG inequalities that

Corollary 5.3. *If $f(\underset{\sim}{S})$ is an increasing function then so is $\nu_A(f|\underset{\sim}{S})$, considered as a function on S_{Λ^c}.*

Proof. There are two ways in which $\nu_A(f|\underset{\sim}{S})$ can depend on S_{Λ^c}: a) an explicit dependence of f on S_{Λ^c} and b) through $\varrho(S_\Lambda|\underset{\sim}{S})$ in (5.2). Since $f(\underset{\sim}{S})$ is increasing the explicit dependence a) certainly satisfies the corollary. To treat b) we differentiate $\nu_A(f|\underset{\sim}{S})$ with respect to S_y, $y \in \Lambda^c$, ignoring any dependence of f on S_{Λ^c}. Using (5.2) we find,

$$\frac{\partial}{\partial S_y}\nu_A(f|\underset{\sim}{S}) = \sum_{x\in\Lambda} J(x,y)[\nu_A(fS_x|\underset{\sim}{S}) - \nu_A(f|\underset{\sim}{S})\nu_A(S_x|\underset{\sim}{S})] \geqq 0,$$

since S_x is an increasing function and $J(x, y) \geqq 0$.

It follows from Corollary 5.3 that for an increasing function f

$$\nu_A(f|\underset{\sim}{S}^{(1)}) \leqq \nu_A(f|\underset{\sim}{S}^{(2)}) \leqq \nu_A(f|\underset{\sim}{S}^{(3)}) \tag{5.4}$$

where $S^{(i)}_{\Lambda^c}$, $i=1,2,3$, are configurations of spins in Λ^c such $S^{(1)}_x \leqq S^{(2)}_x \leqq S^{(3)}_x$, $x \in \Lambda^c$.

Let us now introduce the following increasing functions:

$$\sigma_x = \sigma_x(S_x) = \begin{cases} S_x/\lambda, & |S_x| \leqq \lambda, \\ \operatorname{sgn} S_x, & |S_x| \geqq \lambda, \end{cases} \quad \lambda \geqq 1,$$

$$\varrho_x = \tfrac{1}{2}(1+\sigma_x), \tag{5.5a}$$

$$R_\Delta = \prod_{i=1}^{n} \varrho_{x_i}, \; \Sigma_\Delta = \sum_{i=1}^{n} S_{x_i}, \; Q_\Delta = \Sigma_\Delta - R_\Delta, \tag{5.5b}$$

where $x_i \in \Delta$; not necessarily distinct (note that $0 \leqq R_\Delta \leqq 1$).

It then follows from (5.4), cf. [6], that

$$0 \leqq \nu_A(R|\underset{\sim}{S}^+) - \nu_A(R|\underset{\sim}{S}^-) \leqq \sum_{i=1}^{n} [\nu_A(S_{x_i}|\underset{\sim}{S}^+) - \nu_A(S_{x_i}|\underset{\sim}{S}^-)], \; x_i \in \Delta \subset \Lambda, \tag{5.6}$$

whenever $S^+_y \geqq S^-_y$, $y \in \Lambda^c$ and we have dropped the subscript Δ from R since Δ will be kept fixed from now on.

It is seen from (5.4) and (5.5) that iff $\nu_A(S_x|\underset{\sim}{S}^+) - \nu_A(S_x|\underset{\sim}{S}^-) \to 0$ as $\Lambda \to \infty$, $x \in Z^\nu$, then $\lim_{\Lambda\to\infty} \nu_A(R|\underset{\sim}{S})$ is independent of the b.c. $\hat{S}_{\Lambda^c}$ whenever $S^-_{\Lambda^c} < S_{\Lambda^c} < S^+_{\Lambda^c}$ for sufficiently large Λ. Since the functions R form a total set (by letting $\lambda \to \infty$) for all cylinder functions $f(S_\Delta)$ and Δ is arbitrary the same will be true for the infinite volume limit of $\nu_A(f|\underset{\sim}{S})$. This was the method used in [6] to prove the uniqueness of the equilibrium state for a spin one Ising ferromagnet ($S_x = \pm 1$), in a uniform magnetic field h and $J(x,y)=J(x-y)$ whenever the free energy density $F(h)$ is differentiable. Their proof consisted of showing that the differentiability of $F(h)$ implies $\nu_+(S_x)=\nu_-(S_x)$ where $\nu_\pm$ are the equilibrium states obtained with $\pm$b.c., $\{S^\pm_{\Lambda^c}; S^+_y = -S^-_y = 1,\ y\in\Lambda^c\}$. The analysis of [6] carries over unchanged to the case of compact spins, $\mu(dS)=0$ for $|S|>1$.

In the unbounded case, also, extremal measures can be introduced which play the role of $\nu_\pm$. We assume hereafter that the hypotheses of Section 1 hold, then by Lemma 3.1 we can fix a so that $\nu[\bar{X}(a)]=1$, when ν is any tempered equilibrium measure, see Theorem 4.4 and 4.5. In this class there are extremal states, determined for each Λ by the values of the external spins:

$$S^\pm_x = \pm a(\log|x|)^{\frac{1}{2}}, x\notin\Lambda .$$

We write

$$\nu_\Lambda(dS_\Lambda|\underline{S}^\pm)=\nu_\Lambda(dS_\Lambda|\pm) . \tag{5.7}$$

Since $\bar{X}(a)$ has full measure we will find that the interesting measures are between $\nu_\Lambda(dS_\Lambda|+)$ and $\nu_\Lambda(dS_\Lambda|-)$ as in the unbounded case. Also as in the bounded case we have the + and − limiting states:

Theorem 5.4. *Let the hypotheses of Section 1 hold and let the interaction be ferromagnetic. Let $\Lambda_n\to\mathbb{Z}^\nu$ in the Van Hove sense. Then there are equilibrium measures $\nu(d\underline{S}|\pm)$ which are limits (in the sense of Eq. (4.1)) of $\nu_\Lambda(dS_\Lambda|\pm)$.*

Proof. It is sufficient to prove that for any sequence $\Lambda_n\to\mathbb{Z}^\nu$ the limits for $\nu_{\Lambda_n}(dS_{\Lambda_n}|\pm)$ exist. Let $\Lambda_n\uparrow\mathbb{Z}^\nu$ be fixed. We consider Δ and R_Δ as in Equation (5.5). By Theorem 4.1 and 4.3 there is a subsequence $\Lambda_{n'}$ such that $\nu_{\Lambda_{n'}}\to\nu$ and

$$\limsup \nu_{\Lambda_n}(R_\Delta|+)=\nu(R_\Delta)\leqq 1 . \tag{5.8}$$

Given $\varepsilon>0$ we can find Λ_m, m sufficiently large, so that [see Eq. (3.12)]

$$\begin{aligned}\nu(R_\Delta) &= \int \nu(d\hat{\underline{S}})\int \nu_{\Lambda_m}(dS_{\Lambda_m}|\hat{\underline{S}})R_\Delta(S_\Delta)\\ &\leqq \nu_{\Lambda_m}(R_\Delta|+)+\varepsilon\end{aligned} \tag{5.9}$$

where

$$1-\varepsilon\leqq\nu[\{\underline{S}|S^2_x\leqq a\log|x| \quad \text{for} \quad x\notin\Lambda_m\}] . \tag{5.10}$$

By Equations (5.9) and (5.8)

$$\nu_{\Lambda_m}(R_\Delta|+)\geqq -\varepsilon+\limsup \nu_{\Lambda_n}(R_\Delta|+) .$$

Since ε is arbitrary the existence of $\nu(d\underline{S}|+)$ is proven. The analogous holds for $\nu(d\underline{S}|-)$ and so the thesis is proven. By standard arguments [6, 8] we have

Corollary 5.5. *The states $\nu(d\underline{S}|\pm)$ defined in Theorem 5.4 are translationally invariant.*

We are now able to prove the following

Theorem 5.6. *Let the hypotheses in Section 1 hold. Let the interaction be ferromagnetic. Define $M(h)$ as the set of tempered equilibrium states at magnetic field h and $F(h)$ as the corresponding free energy defined via Theorem 2.7. Then if $F(h)$ is differentiable at h, $M(h)$ contains only one measure.*

Proof. Equation (5.9)–(5.10) and Theorem 4.4 applied to $\nu\in M(h)$ show that for any R_Λ

$$\nu(R_\Lambda|-)\leqq\nu(R_\Lambda)\leqq\nu(R_\Lambda|+)\,. \tag{5.11}$$

We also have as a consequence of Equation (5.6) and of Theorem 5.4 that

$$\nu(R_\Lambda|+)-\nu(R_\Lambda|-)\leqq\sum_{i=1}^{n}\left[\nu(S_{x_i}|+)-\nu(S_{x_i}|-)\right],\,x_i\in\Lambda\,. \tag{5.12}$$

Using now Corollary 5.5 in Equation (5.12) we obtain for $x\in Z^\nu$

$$\begin{aligned}
&\nu(R_\Lambda|+)-\nu(R_\Lambda|-)\leqq n[\nu(S_x|+)-\nu(S_x|-)]\\
&=|\Lambda|^{-1}n\sum_{x\in\Lambda}[\nu(S_x|+)-\nu(S_x|-)]\\
&\leqq n\{\textstyle\int\nu(d\hat{\underline{S}}|+)\int\nu_\Lambda(dS_\Lambda|\hat{\underline{S}})[|\Lambda|^{-1}\sum_{x\in\Lambda}S_x]-\int\nu(d\hat{\underline{S}}|-)\int\nu_\Lambda(dS_\Lambda|\hat{\underline{S}})\\
&\quad[|\Lambda|^{-1}\sum_{x\in\Lambda}S_x]\}\\
&=n\left[\frac{d}{dh}F(\Lambda;\nu(d\underline{S}|+);h)-\frac{d}{dh}F(\Lambda;\nu(d\underline{S}|-);h)\right]
\end{aligned} \tag{5.13}$$

where F is defined in Definition 3.3 (general b.c.) and its dependence on h has been made explicit. Taking the limit $\Lambda\uparrow\mathbb{Z}^\nu$ it follows from the independence of $F(h)$ on the boundary conditions, and the convexity of F as a function of h [8, 6] that the right side of Equation (5.13)$\to 0$ whenever $F(h)$ is differentiable. Q.E.D.

Remark. It follows from the convexity of $F(h)$ that it will be differentiable, and there will thus be a unique equilibrium state, except possibly at a countable number of values of h. If furthermore the system is of the Lee-Yang type, [1, 12] e.g. $U(S)$ and $\mu(dS)$ even and $\int\exp[hS-U(S)]\mu(dS)\neq 0$ for $\mathrm{Re}\,h\neq 0$, then for real h only $h=0$ can be a place of non-analyticity of $F(h)$ and therefore of non-uniqueness.

(Note that in proving Theorem 5.6 we did not require that U and μ be even, i.e. the system need not have any symmetry for $h=0$.)

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Appendix to Section 2

Proof of Lemma 2.5. By Equation (2.2) we have $|g_j^\Lambda(\Delta_1,\Delta_2)|\geqq j$ so that using Equation (1.7)

$$\nu_\Delta[g^c(\Delta_1,\Delta_2)]\leqq\sum_i\nu_\Delta[g_i^c(\Delta_1,\Delta_2)]\leqq\sum_j\exp[-(\gamma N^2-\delta)j]$$

for $N^2\geqq\delta/\gamma$. The r.h.s. goes to zero as N increases so that the thesis is proven for N sufficiently large.

Proof of Lemma 2.6. We first prove the following Lemma A2.1. The proof of Lemma 2.6 is then obtained as a corollary.

Lemma A2.1. *Let the assumptions of Section 1 hold. Let $\Lambda=\Delta_1\cup\Delta_2$, $\Delta_1\cap\Delta_2=\emptyset$. Then*

$$\tfrac{1}{2}Z(\Delta_1)Z(\Delta_2)\exp\left[-N^2\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)\right]\leqq Z(\Lambda)\leqq 2Z(\Delta_1)Z(\Delta_2)$$
$$\exp\left[N^2\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)\right].$$

Proof. Let $g=g^\Lambda(\Delta_1,\Delta_2)\cap g^\Lambda(\Delta_2,\Delta_1)$ then using Equations (2.4), (2.5) we find

$$\begin{aligned}Z(\Lambda)&=\int_{g^c}\mu(dS_\Lambda)\exp[-U(S_\Lambda)]+\int_g\mu(dS_\Lambda)\exp[-U(S_\Lambda)]\\ &\leqq\tfrac{1}{2}Z(\Lambda)+\int_g\mu(dS_{\Delta_1})\mu(dS_{\Delta_2})\exp[-U(S_{\Delta_1})-U(S_{\Delta_2})]\\ &\quad\exp N^2\left[\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)\right]\\ &\leqq\tfrac{1}{2}Z(\Lambda)+\exp N^2\left[\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)\right]\int\mu(dS_{\Delta_1})\mu(dS_{\Delta_2})\\ &\quad\exp[-U(S_{\Delta_1})-U(S_{\Delta_2})]\end{aligned}$$

so that the upper bound is proven. To obtain the lower bound we write

$$\begin{aligned}Z(\Lambda)&\geqq\int_g\mu(dS_\Lambda)\exp[-U(S_\Lambda)]\geqq\int_g\mu(dS_{\Delta_1})\mu(dS_{\Delta_2})\\ &\quad\exp\left[-U(S_{\Delta_1})-U(S_{\Delta_2})-N^2\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)\right].\end{aligned}$$

Thus

$$\begin{aligned}&\exp\left[N^2\sum_{x\in\Delta_1}\sum_{y\in\Delta_2}\Psi(|x-y|)\right]Z(\Lambda)\geqq\int_g\mu(dS_{\Delta_1})\mu(dS_{\Delta_2})\\ &\exp[-U(S_{\Delta_1})-U(S_{\Delta_2})]\geqq Z(\Delta_1)Z(\Delta_2)-\int_{g^c}\mu(dS_{\Delta_1})\\ &\mu(dS_{\Delta_2})\exp[-U(S_{\Delta_1})-U(S_{\Delta_2})]\end{aligned}$$

with the notation of Equations (2.2), (2.3) we have

$$g^c=g^c(\Delta_1,\Delta_2)\cup g^c(\Delta_2,\Delta_1)$$

hence

$$\begin{aligned}&\int_{g^c}\mu(dS_{\Delta_1})\mu(dS_{\Delta_2})\exp[-U(S_{\Delta_1})-U(S_{\Delta_2})]\\ &\leqq\int\mu(dS_{\Delta_2})\exp[-U(S_{\Delta_2})]\{\textstyle\int_{g^c(\Delta_1,\Delta_2)}\mu(dS_{\Delta_1})\exp[-U(S_{\Delta_1})]\}\\ &\quad+\int\mu(dS_{\Delta_1})\exp[-U(S_{\Delta_1})]\{\textstyle\int_{g^c(\Delta_2,\Delta_1)}\mu(dS_{\Delta_2})\exp[-U(S_{\Delta_2})]\}\\ &\leqq\tfrac{1}{4}Z(\Delta_1)Z(\Delta_2)+\tfrac{1}{4}Z(\Delta_2)Z(\Delta_1)\end{aligned}$$

where Lemma 2.5 has been applied twice with $\Delta=\Delta_1$ and $\Delta=\Delta_2$.

Proof of Lemma 2.6. It is obtained from the previous Lemma by induction on the number n of the decomposition. For $n=2$ we are in the case of Lemma A2.1. Therefore we assume the Lemma true for $n-1$ and we have to prove it for n. We consider the decomposition in $n-1$ sets given by $\Delta_1 \cup \Delta_2, \Delta_3, \ldots$. Then we have by hypothesis and Lemma A2.1 applied to $\Delta_1 \cup \Delta_2$

$$\begin{aligned}
Z(\Lambda) &\geqq \tfrac{1}{2} Z(\Delta_1 \cup \Delta_2) \exp\left[-\tfrac{1}{2}N^2 \sum_{x\in\Delta_1\cup\Delta_2} \sum_{y\notin\Delta_1\cup\Delta_2} \Psi(|x-y|)\right] \\
&\quad \prod_{i=3}^{n} \left\{\tfrac{1}{2} Z(\Delta_i) \exp\left[-\tfrac{1}{2}N^2 \sum_{x\in\Delta_i} \sum_{y\notin\Delta_i} \Psi(|x-y|)\right]\right\} \\
&= \tfrac{1}{2}\tfrac{1}{2} Z(\Delta_1) Z(\Delta_2) \exp\left[-N^2 \sum_{x\in\Delta_1} \sum_{y\in\Delta_2} \Psi(|x-y|)\right] \\
&\quad \exp\left[-\tfrac{1}{2}N^2 \left(\sum_{x\in\Delta_1} + \sum_{y\in\Delta_2}\right) \sum_{y\notin\Delta_1\cup\Delta_2} \Psi(|x-y|)\right] \\
&\quad \prod_{i=3}^{n} \tfrac{1}{2} Z(\Delta_i) \exp\left[-\tfrac{1}{2}N^2 \sum_{x\in\Delta_i} \sum_{y\notin\Delta_i} \Psi(|x-y|)\right].
\end{aligned}$$

We proceed analogously for the upper bound, therefore the corollary is proven.

Appendix to Section 3

Proof of Theorem 3.1, General B.C. We divide the potentials for which the superstability condition Definition 1.4(b) holds into two classes. The first class consists of potentials whose translationally invariant one site energy $U(S_x)$ satisfies the condition

$$U(S_x) \geqq \tilde{A} S_x^4 - \tilde{c}\,. \tag{A3.1}$$

It follows then from the regularity condition Definition 1.4(c) on the interaction that there are A and c such that for any bounded Λ

$$U(S_\Lambda) \geqq \sum_{x\in\Lambda} (A S_x^4 - c)\,. \tag{A3.2}$$

The second class consists of potentials for which Equation (A3.1) does not hold for any A, c. For this class we can find A, c, A^+, c^+ such that for any Λ

$$\sum_{x\in\Lambda} (A S_x^2 - c) \leqq U(S_\Lambda) \leqq \sum_{x\in\Lambda} (A^+ S_x^4 - c^+)\,. \tag{A3.3}$$

First Class of Potentials, Equation (A3.2) Holds. As usual we look for upper and lower bounds of the free energy. We have

$$\begin{aligned}
Z(\Lambda_n|\underline{S}) &= \int \mu(dS_{\Lambda_n}) \exp[-U(S_{\Lambda_n}) - W(S_{\Lambda_n}|S_{\Lambda_n^c})] \\
&\leqq \int \mu(dS_{\Lambda_n}) \exp[-U(S_{\Lambda_n}) + I(S_{\Lambda_n})] \exp\left[\tfrac{1}{2} \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) S_y^2\right] \\
&\leqq Z^+(\Lambda_n) \exp\left[\tfrac{1}{2} \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) S_y^2\right]
\end{aligned}$$

where $Z^+(\Lambda_n)$ is defined by the last two inequalities and

$$I(S_{\Lambda_n}) = \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \tfrac{1}{2}\Psi(|x-y|)S_x^2$$

$$\begin{aligned} F(\Lambda_n|\nu) &\leqq |\Lambda_n|^{-1} \ln[Z^+(\Lambda_n)] + (2|\Lambda_n|)^{-1} \int \nu(d\underline{S}) \left\{ \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|)S_y^2 \right\} \\ &\leqq |\Lambda_n|^{-1} \ln Z^+(\Lambda_n) + (2|\Lambda_n|)^{-1} \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) \int \nu(d\underline{S}) S_y^2 \\ &\leqq |\Lambda_n|^{-1} \ln Z^+(\Lambda_n) + k(2|\Lambda_n|)^{-1} \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) \end{aligned} \tag{A3.4}$$

where

$$\infty > k > \int \nu(d\underline{S}) S_y^2 \quad \text{for every} \quad y \in \mathbb{Z}^\nu .$$

Fatou's lemma has been used in the second inequality in Equation (A3.4). It follows therefore from Equation (A3.4) that

$$\limsup F(\Lambda_n|\nu) \leqq \limsup |\Lambda_n|^{-1} \ln Z^+(\Lambda_n) . \tag{A3.5}$$

We will now prove that the r.h.s. of Equation (A3.5) is bounded by F. The partition functions $Z^+(\Lambda_n)$ are determined by the energies

$$U^+(S_\Lambda) = U(S_\Lambda) - \tfrac{1}{2} \sum_{x\in\Lambda} \sum_{y\notin\Lambda} \Psi(|x-y|)S_x^2 = U(S_\Lambda) - I(S_\Lambda) . \tag{A3.6}$$

U^+ is not translationally invariant, while the energy U is. Nevertheless because we assumed Equation (A3.2) to hold, all the other conditions in Definition 1.4 on the interaction are fulfilled by U^+. Therefore Theorem 1.1 can be stated for the finite volume Gibbs measures ν_Λ^+ corresponding to the energy U^+. Then, with the notation employed in Theorem 1.1, we have that there are $\gamma^+ > 0$ and δ^+ so that

$$\nu_\Lambda^+[B(\Delta, N^2)] \leqq \exp[-(\gamma^+ N^2 - \delta^+)|\Delta|] \tag{A3.7}$$

for any bounded Λ and $\Delta \subset \Lambda$, $N^2 > 0$. [g_n below denotes $g(\Lambda_n, \Lambda_n^c)$ as defined in Eq. (2.3).]

$$\begin{aligned} Z^+(\Lambda_n) &= \left(\int_{g_n} + \int_{g_n^c} \right) \{\mu(dS_{\Lambda_n}) \exp U^+(S_{\Lambda_n})\} \\ &\leqq \int_{g_n} \mu(dS_{\Lambda_n}) \exp[-U(S_{\Lambda_n}) + I(S_{\Lambda_n})] + \tfrac{1}{2} Z^+(\Lambda_n) \\ &\leqq 2Z(\Lambda_n) \exp \tfrac{1}{2} N^2 \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) \end{aligned}$$

where $Z(\Lambda_n)$ is the zero b.c. partition function. Use of this inequality in Equation (A3.5) then gives the required upper bound. For the lower bound the procedure is analogous and so the thesis is proven (in the case we examined).

Second Class of Potentials, Equation (A3.3) Holds. The difficulty here lies in the fact we cannot evaluate directly the interaction energy in $Z(\Lambda_n|\underline{S})$ because the corresponding energy U^+ need not be superstable anymore. It is therefore necessary to prove first that configurations with large spins in a neighborhood of the boundaries $\partial\Lambda_n$ have small probability and therefore do not contribute in the thermodynamic limit. For the remaining configurations we can bound the interaction via Definition 1.4(c) and then proceed as in the first case. We need some notation: given the region Λ_n its "contour" $\bar{\partial}\Lambda_n$ is defined as

$$\bar{\partial}\Lambda_n \cap \Lambda_n = \emptyset \qquad \bar{\Lambda}_n = \Lambda_n \cup \bar{\partial}\Lambda_n\,, \tag{A3.8a}$$

$$\sum_{y\notin\bar{\Lambda}_n} \Psi(|x-y|) < 2A \quad \text{for every} \quad x\in\Lambda_n \tag{A3.8b}$$

where A is the superstability coefficient appearing in Equation (A3.3). Since Λ_n increases to $\mathbb{Z}^\nu$ in the Van Hove sense, Definition 2.3, $\bar{\partial}\Lambda_n$ can and will be chosen so that.

$$\lim \frac{|\bar{\partial}\Lambda_n|}{|\Lambda_n|} = 0\,. \tag{A3.9}$$

As usual we look for lower and upper bounds. We have

$$\begin{aligned} Z(\Lambda_n|\underline{S}) &= Z(\Lambda_n|\underline{S}) \exp[-U(S_{\bar{\partial}\Lambda_n})] \exp[U(S_{\bar{\partial}\Lambda_n})] \\ &\leqq \int \mu(dS'_{\Lambda_n}) \exp\Big[-U(S'_{\Lambda_n}\cup S_{\bar{\partial}\Lambda_n}) + \tfrac{1}{2}\sum_{x\in\Lambda_n}\sum_{y\notin\bar{\Lambda}_n}\Psi(|x-y|)S_x^2\Big] \\ &\quad \exp\Big[\tfrac{1}{2}\sum_{x\in\Lambda_n}\sum_{y\notin\bar{\Lambda}_n}\Psi(|x-y|)S_y^2\Big] \exp[U(S_{\bar{\partial}\Lambda_n})] \\ &\leqq \hat{Z}(S_{\bar{\partial}\Lambda_n}) \exp\Big[\tfrac{1}{2}\sum_{x\in\Lambda_n}\sum_{y\notin\Lambda_n}\Psi(|x-y|)S_y^2\Big] \exp U(S_{\bar{\partial}\Lambda_n}) \end{aligned} \tag{A3.10}$$

where

$$\hat{Z}(S_{\bar{\partial}\Lambda_n}) = \int \mu(dS'_{\Lambda_n})\lambda(dS'_{\bar{\partial}\Lambda_n}|S_{\bar{\partial}\Lambda_n}) \exp[-U(S'_{\Lambda_n}) + I'(S'_{\Lambda_n})]$$

$$\lambda(dS'_{\bar{\partial}\Lambda_n}|S_{\bar{\partial}\Lambda_n}) = \prod_{x\in\bar{\partial}\Lambda_n} \lambda_x(dS'_x|S_x) \tag{A3.11a}$$

$$\int \lambda_x(dS'_x|S_x) f(S_x) = f(S'_x=0) + f(S'_x=S_x) \tag{A3.11b}$$

$$I'(S'_{\Lambda_n}) = \tfrac{1}{2}\sum_{x\in\Lambda_n}\sum_{y\notin\Lambda_n}\Psi(|x-y|)S_x'^2\,. \tag{A3.11c}$$

With the above choice λ satisfies the same properties as the intrinsic free measure μ uniformly in $S_{\bar{\partial}\Lambda_n}$; we will then extend Theorem 1.1 to this case. Firstly we have from Equation (A3.10) by use of Equation (A3.3)

$$\begin{aligned} F(\Lambda_n|\nu) &\leqq |\Lambda_n|^{-1}\int \nu(d\underline{S}) \ln \hat{Z}(S_{\bar{\partial}\Lambda_n}) + (2|\Lambda_n|)^{-1}\sum_{x\in\Lambda_n}\sum_{y\notin\bar{\Lambda}_n} \\ &\quad \Psi(|x-y|)\int \nu(d\underline{S})S_y^2 + |\Lambda_n|^{-1}\sum_{x\in\bar{\partial}\Lambda_n} A^+ \int \nu(d\underline{S})S_x^4 \\ &\leqq |\Lambda_n|^{-1}\int \nu(d\underline{S}) \ln \hat{Z}(S_{\bar{\partial}\Lambda_n}) + K|\Lambda_n|^{-1}\sum_{x\in\Lambda_n}\sum_{y\notin\bar{\Lambda}_n}\Psi(|x-y|) \\ &\quad + A^+K'|\bar{\partial}\Lambda_n|/|\Lambda_n| \end{aligned} \tag{A3.12}$$

where K is the same as in Equation (A3.4) and, by the regularity of the measure ν,

$$\infty > K' > \int \nu(d\underline{S}) S_x^4 \quad \text{for every} \quad x \in \mathbb{Z}^\nu .$$

Therefore from Equation (A3.12)

$$\limsup F(\Lambda_n|\gamma) \leqq \limsup |\Lambda_n|^{-1} \int \nu(d\underline{S}) \ln \hat{Z}(S_{\bar{\partial}\Lambda_n}) . \tag{A3.13}$$

We will now prove that the r.h.s. of Equation (A3.13) is bounded by F. As after Equation (A3.5) we introduce the new energy

$$\tilde{U}^+(S_{\bar{\Lambda}_n}) = U(S_{\bar{\Lambda}_n}) - I'(S_{\Lambda_n})$$

and correspondingly the Gibbs measure

$$\tilde{\nu}^+_{\Lambda_n}(dS'_{\bar{\Lambda}_n}|S_{\bar{\partial}\Lambda_n}) = \hat{Z}(S_{\bar{\partial}\Lambda_n})^{-1} \int \tilde{\mu}(dS'_{\bar{\Lambda}_n}|S_{\bar{\partial}\Lambda_n}) \exp[-\tilde{U}^+(S'_{\bar{\Lambda}_n})] , \tag{A3.14a}$$

$$\tilde{\mu}(dS'_{\bar{\Lambda}_n}|S_{\bar{\partial}\Lambda_n}) = \mu(dS'_{\Lambda_n}) \lambda(dS'_{\bar{\partial}\Lambda_n}|S_{\bar{\partial}\Lambda_n}) . \tag{A3.14b}$$

Since the "free" measure $\tilde{\mu}$ and the new energy $\tilde{U}^+$ satisfy the conditions in Definitions 1.3 and 1.4 uniformly in $\underline{S}$ the estimates of Theorem 1.1 can be extended to the new Gibbs measure $\tilde{\nu}^+_{\Lambda_n}$. As a consequence the analogous of Lemma 2.6 can be proven also in this case and so we obtain uniformly in $S_{\bar{\partial}\Lambda_n}$

$$\hat{Z}(S_{\bar{\partial}\Lambda_n}) \leqq 2Z'(\Lambda_n) Z''(S_{\bar{\partial}\Lambda_n}) \exp\left[N^2 \sum_{x\in\Lambda_n} \sum_{y\in\bar{\partial}\Lambda_n} \Psi(|x-y|)\right] , \tag{A3.15a}$$

$$Z'(\Lambda_n) = \int \mu(dS_{\Lambda_n}) \exp[-U(S_{\Lambda_n}) + I'(S_{\Lambda_n})] , \tag{A3.15b}$$

$$Z''(S_{\bar{\partial}\Lambda_n}) = \int \lambda(dS'_{\bar{\partial}\Lambda_n}) \exp[-U(S'_{\bar{\partial}\Lambda_n})] \leqq \exp(\alpha|\bar{\partial}\Lambda_n|) \tag{A3.15c}$$

for some sufficiently large α independently of $S_{\bar{\partial}\Lambda_n}$ and $\bar{\partial}\Lambda_n$. By the estimate in Equation (A3.15) we reduce Equation (A3.13) to the analogous of Equation (A3.5) and so we prove that the rhs of Equation (A3.13) is bounded by F.

Lower bound. Given Λ_n we write

$$\Lambda_n = \bar{\Delta}_n = \Delta_n \cup \bar{\partial}\Delta_n$$

where $\bar{\partial}\Delta_n$ is defined as in Equations (A3.8), (A3.9). By use of Definition 1.4(d) we have:

$$\begin{aligned} Z(\Lambda_n|\underline{S}) &\geqq \int \mu(dS'_{\Lambda_n}) \int_{\Sigma\bar{\partial}\Delta_n} \mu(dS'_{\bar{\partial}\Delta_n}) \exp[-U(S_{\Lambda_n}) - W(S'_{\Lambda_n}|S_{\Lambda_n^c})] \\ &\geqq \int \mu(dS_{\Delta_n}) \int_{\Sigma\bar{\partial}\Delta_n} \mu(dS_{\bar{\partial}\Delta_n}) \exp[-U(S_{\Delta_n}) - I'(S_{\Delta_n})] \\ &\quad \exp\left[-\tfrac{1}{2} \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) S_y^2 - \tfrac{1}{2} \sum_{x\in\bar{\partial}\Delta_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|) S^2\right] \end{aligned} \tag{A3.17}$$

where S^2 is the sup of S_x^2 in Σ.

By Equation (A3.17) we are essentially reduced to the case in which Equation (A3.2) holds, in fact the energies in Equation (A3.17) are superstable and the procedure is completely analogous to the previous case.

Proof of Theorem 3.1. Pure B.C. The proof is analogous to the one for general b.c. The only difference being that estimates of the kind

$$\int \nu(d\underline{S})S_x^m < k \quad \text{for} \quad m=2,4$$

used in the latter, now become

$$\int \nu(d\underline{S})S_x^m = a^{m/2} \ln^{m/2}|x| \qquad m=2,4\,.$$

However the condition Equation (1.4) on $\Psi(|x-y|)$ allows the same estimates as before.

Proof of Theorem 3.1. Periodic B.C. By Equation (3.15) we have

$$-\tfrac{1}{2}\sum_{x\in\Gamma}\sum_{y\notin\Gamma}\Psi(|x-y|)(S_x^2+S_y^2)+U(S_\Gamma)\leqq U_p(S_\Gamma)\leqq\tfrac{1}{2}\sum_{x\in\Gamma}\sum_{y\notin\Gamma}$$
$$\Psi(|x-y|)(S_x^2+S_y^2)+U(S_\Gamma)\,.$$

Since we assumed Equation (3.16) to hold we can apply the estimates of Theorem 1.1 to $U_p(S_\Gamma)$. The proof is then straigthforward.

Appendix to Section 4

Proof of Theorem 4.1. We fix Δ then $\Lambda(\Delta)$ is such that

$$2|x-y|\geqq|x| \quad \text{for} \quad x\notin\Lambda(\Delta) \quad \text{and} \quad y\in\Delta\,. \tag{A4.1}$$

For the notation we refer to [4, 5]. Let $\underline{S}\in X(a)$ then we choose the function ψ in Proposition 2.1 of Ref. [4] as

$$\psi(r)=b\ln_+ r \qquad \ln_+ r=\max\{1,\ln r\} \tag{A4.2}$$

with b larger than a and fixed as in Equation (A4.7) below. Note that the choice of ψ in Equation (A4.2) is allowed by the assumption Equation (1.4). Any $\underline{S}\in X(a)$ is in some set either of the form

$$\left\{\underline{S}\,\Big|\,\sum_{x\in[q]}S_x^2<\psi_q V_q \qquad \forall q\geqq P\right\} \tag{A4.3a}$$

or

$$\left\{\underline{S}\,\Big|\,\text{there is a largest } q\geqq P \text{ such that } \sum_{x\in[q]}S_x^2\geqq\psi_q V_q\right\}. \tag{A4.3b}$$

b will be chosen larger than $2a$ so that the statement remains true for translations of the origin of $\mathbb{Z}^\nu$ to any point in Δ, see Equation (A4.1).

As in [5] we split up ϱ_Λ into two parts ϱ'_Λ, ϱ''_Λ. For ϱ'_Λ we can proceed as in [5]. ϱ''_Λ is expressed as a sum over q: still no difference arises when the cube q (centered somewhere in Δ) is in Λ. For the other cases, in [5] there appears a factor $\exp[-U(S_{[q]})]$ while we have $\exp[-U(S_\Lambda)]$ therefore we lack a factor

$$\exp\left[-\sum_{x\in[q]/\Lambda}(AS_x^2-c)\right]\leqq\exp\left[-\sum_{x\in[q]/\Lambda}(2aA\ln|x|-c)\right]$$
$$\leqq\exp\{-V_q[2aA\psi_q/b-c]\}\,. \tag{A4.4}$$

In [5] the sum on q was of the form

$$\exp\left[-\sum_{x\in\Lambda}\gamma S_x^2\right]\sum_{q\geqq P}\exp\{-c''\psi_{q+1}V_{q+1}+D''V_{q+1}\};\ c''>0\,. \tag{A4.5}$$

In our case therefore we have

$$\exp\left[-\sum_{x\in\Lambda}\gamma S_x^2\right]\sum_{q\geqq P}\exp[-c''\psi_{q+1}V_{q+1}+D''V_{q+1}+(2aA/b)\psi_{q+1}V_{q+1}] \tag{A4.6}$$

which is again of the form of Equation (A4.5) if

$$c''>2aA/b$$

and therefore b has to be chosen so that

$$b>2a \quad \text{and} \quad b>2aA/c''\,. \tag{A4.7}$$

Proof of Theorem 4.4. For $N^2<\infty$ and Λ bounded we introduce the operator $\tau_\Delta^{(N^2,\Lambda)}$ on the bounded Borel cylindrical functions, $f\in\mathscr{I}$

$$(\tau_\Delta^{(N^2,\Lambda)}f)(\hat{S})=\int\mu(dS_\Delta)\chi\left[\sum_{x\in\Delta}S_x^2<N^2\right]\exp[-U(S_\Delta)-W(S_\Delta|\hat{S}_\Lambda)]f(S_\Delta,\hat{S}_{\Delta^c})$$
$$\left\{\int\mu(dS_\Delta)\chi\left[\sum_{x\in\Delta}S_x^2<N^2\right]\exp[-U(S_\Delta)-W(S_\Delta|\hat{S}_\Lambda)]\right\}^{-1} \tag{A4.8}$$

with range on the same space.

Let the measure ν be the limit of finite volume Gibbs measures ν_n, where n refers to the bounded region Λ_n. We have to prove that for every $f\in\mathscr{I}$, $\tau_\Delta f$ is ν-integrable and that

$$\nu(\tau_\Delta f)=\nu(f)\,. \tag{A4.9}$$

We have for every N^2, Λ

$$\lim\nu_n[\tau_\Delta^{(N^2,\Lambda)}f]=\nu[\tau_\Delta^{(N^2,\Lambda)}f] \tag{A4.10}$$

and by the (Lebesgue) dominated convergence theorem [f is bounded] we have that $\tau_\Delta f$ is ν-measurable and

$$\lim_{\substack{N^2\to\infty\\ \Lambda\to\mathbb{Z}^\nu}}\nu[\tau_\Delta^{(N^2,\Lambda)}f]=\nu(\tau_\Delta f)\,. \tag{A4.11}$$

The idea of the proof is the following: we have by Equations (A4.10), (A4.11) that

$$\begin{aligned}|\nu(f)-\nu(\tau_\Delta f)|&\leqq\varepsilon+|\nu(f)-\nu(\tau_\Delta^{(N^2,\Lambda)}f)|\\ &\leqq 3\varepsilon+|\nu_n(\tau_\Delta f)-\nu_n(\tau_\Delta^{(N^2,\Lambda)}f|\end{aligned} \tag{A4.12}$$

for N^2, Λ, and n sufficiently large. The proof is then completed if we can show that

$$|\nu_n(\tau_\Delta f)-\nu_n(\tau_\Delta^{(N^2,\Lambda)}f|<\varepsilon\,. \tag{A4.13}$$

More precisely we ask that given $\varepsilon>0$ there exist Λ_ε, N_ε^2, n_ε such that Equation (A4.13) holds for all $n>n_\varepsilon$, $N>N_\varepsilon$, $\Lambda\supset\Lambda_\varepsilon$. In proving Equation (A4.13) we shall for the sake of simplicity, consider ν_n as the pure b.c. Gibbs measure, the general case can be treated analogously.

The estimates we need are collected in the following:

Lemma A4.1. *Let ν_n be the pure b.c. Gibbs measure, and let the assumption in Section 1 hold. Then the following is true:*

(i) *given $\varepsilon>0$ and Δ bounded in $\mathbb{Z}^\nu$ there are N^2 and $\Lambda(\varepsilon,\Delta)$ such that*

$$\nu_\Lambda\left[\left\{\underline{S}\,|\sum_{x\in\Delta} S_x^2 \geqq N^2\right\}\right] < \varepsilon \quad \textit{for} \quad \Lambda \supset \Lambda(\varepsilon,\Delta);$$

(ii) *given Δ bounded and $\varepsilon>0$ there exists $\Lambda'(\varepsilon,\Delta)$ such that*

$$\nu_\Lambda[\chi_\Delta(\Lambda')] \geqq 1-\varepsilon \qquad \Lambda \supset \Lambda'(\varepsilon,\Delta) \supset \Delta$$

where $\chi_\Delta(\Lambda')$ is the characteristic function of the set

$$\left\{\underline{S}\,|\textit{if } q \textit{ is the largest integer such that } \sum_{x\in[q]} S_x^2 \geqq \psi_q V_q,\right.$$

then $\left.\Delta \subset [q+1] \subset \Lambda'\right\}$;

(iii) *given Δ bounded, $\varepsilon>0$, N^2 there is $\Lambda''(\varepsilon,\Delta,N^2) \supset \Lambda'(\varepsilon,\Delta)$ such that*

$$\left\{\exp\left[\pm \sum_{x\in\Delta}\sum_{y\notin\Lambda''} \Psi(|x-y|)\tfrac{1}{2}(S_x^2+S_y^2)\right]\right\}^{\pm 1} \leqq 1+\varepsilon$$

for all the configurations $\underline{S}\in\chi_\Delta(\Lambda'')$ and such that

$$\sum_{x\in\Delta} S_x^2 \leqq N^2.$$

Proof. (i) is proven in Theorem 4.1. (ii) is also proven in Theorem 4.1. (iii) is a consequence of Lemma 2.2(b) and Lemma 2.4(b) of Ref. [4].

We now proceed in the proof of Equation (A4.13). Given $\varepsilon>0$ we fix Λ_ε, N_ε, n_ε as in Equations (A4.17)–(A4.19). We will obtain upper and lower bounds for $(\nu_\Lambda\, \tau_\Delta f)$ in which

$$\nu_n(\tau_\Delta^{(N^2,\Lambda)} f)$$

appears. For the sake of brevity we employ the following notation

$$W(S_\Delta|\hat{S}_{\Delta^c}) = W(S_\Delta|\hat{\underline{S}}), \tag{A4.14}$$

$$\int \mu(dS_\Delta) g(S_\Delta, S_{\Delta^c}) \chi\left(\left\{S_\Delta|\sum_{x\in\Delta} S_x^2 \leqq N^2\right\}\right) = \int_{N^2} \mu(dS_\Delta) g(S_\Delta, S_{\Delta^c}) \tag{A4.15}$$

for any $g\in\mathscr{I}$.

We then have, with $\nu_{\Lambda_n} = \nu_n$

$$\begin{aligned} \int \nu_n(d\hat{\underline{S}})(\tau_\Delta f) = {} & \int \nu_n(d\hat{\underline{S}})\left[\int_{N^2} \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{\underline{S}})] f(S_\Delta, \hat{S}_{\Delta^c})\right] \\ & \left[\int \mu(dS_\Delta) \exp - U(S_\Delta) - W(S_\Delta|\hat{\underline{S}})\right]^{-1} \\ & + \int \nu_n(d\hat{\underline{S}})\left[\int \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{\underline{S}})] f(S_\Delta, \hat{S}_{\Delta^c})\right. \\ & \left.\chi\left[\sum_{x\in\Delta} S_x^2 \geqq N^2\right]\right] \left\{\int \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{S})]\right\}^{-1}. \end{aligned} \tag{A4.16}$$

The second term in the rhs of Equation (A4.16) is not larger than ($|f| \leqq 1$)

$$\begin{aligned} & \int \nu_n[d(\hat{S}_{\Delta^c}, \emptyset)] \int \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{\underline{S}})] \chi\left\{\sum_{x\in\Delta} S_x^2 \geqq N^2\right\} \\ & = \nu_n\left[\left\{\underline{S}\,\Big|\sum_{x\in\Delta} S_x^2 \geqq N^2\right\}\right] < \varepsilon \quad \text{for} \quad \Lambda_n \supset \Lambda(\varepsilon,\Delta) \end{aligned} \tag{A4.17}$$

where (i) of Lemma A4.1 has been used and the notation

$$\nu_n(d(\hat{S}_{\Lambda^c}, Q)) = \mu(d\hat{S}_{\Lambda_n/\Lambda}) \exp[-U(\hat{S}_{\Lambda_n/\Lambda}) - W(\hat{S}_{\Lambda_n/\Lambda}|\hat{S}_{\Lambda_n^c})] Z(\Lambda_n, \hat{\underline{S}})^{-1} \tag{A4.18}$$

has been employed. Therefore by Equation (A4.17) from Equation (A4.16)

$$\begin{aligned}\int \nu_n(d\hat{\underline{S}})\tau_\Delta f \leqq \varepsilon + \int \nu_n(d\hat{S})\chi_\Delta(\Lambda'', \hat{\underline{S}}) \left[\int_{N^2} \mu(dS) \exp[\cdot] f\right] \\ \left[\int_{N^2} \mu(dS_\Delta) \exp[\cdot]\right]^{-1} + \int \nu_n(d\underline{S})[1 - \chi_\Delta(\Lambda'', \underline{S})]\end{aligned} \tag{A4.19}$$

where $\chi_\Delta(\Lambda'', \underline{S})$ is the characteristic function as introduced in Lemma A4.1(ii) and we used the inequality

$$1 \geqq \int_{N^2} \mu(dS_\Delta) \exp[\cdot] f \{\int_{N^2} \mu(dS_\Delta) \exp[\cdot]\}^{-1}.$$

Therefore for $\Lambda_n \supset \Lambda''(\varepsilon, \Delta)$, see Lemma 4.1(ii), we have

$$\begin{aligned}\nu_n(\tau_\Delta f) \leqq 2\varepsilon + \int \nu_n(d\hat{\underline{S}})\chi_\Delta(\Lambda'', \hat{\underline{S}}) \int_{N^2} \mu(dS_\Delta) \exp[\cdot] f \\ \{\int_{N^2} \mu(dS_\Delta) \exp[\cdot]\}^{-1}.\end{aligned} \tag{A4.20}$$

We now finally consider $\Lambda_n \supset \Lambda \supset \Lambda''(\varepsilon, \Delta, N^2)$, see Lemma 4.1(iii)

$$\begin{aligned}\nu_n(\tau_\Delta f) &\leqq 5\varepsilon + \int \nu_n(d\hat{\underline{S}})\chi_\Delta(\Lambda'', \underline{S}) \int_{N^2} \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{S}_\Lambda) f] \\ &\quad \{\int_{N^2} \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{S}_\Lambda)]\}^{-1} \\ &\leqq 5\varepsilon + \int \nu_n(d\underline{S})(\tau_\Delta^{(N^2, \Lambda)} f).\end{aligned} \tag{A4.21}$$

Lower Bound. We have, with the same notation as in the upper bound:

$$\begin{aligned}\int \nu_n(d\hat{\underline{S}})\tau_\Delta f &\geqq \int \nu_n[d(\hat{S}_{\Delta^c}, \emptyset)] \int_{N^2} \mu(dS_\Delta) \exp[-U(S_\Delta) - W(S_\Delta|\hat{S}_{\Delta^c})] f \\ &\geqq \int \nu_n d(\hat{S}_{\Delta^c}, \emptyset)\chi_\Delta(\Lambda'', \hat{\underline{S}}) \int_{N^2} \mu(dS_\Delta) \exp[U(S_\Delta) \\ &\qquad - W(S_\Delta|\hat{S}_\Lambda)](1-\varepsilon) f \\ &\geqq -\varepsilon + \int \nu_n[d(\hat{S}_{\Delta^c}, \emptyset)]\chi_\Delta(\Lambda'', \hat{\underline{S}}) \int_{N^2} \mu(dS_\Delta) \exp[\cdot] f \\ &\geqq -\varepsilon + \int \nu_n[d(\hat{S}_{\Delta^c}, \emptyset)]\chi_\Delta(\Lambda'', \hat{\underline{S}}) \int_{N^2} \mu(dS_\Delta) \exp[-U(S_\Delta) \\ &\qquad - W(S_\Delta|\hat{S}_\Lambda)](\tau_\Delta^{(N^2, \Lambda)} f) \\ &\geqq -2\varepsilon + \int \nu_n(d(\hat{S}_{\Delta^c}, \emptyset))\chi_\Delta(\Lambda'', \hat{\underline{S}}) \int \mu(dS_\Delta) \exp[-U(S_\Delta) \\ &\qquad - W(S_\Delta|\hat{\underline{S}})](\tau_\Delta^{(N^2, \Lambda)} f) - \int \nu_n[d(S_{\Delta^c}, \emptyset)]\chi_\Delta(\Lambda'', \underline{S}) \int \mu(dS_\Delta) \\ &\qquad \exp[\cdot]\chi\left[\left\{S_\Delta | \sum_{x\in\Delta} S_x^2 \geqq N^2\right\}\right] \\ &\geqq -3\varepsilon + \int \nu_n(d\hat{\underline{S}})\chi_\Delta(\Lambda'', \hat{\underline{S}})(\tau_\Delta^{(N^2, \Lambda)} f) \\ &= -4\varepsilon + \int \nu_n(d\underline{S})(\tau_\Delta^{(N^2, \Lambda)} f).\end{aligned}$$

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Note Added in Proof. In Theorem 3.1 the following additional hypothesis is needed: the sequence $\Lambda_n \to \mathbb{Z}^\nu$ as in Definition 2.3 in such a way that

$$\lim_{n\to\infty} |\Lambda_n|^{-1} \left\{ \sum_{x\in\Lambda_n} \sum_{y\notin\Lambda_n} \Psi(|x-y|)(\ln_+ |y|)^2 \right\} = 0 .$$