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Divergent Susceptibility of Isotropic Ferromagnets*

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We prove that the susceptibility of an isotropic, two-component, classical vector spin system, e.g., the rigid rotator, diverges in three or four dimensions as the magnetic field $h\to 0$, at all temperatures for which there is a spontaneous magnetization. The divergence is at least as strong as $h^{1/2}$ in three and as $|\ln h|$ in four dimensions. We also obtain bounds on the rates of exponential decay of the parallel- and transverse-pair correlation functions.

We consider a system of generalized, classical, D-dimensional, vector spins, $\vec{S}_r=(S_r^1,\ldots,S_r^D)$, with pair interactions, on a cubical lattice in ν dimensions, $r\in\mathfrak{z}^\nu$. The energy of this system in a region $\Lambda\subset\mathfrak{z}^\nu$ is

$$H_\Lambda=-\tfrac{1}{2}\sum_{q,r}\sum_{i=1}^{D}J^i(q-r)S_q^iS_r^i-\sum_q hS_q^1,\qquad q,\, r\in\Lambda\,,\tag{1}$$

where h is an external magnetic field, and we assume for simplicity that the interaction has a finite range, $J^i(r)=0$ for $|r|>R$. The joint probability density in the canonical ensemble with reciprocal temperature $T^{-1}=\beta$ is

$$\mu_\Lambda(\{\vec{S}\})=Z^{-1}(\beta,h;\Lambda)\exp(-\beta H_\Lambda)\prod_{q\in\Lambda}\rho(S_q)\,,\tag{2}$$

where $S_q=|\vec{S}_q|$, $\rho(S)d\vec{S}$ is a finite positive measure on $\mathfrak{R}^\nu$, and Z is the partition function which normalizes μ_Λ.

In the case normally considered in statistical mechanics[1] $\rho(S)$ is assumed to be concentrated on the unit D-dimensional sphere, $\rho(S)=\delta(S-1)$, and $D=1,2,3$ correspond respectively to the Ising, rigid-rotator, and classical Heisenberg models. In the Euclidian version of quantum field theory however[2] it is natural to consider $\rho(S)=\exp[-u_n(S)]$, where $u_n(S)$ is an even polynomial of degree n; $u_4(S)=aS^4-bS^2$, $a,b>0$, being the simplest interesting case.[3]

When the interactions are isotropic, $J^i(r)=J(r)$, $i=1,2,\ldots,l$, $2\le l\le D$, then the system, in the absence of an external magnetic field ($h=0$), possesses a continuous symmetry group $O(l)$. It is widely believed (on the basis of series analysis, spin-wave theory, spherical-model behavior)[1,4] that for isotropic systems with ferromagnetic interactions, $J(r)\ge|J^k(r)|$, the susceptibility $\chi(T,h)=\partial m(T,h)/\partial h$ diverges as $h\to 0$, at least in three and four dimensions, whenever there is a spontaneous magnetization

$$\lim_{h\to 0^+} m(T,h)=m^*(T)>0\,.\tag{3}$$

Here $m(T,h)$ is the expectation value of S_q^1 in the thermodynamic ($\Lambda\to\infty$) limit. In two dimensions, $\nu=2$, where Mermin proved[5,6] that for the isotropic system $m^*(T)=0$ for $T\ne 0$, it is nevertheless thought[1,4] that the susceptibility diverges, as $h\to 0$, at low temperatures. For $\nu\ge 3$, it is expected that there is a critical temperature $T_c>0$ such that (3) holds for $T<T_c$. This has however only been proven[7] for the anisotropic case $J^1(r)>|J^i(r)|$, $i=2,\ldots,D$. (A simple elegant proof, based on inequalities, has recently been given by Kunz[8] for the anisotropic two-component system.)

The physical origin for the expected divergence in the susceptibility is that the long-wavelength fluctuations in the magnetization have, in the isotropic case, vanishingly small energy as $h \to 0$. This corresponds to a weak (nonexponential) decay of the spin-spin correlations in the limit $h = 0^+$ which gives rise to an infinite susceptibility.[4] In contrast, Ising-spin systems ($D=1$) are known to have exponentially decaying correlations and thus finite susceptibilities at sufficiently low temperatures in all dimensions and at all temperatures $T \neq T_c$ in two dimensions (with nearest-neighbor interactions).[9] The same is presumably also true for $D \neq 1$ whenever there is any anisotropy.

The existence of such divergent long-wavelength fluctuations for the spin components transverse to the magnetic field was indeed proven by Mermin[6] when there is a spontaneous magnetization. This is entirely analogous to the appearance of zero-mass Goldstone bosons in field theory.[10] However until recently there was no way of relating this divergence in the transverse susceptibility to the parallel susceptibility.[2] The necessary tool for $D=2$ was provided recently by Dunlop and Newman[3] who proved for this case that the correlation function for the parallel components of the spins cannot decay faster than the square of the transverse-spin correlations.

In this note we use Dunlop and Newman's result together with Mermin's inequality[6] to prove rigorously that for $D=2$ the existence of a spontaneous magnetization does indeed imply that the susceptibility diverges at least as fast as $h^{1/2}$ in three dimensions and as $|\ln h|$ in four dimensions. We also make use of the existence of a Lee-Yang theorem for these systems[3,11] to obtain bounds on how fast the exponential decay rates (field-theory masses) have to approach zero as $h \to 0$. [The need for a rigorous proof in place of the heuristic arguments is evidenced by the fact that, as shown by Thompson,[12] at temperatures above $T_c(1)$, the critical temperature of the $D=1$ system with the same J, the correlations of a D-component vector spin system with ferromagnetic interactions, and hence also the susceptibility, are bounded by those of the $D=1$ (Ising) system for $h=0$.]

We shall restrict ourselves from now on to the case $D=2$, $J^1(r)=J^2(r) \geq 0$, $h>0$, and (for simplicity) take Λ to be a cube of sides $2L+1$, $L > 2R$, centered on the origin, $|\Lambda| = (2L+1)^\nu$, with periodic boundary conditions on the interaction. We define

$$U_j(r;T,h;\Lambda) = \langle S_0^j S_r^j\rangle - \langle S_0^j\rangle^2 = |\Lambda|^{-1}\sum_k \exp[-i\vec{k}\cdot\vec{r}]\tilde{U}_j(k;T,h;\Lambda), \tag{4}$$

where the expectations are taken with respect to the measure (2) with $\int \exp[\lambda S^2]\rho(S)\,dS < \infty$ for all λ. The components of the vector r are integers, $r \in [-L, L]$, and the sum over k is over the first Brillouin zone. Clearly

$$\langle S_q^1\rangle(T,h;\Lambda) = m(T,h;\Lambda), \quad \langle S_q^2\rangle(T,h;\Lambda) = 0, \tag{5}$$

and

$$\beta^{-1}\chi(T,h;\Lambda) = \beta^{-1}\partial m(T,h;\Lambda)/\partial h = \sum_r U_1(r;T,h;\Lambda) = \tilde{U}_1(0;T,h;\Lambda). \tag{6}$$

Theorem 1 (Mermin).—

$$U_2(k;T,h;\Lambda) \geq m^2(T,h;\Lambda)/\beta[Ck^2+hm], \tag{7}$$

where $C = \sum_r |r^2| J(r)\alpha$ and α is a constant chosen so that $\langle (S_q)^2\rangle < \alpha$ for $h < h_0$; h_0 fixed and arbitrary.

Proof: This is Mermin's[6] Eq. (15) when $\rho(S) = \delta(S-1)$. It is obtained by setting $\vec{S}_q = (S_q\cos\theta_q, S_q\sin\theta_q)$, defining $A = \sum_q \exp(i\vec{k}\cdot\vec{q})S_q^2$ and $B = \sum_q \exp(i\vec{k}\cdot\vec{q})\partial H/\partial\theta_q$, and using the Schwartz inequality $\langle |A|^2\rangle \geq |\langle A * B\rangle|^2/\langle |B|^2\rangle$. This theorem extends readily to (isotropic) $D>2$ and arbitrary $\rho(S)$ for which $\langle S^2\rangle$ is bounded. It does not depend on the sign of $J(r)$ or h. As was shown by Mermin, Eq. (7) implies that $m^*(T)=0$ for $\nu = 1, 2$ and $T>0$, since $U_j(r) < \alpha$.

Theorem 2. (Dunlop-Newman).—For $J(r) \geq 0$, $h \geq 0$,

$$[U_2(r;T,h;\Lambda)]^2 \leq U_1(r;T,h;\Lambda)[U_1(r;T,h;\Lambda)+2m^2] \leq 2\alpha U_1(r;T,h;\Lambda). \tag{8}$$

Proof: This is Theorem 13 of Dunlop and Newman[3] and is obtained by a clever manipulation of the Ginibre inequalities.[13] It is unfortunately restricted (at present) to $D=2$ where these inequalities have been proven.[13] It requires $J(r)>0$, $h \geq 0$, but permits different fields and interactions on different sites [with appropriate modifications of (8)].

Combining (6), (7), and (8) we obtain the following.

Theorem 3.—For $J(r)\geqslant 0$, $h\geqslant 0$,

$$\chi(T,h;\Lambda)\geqslant(2\alpha\beta)^{-1}|\Lambda|^{-1}\sum_k m^4(T,h;\Lambda)[Ck^2+hm]^{-2}. \tag{9}$$

Theorem 4 follows now easily from (9).

Theorem 4.—For $h>0$, $J(r)\geqslant 0$,

$$\chi(T,h)\geqslant\begin{cases}\text{const}[m(T,h)]^{2+\nu/2}[h^{\nu/2-2}] \text{ if } \nu=1,2,3,\\ \text{const}[m(T,h)]^4\ln[\text{const}/hm(T,h)] \text{ if } \nu=4,\end{cases} \tag{10}$$

where

$$m(T,h)=\lim_{\Lambda\to\infty}m(T,h;\Lambda),$$

$$\chi(T,h)=\lim_{\Lambda\to\infty}\chi(T,h;\Lambda)=\partial m(T,h)/\partial h.$$

Proof: The existence and interchange of limits is guaranteed[13] for $\rho(S)=\delta(S-1)$ or $\rho(S)=\exp[-u_4(S)]$ by the Lee-Yang theorem[3,11] which says that $m(T,h;\Lambda)$ is analytic in the complex h plane, $\text{Re}h\neq 0$. For other $\rho(S)$ one first has to integrate (9) between h_1 and h_2, take the limit $\Lambda\to\infty$, and then differentiate with respect to h. We note the following.

(i) The same arguments also show[13] that $m(T,h)$ is independent of boundary conditions as long as these are reasonable so that the use of periodic boundary conditions does no harm. By reasonable we mean that the magnitude of the spins outside Λ be uniformly bounded (to insure the existence of the thermodynamic limit of the free energy). This is automatically true for $\rho(S)=\delta(S-1)$. Free boundary conditions are always reasonable.

(ii) It is expected that when the Lee-Yang theorem holds, then the correlation functions $\langle S_{q_1}^{i_1},\ldots,S_{q_m}^{i_m}\rangle(T,h;\Lambda)$ would have for $\text{Re}h\neq 0$ (a) a unique thermodynamic limit ($\Lambda\to\infty$) independent of (reasonable) boundary conditions, (b) these infinite-volume correlations would be analytic in h and real analytic in T, and (c) that the truncated functions, such as $U_i(r;T,h)$, would have an exponential decay, i.e., $U_i(r;T,h)\leqslant\text{const}\times\exp(-\kappa_i|r|)$. This is the case for the spin-$\frac{1}{2}$ Ising system, $D=1$ and $\rho(S)=\delta(S-1)$, where (a) was shown[14] to hold for (and only for) those values of T and h for which $m(T,h)$ is continuous in h, and (b) and (c) follow essentially[15] from the convergence of the power series in $z=\exp(-2\beta h)$ for the correlation functions. For the more general case only weaker results are available at the present time.[16,17] In particular (b) and (c) hold in the case considered here, for the $U_i(r;T,h)$ obtained from periodic (and free) boundary conditions.[17] Another theorem now follows from Eqs. (7) and (8).

Theorem 5.—For $D=2$, $\rho(S)=\delta(S-1)$ or $\rho(S)=\exp[-u_4(S)]$, $J(r)=0$ for $|r|>1$, and $m^*(T)>0$, then $\kappa_i(T,h)\leqslant\text{const}\times h^{1/\nu}$ as $h\to 0^+$.

The results of Ref. 16 would however prevent a decay of $\kappa_i(T,h)$ [if indeed $\kappa_i(T,h)$ is greater than 0 for Reh sufficiently large] faster than linear in h.

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Giant Quadrupole Resonance in Deformed Nuclei*

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With use of the ordinary quadrupole-quadrupole (Q-Q) interaction, a total splitting of the $K=0,1,2$ components in the giant quadrupole resonance (GQR) of about 6 MeV is predicted for deformed ^{154}Sm. A 0.8 ± 0.3-MeV broadening of the GQR is observed for ^{154}Sm relative to spherical ^{144}Sm with inelastic α scattering. A model requiring rigorous self-consistency is proposed which modifies the Q-Q interaction, reduces the predicted splitting to about 2 MeV, and removes inconsistencies in the strength of the interaction for $K=0$ and $K=2$ components in low-lying β and γ bands.

The splitting of the giant dipole resonance in deformed nuclei is a well-established phenomenon.[1] Qualitatively it may be understood in terms of either a macroscopic or a single-particle (Nilsson model) view of the nucleus and more sophisticated models have been proposed in order to quantitatively describe the numerous experimental results. There has been little reported theoretical work on the effects of deformation on the giant quadrupole resonance (GQR) however. In the harmonic-oscillator model, the high-frequency isoscalar quadrupole mode corresponds to $\Delta N=2$ particle-hole excitations. The existence of this mode has been implied by effective-charge phenomena. Following the development of Mottelson,[2] when a particle moves outside a closed-shell core it distorts the core and induces a quadrupole moment as large as that of the valence particle. When a second particle is added, an effective interaction between the particles is induced because of the distortion of the core. This effective interaction has the form of a quadrupole-quadrupole (Q-Q) interaction and is given by $V(1,2)=-\chi Q(1)\cdot Q(2)$. The strength of the interaction is

$$\chi=\chi_{\rm self}=(4\pi/5)m\varpi_0^2/\langle\textstyle\sum_i r_i^2\rangle_{\rm sph}$$

and is determined by requiring the shapes of the potential and density distributions to be the same (nuclear self-consistency). This must be the case as the average potential is generated by the nucleons themselves moving more or less independently in this potential and interacting through a short-range force. The Q-Q interaction thus derived has been applied quite successfully to low-lying quadrupole collective motion in spherical nuclei.[3] In well-deformed nuclei on the other hand different strengths are required for $K=0$ and $K=2$ to fit the data.[4] For example, $\chi_{K=0,\beta}\cong0.85\chi_{\rm self}$ and $\chi_{K=2,\gamma}=(1.4-1.5)\chi_{\rm self}$, which is indicative of some problem with the model.

Random-phase-approximation (RPA) calculations for ^{144}Sm utilizing the Q-Q interaction give both eigenfrequencies and transition probabilities for the low-lying collective 2^+ states and the GQR[5] in reasonable agreement with the experimental values obtained. In a deformed nucleus such as ^{154}Sm or ^{150}Nd one would expect three GQR components ($K=0$, 1, and 2) due to coupling with the low-lying quadrupole rotational motions. In an axially symmetric system with $\chi=\chi_{\rm self}$ the total splitting of the three peaks would be $2\sqrt{2}\,\epsilon\hbar\omega_0$, implying for ^{154}Sm a total splitting of about 6 MeV [Fig. 1(a)].

The first experimental attempts to observe splitting of the GQR revealed little if any difference[5,6] in the width or shape of the resonance between spherical and deformed Sm isotopes. More recent (e,e') experiments[7] have shown a broader GQR in ^{150}Nd than in ^{142}Nd, but only by ~ 2 MeV. We have undertaken further investigation of the GQR in ^{144,148,154}Sm by inelastic α scattering. A beam of 115-MeV α particles was used to bombard self-supporting metal foils enriched to > 99% in the appropriate isotopes. Spectra from $E+E$ solid-state detector telescopes were recorded in