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On the Exponential Decay of Correlation Functions*

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Abstract. We use the properties of subharmonic functions to prove the following results, First, for any lattice system with finite-range forces there is a gap in the spectrum of the transfer matrix, which persists in the thermodynamic limit, if the fugacity z lies in a region E of the complex plane that contains the origin and is free of zeros of the grand partition function (with periodic boundary conditions) as the thermodynamic limit is approached. Secondly, if the transfer matrix is symmetric (for example, with nearest and next-nearest neighbor interactions in two dimensions), and if infinite-volume Ursell functions exist that are independent of the order in which the various sides of the periodicity box tend to infinity, then these Ursell functions decay exponentially with distance for all positive z in E. (For the Ising ferromagnet with two-body interactions, exponential decay holds for z in E even if the range of interaction is not restricted to one lattice spacing). Thirdly, if the interaction potential decays more *slowly* than any decaying exponential, then so do all the infinite-volume Ursell functions, for almost all sufficiently small fugacities in the case of general lattice systems, and for all real magnetic fields in the case of Ising ferromagnets.

1. Introduction

We investigate here how the asymptotic decay of Ursell and covariance functions is related to the analyticity of the free energy as a function of the thermodynamic parameters of the system, in particular the fugacity z of the magnetic field. It is generally felt that analyticity ought to imply exponential decay of the correlations, at least for finite-range potentials and non-crystalline systems. Indeed in many cases where analyticity can be proven, such as at low fugacities and/or high temperatures (or fugacities corresponding to non-zero external magnetic fields in Ising spin systems with ferromagnetic pair interactions), exponential decay can also be proven as an independent result if the interaction has finite range [1–5]. No general relationship has, however, been found.

In this paper we shall prove that analyticity does indeed imply exponential decay for certain types of system. We require, however,

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that the free energy be analytic, in a region E of the complex z-plane containing the origin, for sufficiently large systems; this is stronger than requiring only that the thermodynamic limit of the free energy be analytic in the region E. This is analogous to the result [4] that an exponential bound for the decay of the Ursell functions for sufficiently large Λ, for all real z in a region E, implies that the thermodynamic free energy is infinitely differentiable for all real z in E.

A second aim of the paper is to prove the *absence* of exponential decay under certain conditions.

The main new idea is the use of the properties of subharmonic functions. An outline of this work has appeared earlier [5].

We first consider lattice systems with interactions of finite range. Here our results are based on the properties of the transfer matrix, the rate of exponential decay being related to the ratio of the moduli of its two largest eigenvalues [6]. We show that the logarithm of this ratio is subharmonic in any region E of the complex z-plane which intersects the real z axis and is free of zeros of the grand partition function for periodic boundary conditions. We also obtain a negative upper bound on this logarithm near $z=0$, which is independent of the size of the system. Using the properties of subharmonic functions, we conclude that if E contains the origin and remains free of zeros as the thermodynamic limit is approached, this logarithm is bounded away from zero (i.e. there is a gap in the spectrum of the transfer matrix) for all z in E as the thermodynamic limit is approached. The proof that the infinite-volume Ursell functions have exponential decay in the direction of transfer can now be completed provided that the transfer matrix is symmetric, which is so if the range of interaction does not exceed one lattice spacing in any direction. (The proof for non-symmetric transfer matrices has so far eluded us.) Exponential decay in all directions will hold if the infinite-volume Ursell functions are independent of the order in which the sides of the periodicity box become infinite.

In the case of attractive two-body interactions (as in the Ising ferromagnet) a different method of proving exponential decay is available. It combines subharmonicity with Lieb and Ruelle's extension [7] of the Lee-Yang circle theorem. Since we do not use the transfer matrix in this case, the range of interaction need not be restricted to one lattice spacing.

We also consider systems with slowly decaying interactions, and show that for them the correlations also decay slowly. We assume that the fugacity z satisfies the condition $0 \leqq z < a$, where a is the radius of a disk in the complex plane within which the expansions of the distribution functions in powers of z are known to converge. To prove the result we show that the negative of the rate of exponential decay of the infinite-volume Ursell functions is a non-positive subharmonic function of z

within this disk. For interactions that decay slower than exponentially, the series expansions show that the rate of exponential decay is 0 at $z=0$, and it follows from the properties of subharmonic functions that it is 0 for almost all z between 0 and a.

In the case of attractive two-body forces (Ising ferromagnets), the inequalities of Griffiths, Hurst and Sherman [8] enable us to extend this result to all positive values of z (i.e. all real values of the magnetic field variable).

2. Subharmonic Functions

A function ψ defined in a region (connected open set) $D \subset \mathbb{R}^2$ and taking values in the set $[-\infty, +\infty)$ is said to be subharmonic [9] if

(a) ψ takes a value other than $-\infty$ for some z in D.

(b) ψ is upper semi-continuous in D.

(c) for any region D_1 whose closure lies in D, any harmonic function h satisfying $h(z) \geqq \psi(z)$ for all z on the boundary of D_1 satisfies $h(z) \geqq \psi(z)$ for all z in D_1.

Every harmonic function is also subharmonic, and if $f(z)$ is analytic in a region D of the complex plane then $\ln|f(z)| = \mathrm{Re}[\ln f(z)]$ is subharmonic in D (Ref. [9], p. 23).

The properties of subharmonic functions that we require are expressed in the following lemma:

Lemma 1. *Let $v_1, v_2, \ldots$ be a sequence of non-positive subharmonic functions on a region D in $\mathbb{C}$, such that*

$$\limsup_{n \to \infty} v_n(z_0) = 0$$

for some $z_0 \in D$. Then for any sufficiently smooth closed arc A *of finite length within D, we have*

$$\limsup_{n \to \infty} v_n(z) = 0$$

for almost all z in A.

Proof. Let C be a sufficiently smooth curve in D which encloses z_0 and includes the arc A. Let $G(z', z)$ be the Green's function for this contour, so that given any continuous real-valued function $f(z)$, defined for $z \in C$, the function

$$w(z') = \int_C G(z', z) f(z)\, dz \tag{2.1}$$

is harmonic inside C and satisfies

$$\lim_{z' \to z} w(z') = f(z) \qquad (z \in C).$$

Since v_n is subharmonic Eq. (2.1) gives, for any f satisfying $f(z) \geqq v_n(z)$ $(z \in C)$, the inequality

$$v_n(z_0) \leqq w(z_0) = \int_C G(z_0, z) f(z)\, dz$$

and hence, by taking the "inf" over all such functions f,

$$v_n(z_0) \leqq \int_C G(z_0, z) v_n(z)\, dz\,. \tag{2.2}$$

Since C is sufficiently smooth the Green's function G is bounded below (Ref. [9], p. 6), by the positive number

$$G_{\min} = \min_{z \in C} G(z_0, z)\,.$$

It follows by (2.2) that

$$\begin{aligned} v_n(z_0) &\leqq G_{\min} \int_C v_n(z)\, dz \\ &\leqq G_{\min} \int_A v_n(z)\, dz\,. \end{aligned} \tag{2.3}$$

Fatou's lemma [10] gives

$$\begin{aligned} \int_A \limsup_{n \to \infty} v_n(z)\, dz &\geqq \limsup_{n \to \infty} \int_A v_n(z)\, dz \\ &\geqq \limsup v_n(z_0)/G_{\min} \qquad \text{by (2.3)} \\ &= 0 \end{aligned}$$

and since $\limsup v_n(z)$ is non-positive the theorem is established.

Corollary to Lemma 1. *If* $v_1, v_2, \ldots$ *is a sequence of non-positive subharmonic functions on a region* D *in* $\mathbb{C}$, *and* $\limsup_{n \to \infty} v_n(z) < 0$ *for almost all* z *on a sufficiently smooth arc in* D, *then* $\limsup v_n(z) < 0$ *for all* z *in* D. *(In particular of* $v_n(z) = v(z) < 0$ *on some are then* $v(z) < 0$ *for all* z *in* D.*)*

3. The Transfer Matrix

In the present section we formulate the transfer matrix method for many-body interactions of arbitrary finite range, and show that in a suitable region of the complex plane the gap between the two largest eigenvalues of the transfer matrix is a subharmonic function of the complex fugacity z.

We consider a lattice gas on a d-dimensional periodic cuboidal lattice

$$\Lambda = \{x : x = (x_1, \ldots x_d) \text{ and } x_i \in \{0, 1, \ldots, L_i - 1\}\}$$

where $L_1, \ldots L_d$ are any positive integers. The configuration of the system is specified by giving the set X of occupied sites.

For each configuration X, the energy of the periodic system is

$$H_\Lambda(X)=\sum_{Y\subset X}\phi_\Lambda(Y)\qquad(X\subset\Lambda)$$

Here the interaction potential ϕ_Λ is a function on the subsets of Λ, given by

$$\phi_\Lambda(\emptyset)=0$$
$$\phi_\Lambda(\{\boldsymbol{x}\})=\phi(\{\boldsymbol{x}\})$$
$$\phi_\Lambda(\{\boldsymbol{x},\boldsymbol{y}\})=\sum_{\boldsymbol{a}\in G(\Lambda)}\phi(\{\boldsymbol{x},\boldsymbol{y}+\boldsymbol{a}\})$$
$$\phi_\Lambda(\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\})=\sum_{\boldsymbol{a},\boldsymbol{b}\in G(\Lambda)}\phi(\{\boldsymbol{x},\boldsymbol{y}+\boldsymbol{a},\boldsymbol{z}+\boldsymbol{b}\}),\qquad\text{etc.}$$

where $G(\Lambda)=\{\boldsymbol{a}=(a_1,\dots,a_d)\colon a_i/L_i\in\mathbb{Z}\}$ and ϕ is a function on the subsets of $\mathbb{Z}^d$ which is independent of Λ. This "free" interaction potential ϕ has the properties

$$\phi(X)\in\{R,+\infty\}\qquad(X\subset\mathbb{Z}^d),$$
$$\phi(X+\boldsymbol{q})=\phi(X)\qquad(q\in\mathbb{Z}^d),$$
$$\phi(X)=0\quad\text{if}\quad d(X)>R,$$

where $X+\boldsymbol{q}$ is the set obtained by translating every number by X by an amount $\boldsymbol{q}$, $d(X)$ is the diameter of the set X, i.e. the largest distance between any pair of members of X calculated using the distance function

$$d(\boldsymbol{x},\boldsymbol{y})=\operatorname*{Max}_i|x_i-y_i|\qquad(x,y\in\mathbb{Z}^d),$$

and R is a positive integer which represents the range of the interactions.

The grand partition function Ξ_Λ is defined by

$$\Xi_\Lambda(z)=\sum_{X\subset\Lambda}\exp(-\beta H_\Lambda(X))$$

where the fugacity z is

$$z=\exp-\beta\phi(\{\boldsymbol{x}\})$$

(which is independent of $\boldsymbol{x}$ because of translational invariance) and $\beta>0$ is the inverse temperature. To simplify matters we shall not indicate the explicit dependence of Ξ_Λ on β or any parameters in ϕ.

The transfer matrix T is a square matrix whose rows and columns are labelled by subsets of a layer of Λ with thickness $R-1$, which we shall call W:

$$W=\{\boldsymbol{x}\colon\boldsymbol{x}\in\Lambda\text{ and }x_1\leqq R-1\}.$$

For any pair I,J of subsets of W, the (I,J) element of T is defined to be 0 if $H_\Lambda(I)$ or $H_\Lambda(J)=+\infty$, and otherwise to be

$$\langle I|T|J\rangle=\exp-\beta\left[\tfrac{1}{2}H_\Lambda(I)+\tfrac{1}{2}H_\Lambda(J)+\sum_{\substack{X\subset I,\\X\neq\emptyset}}\ \sum_{\substack{Y\subset\tau J\\Y\neq\emptyset}}\phi_\Lambda(X\cup Y)\right]$$

where τJ means $J+(R,0,\ldots,0)$, the set obtained by translating the set J a distance R in the $(1,0,\ldots,0)$ direction. Provided L_1 is an integral multiple of R, say $L_1=mR$, with $m\geqq 3$, the grand partition function can be written [11]

$$\begin{aligned}\Xi_\Lambda(z) &= \operatorname{trace}(T^m)\\ &= \sum_{i=1}^{k} \lambda_i^{\,m}\end{aligned} \tag{3.1}$$

where $k=2^{|W|}$ and $\lambda_1,\ldots,\lambda_k$ are the characteristic values of T. We assume these to be arranged so that

$$|\lambda_1|\geqq|\lambda_2|\geqq|\lambda_3|\geqq\cdots.$$

There is a formula similar to (3.1) for values of L that are not integer multiples of R, but we shall not need it.

Since the elements of T depend on z, so do the characteristic values λ_i. They also depend on W (but not on L_1), so that a more complete notation would be $\lambda_i(z)$ or $\lambda_i(z,W)$.

In addition to giving the useful formula (3.1), the characteristic values of T also control the decay of the correlation functions: as noted by Onsager [12] (see also [6]) the rate of exponential decay is given (under suitable conditions on z) by $\ln|\lambda_1/\lambda_2|$. (This quantity is also related to the mass gap in Euclidean field theory: see Section 7 of this paper.) The Perron-Frobenius theorem [13] tells us that if all elements of the matrix T are positive then λ_1 is positive and the rate of decay $\ln|\lambda_1/\lambda_2|$ is also positive. The condition of positive matrix elements can be satisfied if $z>0$ and $\phi(X)<+\infty$ for all X. Even if $\phi(X)=+\infty$ for some X, so that some matrix elements of T are 0, we can still show that λ_1 and $\ln|\lambda_1/\lambda_2|$ are strictly positive, by ordering the subsets I so that those with $H_\Lambda(I)<\infty$ come first. Then T can be written in partitioned form

$$T=\left[\begin{array}{c|c} T_0 & 0\\ \hline 0 & 0\end{array}\right] \tag{3.2}$$

where the characteristic values of T and T_0 are the same apart from some zeros. Moreover, the two largest characteristic values of T_0^2 are $\lambda_1{}^2$ and $\lambda_2{}^2$, and since

$$\begin{aligned}\langle I|T_0{}^2|J\rangle &= \sum_K \langle I|T_0|K\rangle\,\langle K|T_0|J\rangle\\ &\geqq \langle I|T_0|\varnothing\rangle\,\langle\varnothing|T_0|J\rangle\\ &= \exp[-\tfrac12\beta H_\Lambda(I)-\tfrac12\beta H_\Lambda(J)]>0;\end{aligned}$$

we see from the Perron-Frobenius theorem applied to T_0^2 that $\lambda_1^2 > |\lambda_2|^2 \geqq 0$.

By the same theorem applied to T we know that $\lambda_1 \geqq 0$ and hence we can conclude, as required, that $\lambda_1 > 0$ and $\ln|\lambda_1/\lambda_2| > 0$ for all $z > 0$.

Unfortunately, when we consider the thermodynamic limit, we find that these considerations only prove exponential decay for the "one-dimensional" limit, in which L_1 becomes large at fixed cross-section W. It is still possible that $\ln|\lambda_1/\lambda_2|$ might go to zero when one or more of the $L_2, \ldots, L_d$ also become large. In this and the next section we develop a technique for dealing with this question. The first step is to show that $\ln|\lambda_2/\lambda_1|$ is a subharmonic function of z in a suitable Region D of the complex plane. Since $\ln|\lambda_2/\lambda_1|$ is negative for $z > 0$ and non-positive for all z, this subharmonicity would be enough to prove that $\ln|\lambda_2/\lambda_1|$ is negative for all z in D at any fixed W, but not necessarily in the limit of large W. We therefore prove further that, as W becomes large, $\ln|\lambda_2/\lambda_1|$ is bounded away from zero on an arc in D; then Lemma 1 will give the required result that $\ln|\lambda_2/\lambda_1|$ is bounded away from zero throughout the region and, in particular, at all points in the region for which $z > 0$.

Lemma 2. *Let G be a region (connected open set) in the complex z-plane which is simply connected, intersects the positive real axis, and is free of zeros of $\Xi_\Lambda(z)$, the grand partition function for periodic boundary conditions, for all sufficiently large L_1 (at fixed W). Then*

(i) $\ln|\lambda_1(z)|$ *is harmonic in D.*

(ii) $\ln|\lambda_2(z)|$ *is subharmonic in D.*

(iii) $\ln|\lambda_2(z)/\lambda_1(z)|$ *is subharmonic in D.*

Proof. By the Yang-Lee theorem [14], we know that the limit

$$\zeta(z) = \lim_{L_1 \to \infty} \frac{\ln \Xi_\Lambda(z)}{L_1} \tag{3.3}$$

exists for any $z \in D$, and defines an analytic function ζ in D. The limit $L_1 \to \infty$ is taken at constant W, and $\zeta(z)$ therefore depends on W as well as z, but we do not show this dependence in our notation. The branch of the logarithm is chosen to be real at some selected point on the real positive axis (and hence on any segment of the positive real axis that lies within D and includes this point). For large L, this branch is unique once the selected point has been chosen, since $\Xi_\Lambda(z)$ has no zeros in D. If D intersects the positive real axis in two distinct segments, we cannot be sure that $\zeta(z)$ is real on the segment that does not contain the chosen point, but this does not affect our result.

To show that $\ln|\lambda_1(z)|$ is harmonic in D, we show that it is proportional to the real part of the analytic function ζ. Indeed, the substitution of

(3.1) into (3.3) gives

$$\begin{aligned}\mathrm{Re}\,\zeta(z) &= \lim_{m\to\infty} \frac{1}{mR} \ln \left| \sum_{i=1}^{k} \lambda_i(z)^m \right| \\ &= \frac{1}{R} \ln|\lambda_1(z)| + \frac{1}{mR} \lim \ln \left| \sum_{i=1}^{k} \left(\frac{\lambda_i(z)}{\lambda_1(z)} \right)^m \right| .\end{aligned}$$

The first term of the summation in the last line is 1 and the ones after it are arranged in order of decreasing magnitude. For large m we need only consider those for which $|\lambda_i/\lambda_1| = 1$, say the first k' terms ($k' \geqq 1$); (as we shall show their sum does not vanish). The sum of these terms is [15] an almost periodic function of m and therefore includes among its limit points as $m \to \infty$ its value for $m = 0$, which is k'. The sequence $m^{-1} \ln|\Sigma(\lambda_i/\lambda_1)^m|$ therefore includes among its limit points the number $\lim(m^{-1} \ln k') = 0$. But since $z \in D$ we know that this sequence converges, and hence its limit is 0 and our equation reduces to

$$\mathrm{Re}\,\zeta(z) = R^{-1} \ln|\lambda_1(z)| \qquad (z \in D).$$

This completes the proof of Part (i), that $\ln|\lambda_1(z)|$ is harmonic in D.

To prove Part (ii), let z_0 be any point in D, let s be the number of characteristic values with magnitudes at least as great as $\lambda_2(z_0)$, and choose M so that

$$|\lambda_2(z_0)| = \cdots = |\lambda_s(z_0)| > M > |\lambda_{s+1}(z_0)| .$$

Let C denote a contour in the complex λ-plane as shown in Fig. 1.

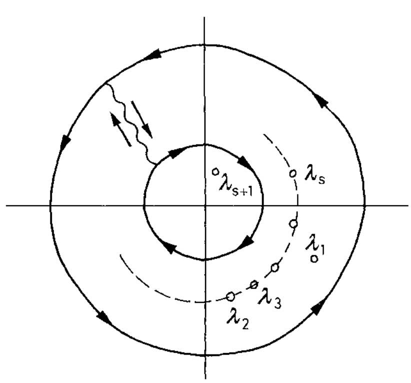

Fig. 1. Contour used in the proof of Lemma 2. In this figure λ_i means $\lambda_i(z_0)$

with the radius of the larger circle chosen greater than $|\lambda_1(z_0)|$, the radius of the smaller circle chosen equal to M, and the two-way connecting path chosen to avoid all the characteristic values $\lambda_i(z_0)$, and consider the expression

$$f(z)=\exp\left\{\frac{1}{2\pi i}\int \ln\lambda \frac{p'(\lambda;z)}{p(\lambda;z)}d\lambda\right\}$$

where

$$p(\lambda;z)=\det(T(z)-\lambda I)$$

is the characteristic polynomial of $T(z)$, $p'(\lambda;z)$ is its derivative with respect to λ, and the branch of $\ln\lambda$ may be chosen arbitrarily. Evaluating the integral by residues, we see that $f(z)$ is equal to the product of the characteristic values of $T(z)$ lying between the two circles, and in particular

$$f(z_0)=\lambda_1(z_0)\dots\lambda_s(z_0).$$

Since the elements of $T(z)$ depend continuously on z, so do its characteristic values; hence the formula

$$f(z)=\lambda_1(z)\dots\lambda_s(z)$$

holds in some neighborhood $N=\{z:|z-z_0|<\delta\}$ of z_0 within which $|\lambda_s(z)|$ stays larger than M, $|\lambda_1(z)|$ stays smaller than the outer radius of C, and no characteristic value lies on the two-way connecting path.

Looking back at the definition of f, we see that the integrand is analytic as a function of z throughout N and therefore $f(z)$ is also analytic in N. It follows that $\ln|f(z)|$ is subharmonic in N, and hence that the function

$$\ln|\lambda_2(z)\dots\lambda_s(z)|=\ln|f(z)|-\ln|\lambda_1(z)|$$

is subharmonic in N since $\ln|\lambda_1(z)|$ is harmonic by (i). Consequently we have, for sufficiently small positive δ,

$$\begin{aligned}\ln|\lambda_2(z_0)| &= \frac{1}{s-1}\ln|\lambda_2(z_0)\dots\lambda_s(z_0)| \\ &\leqq \frac{1}{s-1}\frac{1}{2\pi}\int_0^{2\pi}\ln|\lambda_2(z_0+\delta e^{i\theta})\dots\lambda_s(z_0+\delta e^{i\theta})|\,d\theta \qquad (3.4) \\ &\leqq \frac{1}{2\pi}\int_0^{2\pi}\ln|\lambda_2(z_0+\delta e^{i\theta})|\,d\theta\end{aligned}$$

where the second line comes from the fact that $\ln|\lambda_2\dots\lambda_s|$ is subharmonic in N, and the last from our convention $|\lambda_2|\geqq|\lambda_3|\geqq\cdots$.

The inequality (3.4), being true for any z_0 in D, is sufficient to prove (Ref. [9], p. 7) that $\ln|\lambda_2(z)|$ is subharmonic in D, so that Part (ii) is proved. Part (iii) now follows immediately from parts (i) and (ii), and so the proof of Lemma 2 is complete.

4. The Eigenvalue Gap for Large Cross-Sections

In this section we shall combine the lemma just proved with a further lemma to prove our main result about the transfer matrix, that under suitable conditions the gap in its spectrum stays away from zero as the cross-section becomes very large.

Theorem 1. *Let E be a region (connected open set) in the z-plane which contains the origin $z=0$ and is free of zeros of $\Xi_\Lambda(z)$ when all of $L_1, L_2, \ldots, L_d$ are sufficiently large. Then, for any fixed z_0 in E, the "gap" $\ln|\lambda_1(z_0)/\lambda_2(z_0)|$ in the spectrum of the transfer matrix remains positive (rather than approaching zero) in the limit where any or all of $L_2, \ldots, L_d$ become large.*

Proof. For all sufficiently large $L_1, \ldots, L_d$, the region E is free of zeros of $\Xi_\Lambda(z)$ and we can therefore find a simply-connected region D which satisfies the following conditions

(i) it is a subset of E,

(ii) it contains z_0,

(iii) it contains a segment of the positive real axis.

The Region D then satisfies the conditions of Lemma 2, and so $\ln|\lambda_2/\lambda_1|$ is subharmonic in D.

Let $v_1, v_2, \ldots$ be the sequence of functions on D defined by

$$v_i(z) = \ln|\lambda_2(z, W_i)/\lambda_1(z, W_i)|$$

where $W_1, W_2, \ldots$ is any increasing sequence of cuboidal subsets of $\mathbb{Z}^d$, all having the same thickness R in the $(1, 0, \ldots, 0)$ direction, and with at least one of their widths $L_2, \ldots, L_d$ in the other directions increasing without limit as $i \to \infty$. The functions $v_1, v_2, \ldots$ are all negative and, by Lemma 2 subharmonic on D. Hence, by the corollary to Lemma 1, we shall have proved our theorem if we can prove the next lemma, which shows that $\ln|\lambda_2/\lambda_1|$ has a negative upper bound, independent of $L_2, \ldots, L_d$, on an arc of finite length in D.

Lemma 3. *There exists a positive number a, not depending on W, such that*

$$|z| < a \quad \text{implies} \quad |\lambda_2(z, W)/\lambda_1(z, W)| < |z/a|\,.$$

Proof. The proof falls into two parts. First, we show that if f and g are any two functions, one depending only on the configuration in W and the other only on the configuration in the nth slice in succession after W, then their covariance (computed in the limit $L_1 \to \infty$) must decrease (with increasing n) at least as fast as $|z/a|^n$ when $|z| < a$. Second, we exhibit a pair of such functions f, g whose covariance (in the limit $L_1 \to \infty$) is exactly $|\lambda_2/\lambda_1|^n$. It then follows that $|\lambda_2/\lambda_1| \leq |z/a|$ as required.

For the first part of the proof, let f and g be any real-valued functions on the subsets W, and define their covariance at distance n as

$$\chi(z, n; m, W) = \langle f\tau^n g\rangle - \langle f\rangle \langle g\rangle = \sum_{X\subset\Lambda} \prod_{\Lambda}(X) f(X\cap W)\, g(\tau^{-n}X\cap W) - \sum_{X\subset\Lambda} \prod_{\Lambda}(X) f(X\cap W) \sum_{X\subset\Lambda} \prod_{\Lambda}(X)\, g(\tau^{-n}X\cap W) \tag{4.1}$$

where

$$\prod_{\Lambda}(X) = \Xi_{\Lambda}^{-1} \exp[-\beta H_{\Lambda}(X)]$$

is the equilibrium measure on the sets $X\subset\Lambda$, and $\tau^{\pm n}X$ is the set obtained by translating every element of X by a distance nR in the $(1, 0, \ldots, 0)$ direction, so that $\Lambda = W\cup\tau W\cup\tau^2 W\cup\cdots\cup\tau^{m-1}W$.

The covariance $\chi(z, n; m, W)$ can (see [4]) be expressed in terms of the distribution functions and thus as a finite sum of products of Ursell functions $u(X, z; \Lambda)$ (called cluster functions in [2]) such that (for $m > 2n$) at least one of the Ursell functions in each product refers to a set X with diameter $\geqq (n-1)R$. It was shown by Gallavotti and Miracle-Sole [3] that such Ursell functions have a bound independent of Λ (for sufficiently large Λ) of the form

$$|u(X, z; \Lambda)| \leqq C(X, z')\, |z/a|^n \quad \text{for} \quad |z| \leqq z'$$

where z' is any positive number less than a, and a is a known lower bound on the radius of convergence of the Mayer fugacity expansions. Hence, by expressing the averages in (4.1) in terms of Ursell functions, we can prove that

$$\lim_{m\to\infty} |\chi(z, n; m, W)| \leqq A \left|\frac{z}{a}\right|^n \quad \text{for} \quad |z| \leqq z' \tag{4.2}$$

where A depends on f, g, and z' but is independent of n. When f and g depend on W then A will also depend on W.

For the second part of the proof, we choose f and g so that $\chi(z, n; m, W)$ is equal, in the limit $m\to\infty$, to $|\lambda_2/\lambda_1|^n$; then it will follow, by considering the ratio of the two sides of (4.2) for very large n, that $|\lambda_2/\lambda_1| \leqq |z/a|$. Let P be a non-singular square matrix such that $Q = P^{-1}TP$ is in Jordan normal form with λ_1 in the top left-hand corner and λ_2 next to it on the diagonal. Denoting the columns of P by $|1\rangle, |2\rangle, \ldots$ we have, for z real, since $\lambda_2 \neq \lambda_1$,

$$T|1\rangle = \lambda_1|1\rangle \quad \text{and} \quad T|2\rangle = \lambda_2|2\rangle\,. \tag{4.3}$$

For each $I\subset W$ we define

$$g(I) = \frac{\langle I|2\rangle}{\langle I|1\rangle} \quad \text{if } H_{\Lambda}(I) < \infty \text{ and } = 0 \text{ if not}\,,$$

where $\langle I|$ denotes the row matrix with a 1 in the position labelled by I and zeros everywhere else; thus $\langle I|1\rangle$ is the entry labelled by I in the column matrix $|1\rangle$. We also define

$$f(I)=\frac{\langle 2|I\rangle}{\langle 1|I\rangle} \quad \text{if } H_\Lambda(I)<\infty \text{ and } =0 \text{ if not},$$

where $|I\rangle$ is the transpose of $\langle I|$, and $\langle 1|, \langle 2|, \ldots$ are the rows of P^{-1}, (which are proportional to the transposes of $|1\rangle$, $|2\rangle, \ldots$ if and only if T is symmetric). The Perron-Frobenius theorem guarantees that the denominators in $f(I)$ and $g(I)$ are not zero.

The usual transfer matrix methods [6, 11, 12, 16] give

$$\chi(z,n;m,W)=\frac{\operatorname{tr}(FT^nGT^{m-n})}{\sum_r \lambda_r^m}-\frac{\operatorname{tr}(FT^m)\operatorname{tr}(GT^m)}{\left(\sum_r \lambda_r^m\right)^2}$$

where F and G are diagonal matrices with entries

$$\left.\begin{array}{l}\langle I|F|I\rangle=f(I)\\ \langle I|G|I\rangle=g(I)\end{array}\right\}(I\subset W).$$

These matrices have the properties

$$\begin{aligned}\langle 1|F&=\langle 2|\\ G|1\rangle&=|2\rangle\end{aligned} \tag{4.4}$$

since, for example, $G|1\rangle=\sum_{I\subset W}|I\rangle\, g(I)\,\langle I|1\rangle=\sum_I |I\rangle\langle I|2\rangle=|2\rangle$.

We can evaluate $\chi(z,n;m,W)$ in the limit $m\to\infty$ by using the formula

$$\lim_{m\to\infty}\operatorname{tr}(AT^{m-n})/\operatorname{tr}T^m=\langle 1|A|1\rangle\,\lambda_1^{-n}$$

which holds (if $z>0$) for any $2^{|W|}\times 2^{|W|}$ matrix A and is proved by noting that $\operatorname{tr}(AT^{m-n})$ is the sum of the diagonal elements of $P^{-1}AT^{m-n}P$; this method gives

$$\begin{aligned}\lim_{m\to\infty}\chi(z,n;m,W)&=\langle 1|FT^nG|1\rangle\,\lambda_1^{-n}-\langle 1|F|1\rangle\langle 1|G|1\rangle\\ &=(\lambda_2/\lambda_1)^n \qquad (z>0)\end{aligned} \tag{4.5}$$

by (4.4) and (4.3). Combining (4.5) with (4.2) and considering large values of n as indicated above, we complete the proof that $|\lambda_2/\lambda_1|\leqq z/a$ for all positive $z<a$.

5. Proof of Exponential Decay

To see what is still necessary for a proof of exponential decay, consider the two-point Ursell function $u(\boldsymbol{x},\boldsymbol{y};\Lambda)$ for some fixed z in D. Our result about the gap in the spectrum implies (see below for a proof) that

$$\lim_{m\to\infty}|u(\boldsymbol{x},\boldsymbol{y};\Lambda)|\leqq Ae^{-\kappa|x_1-y_1|} \tag{5.1}$$

with κ independent of W (i.e. of $L_2,\ldots,L_d$). To deduce that the infinite-volume Ursell functions decay exponentially as x_1-y_1 becomes large, we want to take a limit $W \to \infty$ (i.e. $L_2, \ldots, L_d \to \infty$) on both sides of (5.1) and we therefore want to ensure that A remains bounded in this limit. This will be done in Theorem 2. Unfortunately, our proof only works if the transfer matrix is symmetric, and to ensure this we restrict the range of interaction by taking $R=1$. For example, in two dimensions this restricts us to nearest and next-nearest neighbor interactions (and hence to no more than four-body interactions).

We also want to prove exponential decay when $|x_1 - y_1|$ stays finite but some other component of the separation, say $|x_2 - y_2|$, becomes large. Provided the Ursell functions are independent of the order in which the sides $L_1, L_2, \ldots$ of our periodicity box tend to infinity, we can prove exponential decay in this case by reversing the roles of the L_1 and L_2 directions throughout, using a transfer matrix that transfers in the L_2 direction. On the other hand, if the Ursell functions do depend on the way the limit is taken, our theorem only proves exponential decay in a direction parallel to the direction of transfer. We know, since the Ursell functions are bounded on the real z-axis (cf. [17]), that for any increasing sequence of boxes with $L_1, \ldots, L_d \to \infty$ there will be a subsequence on which the Ursell functions approach a limit, but this limit need not be the same for a sequence that starts with $L_1 \to \infty$ as for one that starts with $L_2 \to \infty$.

Theorem 2. *Let $\Xi_\Lambda(z)$ and E be as in Theorem 1, but with the range of interaction R restricted to be* 1: *and let z_0 be any positive element of E. Let $W_1, W_2, \ldots$ be an increasing sequence of "cross-sections" [i.e. cuboidal subsets of $\mathbb{Z}^d$ having the form $\{(1, x_2, \ldots, x_d): 0 \leqq x_i \leqq L_i - 1\}$] and select a subsequence on which the limit*

$$u(X) = \lim_{W \to \infty} \lim_{L_1 \to \infty} u(X, z_0; \Lambda)$$

exists. Then $u(X)$ decays exponentially in the $(1, 0, \ldots, 0)$ *direction; that is*

$$\limsup_{d_1(X) \to \infty} \frac{\ln |u(X)|}{d_1(X)} < 0 \quad \text{if} \quad |X| \geqq 2 \tag{5.2}$$

where $d_1(X)$ is the diameter of the projection of X on the $(1, 0, \ldots, 0)$ *axis.*

Corollary. *If $u(X) = \lim\limits_{L_1, \ldots, L_d \to \infty} u(X, z_0; \Lambda)$ exists, then Eq.* (5.2) *holds with $d(X)$ replacing $d_1(X)$.*

Proof. We can [4] write $u(X, z_0; \Lambda)$ as a finite sum of terms of the form

$$\pm \varrho(X'; \Lambda) [\varrho(Y_1 \cup Y_2; \Lambda) - \varrho(Y_1; \Lambda) \varrho(Y_2; \Lambda)]$$

where (X', Y_1, Y_2) is a partition of X such that the largest $(1, 0, \ldots, 0)$ coordinate in Y_1 differs from the smallest in Y_2 by at least $d(X)/(s-1)$, and $\varrho(\ldots)$ denotes the usual distribution functions (often called correlation functions). Since $|\varrho(X'; \Lambda)| \leqq 1$ when $z_0 > 0$, it is sufficient to show that the quantity in square brackets is bounded above by $|\lambda_2/\lambda_1|^n$ where $n \geqq d_1(X)/(s-1)$. The proof of this fact is given in the following lemma.

Lemma 4. *If Y_1, Y_2 are two finite subsets of $\mathbb{Z}^d$, such that every element of Y_1 has a smaller $(1,0,\ldots,0)$-coordinate than every element of Y_2, then*

$$|\varrho(Y_1 \cup \tau^n Y_2; \Lambda) - \varrho(Y_1; \Lambda)\,\varrho(\tau^n Y_2; \Lambda)| \leqq (\lambda_2/\lambda_1)^n .$$

Proof. Let Y_1 occupy the first k_1 slices of Λ and Y_2 the next k_2, i.e.

$$Y_1 \subset W \cap \tau W \cap \cdots \cap \tau^{k_1-1} W$$

$$Y_2 \subset \tau^{k_1} W \cap \tau^{k_1+1} W \cap \cdots \cap \tau^{k_1+k_2-1} W .$$

Then we have

$$\begin{aligned}\varrho(Y_1;\Lambda) &= \sum_{\substack{X\subset\Lambda\\ X\supset Y_1}} e^{-\beta H_\Lambda(X)} \Big/ \sum_{X\subset\Lambda} e^{-\beta H_\Lambda(X)} \\ &= \mathrm{tr}(F T^{m-k_1})/\mathrm{tr}\, T^m \to \langle 1|F|1\rangle\, \lambda_1^{-k_1} \quad \text{as} \quad m\to\infty\end{aligned}$$

where $F = F_0 T F_1 T \ldots F_{k_1-2} T F_{k_1-1} T$ and F_i is a diagonal matrix with

$$\langle I|F_i|I\rangle = \begin{cases} 1 & \text{if} \quad \tau^i I \supset (Y_1 \cap \tau^i W) \\ 0 & \text{if not} . \end{cases}$$

Similarly

$$\varrho(T^n Y_2;\Lambda) = \mathrm{tr}(T^{k_1+n} G T^{m-k_1-k_2-n})/\mathrm{tr}\, T^m \to \langle 1|G|1\rangle\, \lambda_1^{-k_2}$$

where

$$G = G_{k_1} T G_{k_1+1} \ldots T G_{k_1+k_2-1} T$$

and G_i is a diagonal matrix with

$$\langle I|G_i|I\rangle = \begin{cases} 1 & \text{if} \quad \tau^i I \supset Y_2 \\ 0 & \text{if not} . \end{cases}$$

Finally we have

$$\begin{aligned}\lim_{m\to\infty} \varrho(Y_1 \cup \tau^n Y_2;\Lambda) &= \lim_{m\to\infty} \mathrm{tr}(F T^n G T^{m-k_1-k_2-n})/\mathrm{tr}\, T^m \\ &= \langle 1|F T^n G|1\rangle / \lambda_1^{k_1+k_2+n} \\ &= \sum_r \langle 1|F|r\rangle\, \lambda_r^n \langle r|G|1\rangle / \lambda_1^{k_1+k_2+n}\end{aligned}$$

since T, being symmetric, has a complete set of eigenvectors. Hence

$$\begin{aligned}&\Big|\lim_{m\to\infty} [\varrho(Y_1\cup\tau^n Y_2;\Lambda) - \varrho(Y_1;\Lambda)\,\varrho(\tau^n Y_2;\Lambda)]\Big| \\ &\quad = \Big|\sum_{r\neq 1} \langle 1|F|r\rangle \langle r|G|1\rangle\, \lambda_r^n/\lambda_1^{k_1+k_2+n}\Big| \qquad (5.3)\\ &\quad \leqq \Big\{\sum_{r\neq 1} [\langle 1|F|r\rangle]^2 \sum_{r\neq 1} [\langle r|G|1\rangle]^2\Big\}^{1/2} |\lambda_2{}^n/\lambda_1{}^{k_1+k_2+n}|\end{aligned}$$

by the Schwarz inequality.

To estimate the right-hand side of (5.3) we introduce, for each $k\times k$ real matrix A, the usual norm $\|A\| = \max_{p,q} |\langle p|A|q\rangle|$ where $\langle p|$ and $|q\rangle$

are row and column vectors of unit Euclidean length. This norm has the property $\|AB\| \leqq \|A\| \|B\|$. Since T is symmetric, the matrix P which diagonalizes T may be chosen to be real orthogonal, and then each left eigenvector $\langle r|$ is the transpose of the corresponding right eigenvector $|r\rangle$. Hence we obtain

$$\begin{aligned}\sum_{r \neq 1} |\langle 1|F|r\rangle|^2 &\leqq \sum_r \langle 1|F|r\rangle \langle r|F|1\rangle \\ &= \langle 1|F^2|1\rangle \leqq \|F\|^2 \\ &\leqq \|F_0\| \|T\| \|F_1\| \|T\| \dots \|F_{k_1-1}\| \|T\| \\ &\leqq \lambda_1^{k_1}\end{aligned}$$

since F is symmetric and $\|T\| = \lambda_1$, $\|F_i\| = 1$. Using this and the analogous estimate for $\sum_{r \neq 1} |\langle r|G|1\rangle|$ in (5.2) we obtain

$$\begin{aligned}\lim_{m\to\infty} [\varrho(Y_1 \cup \tau^n Y_2; \Lambda) - \varrho(Y_1; \Lambda)\,\varrho(\tau^n Y_2; \Lambda)] &\leqq \lambda_1^{k_1} \lambda_1^{k_2} |\lambda_2^n / \lambda_1^{k_1+k_2-n}| \\ &= |\lambda_2/\lambda_1|^n .\end{aligned}$$

This completes the proof of the lemma.

A more concise version of the argument based on (5.3) is this:

$$\begin{aligned}&\left|\lim_{m\to\infty} [\varrho(Y_1 \cup \tau^n Y_2; \Lambda) - \varrho(Y_1; \Lambda)\,\varrho(\tau^n Y_2; \Lambda)\right| \\ &\qquad = |\langle 1| F(T - |1\rangle \lambda_1 \langle 1|)^n G|1\rangle / \lambda_1^{k_1+k_2+n}| \\ &\qquad \leqq \|F\| \, \|T - |1\rangle \lambda_1 \langle 1|\, \|^n \, \|G\| / \lambda_1^{k_1+k_2+n} \\ &\qquad \leqq |\lambda_2/\lambda_1|^n\end{aligned}$$

since $\|T - |1\rangle \lambda_1 \langle 1|\, \| = |\lambda_2|$. This shows why we need a symmetric T; for if $T - |1\rangle \lambda_1 \langle 1|$ were non-symmetric then its norm could greatly exceed its largest eigenvalue λ_2.

6. Potentials of Longer Range

In what follows we shall apply our lemmas about subharmonic functions directly to the infinite-volume Ursell functions (or more general covariances) to study their decay properties. The idea is contained in the following lemma:

Lemma 5. *Let S be any positive integer and $f(X; z)$ be a function of $X = \{x^{(1)}, \dots, x^{(s)}\}$ with $x^{(i)} \in Z^d$ or $\mathbb{R}^d$, and of $z \in D$ where D is a region in the complex plane, such that*

(i) *for any fixed X, the function $f(X, z)$ is analytic in z throughout D,*

(ii) $|f(X; z)| \leqq 1$.

Let $S(X)$ be some positive-valued function of X which goes to infinity as $d(X) \to \infty$ [e.g. $S(X)$ could be $d(X)$ itself, or the length of the shortest path connecting all points in X]. Finally, let A be any smooth arc in D.

Then

(a) if $\limsup_{S(X)\to\infty} \frac{\ln|f(X;z)|}{S(X)} < -\kappa$ for almost all $z \in A$

where κ *is a positive number, then*

$$\limsup_{S(X)\to\infty} \frac{\ln|f(X;z)|}{S(X)} < 0 \quad \textit{for all } z \in D\,,$$

(b) if $\limsup_{d(X)\to\infty} \frac{\ln|f(X;z_0)|}{S(X)} = 0$ *for some* $z_0 \in D$

then

$$\limsup_{S(X)\to\infty} \frac{\ln|f(X;z)|}{S(X)} = 0 \quad \textit{for almost all } z \in A\,.$$

Proof. Since $\ln|f(X;z)|$ is a non-positive subharmonic function of z for $z \in D$, the Statements (a) and (b) are direct consequence of Lemma 1 and its corollary.

To apply this lemma, we need some kind of upper bound on the infinite-volume Ursell function $u(X, z)$ or whatever other function we are interested in. For example suppose we know an upper bound $|u(X,z)| \leqq M(X)\,(z \in D)$ such that $\ln M(X)/S(X) \to 0$ as $d(X) \to \infty$. Then our lemma applied to the function $f(X;z) = u(X;z)/M$, tells us that if $u(X,z)$ decays exponentially as $\exp[-\kappa S(X)]$ on some arc in D then it does so throughout D. Conversely, if there is a z in D such that $u(X,z)$ decays more slowly than $\exp[-\kappa S(X)]$, however small we choose κ, then it decays slower than exponentially at almost all points on any arc A in D; in particular, it decays slowly at almost all points on the real axis contained in D.

Ising Spin System with Ferromagnetic Pair Interaction

Our first application of the lemma is to a lattice system in which $\phi(X) \leqq 0$ when X contains two sites and $\phi(X) = 0$ if X contains more than two sites. This system is isomorphic to an Ising spin system with ferromagnetic pair interactions $J(\mathbf{x},\mathbf{y}) = -\frac{1}{4}\phi(\mathbf{x},\mathbf{y})$ and external field $h = (\ln z)/2\beta - \frac{1}{4}\sum_{\mathbf{x}} \phi(\mathbf{x},\mathbf{0})$, i.e. $z = z_0 e^{2\beta h}$ where $z_0 = \exp[-\frac{1}{2}\beta \sum_{\mathbf{x}} \phi(\mathbf{x},\mathbf{0})]$. The lemma enables us to dispense with the condition $R = 1$, which we had to impose in Theorem 2 to make the transfer matrix symmetric.

It is known [21] that the infinite-volume Ursell functions exist for this system and are independent of boundary conditions for all positive β when $h \neq 0$ and for all positive $\beta < \beta_c$ when $h = 0$, where β_c is the reciprocal of the temperature at which spontaneous magnetization sets in. It is also known [1] that when the interaction has finite range the Ursell functions decay exponentially for all positive β when $h \neq 0$. For $h = 0$ Gallavotti and Miracle-Sole [3] proved exponential decay of the correlations at sufficiently small values of β, i.e. $\beta<\beta'$ where β' is much smaller than β_c. It is now possible by the use of our lemma to extend this result to larger values of β, those satisfying $\beta<\beta''$, where β'' is the smallest reciprocal temperature at which the point $h=0$ (i.e. $z=z_0$) is a limit point of zeros of Ξ_Λ. One would expect to find $\beta''=\beta_c$, though this has never been proven; at any rate the Gallavotti-Miracle result shows that $\beta''\geqq\beta'$. To accomplish this extension, we need a bound of the form $|u(X,z)|<M(|X|)$ for $z\in D$ where D is a region containing the point $z=z_0$. Such a bound can be obtained either from the work of Lebowitz and Penrose [1] or more elegantly [18] from a theorem of Lieb and Ruelle [7]. We shall not go into any detail here since a similar result, giving stronger decay properties of the Ursell functions, was recently obtained by Duneau, Iagolnitzer and Souillard [19] using a method which does not use subharmonicity arguments. Their method does, however, use the Lieb-Ruelle theorem for proving that the Ursell functions are bounded in a suitable domain. We refer the reader to their paper for further details.

As an application of Part (b) of our Lemma we shall prove the *absence* of exponential decay of the Ursell functions at small values of z for lattice systems for which the two-body part of the interaction potential decays with distance more slowly than any exponential – for example an interaction proportional to r^{-6}. (In our earlier note [5] we stated that the two-body interaction potential must have constant sign at large distances, but this restriction is unnecessary.) The result can also be proved for continuous systems (for pair interactions, at least) but we shall, for simplicity, only consider explicitly the lattice case and the two-point Ursell function here.

Theorem 3. *In a lattice system for which the two-body part of the interaction potential satisfies*

$$\limsup_{d(\mathbf{x},\mathbf{y})\to 0}\frac{\ln|\phi(\mathbf{x},\mathbf{y})|}{d(\mathbf{x},\mathbf{y})}=0$$

where $d(\mathbf{x},\mathbf{y})$ *means* $\max_i|x_i-y_i|$, *there exists a positive number a such that the infinite-volume two-point Ursell function* $u(\{\mathbf{x},\mathbf{y}\},z)$ *satisfies*

$$\limsup_{d(\mathbf{x},\mathbf{y})\to\infty}\frac{\ln|u(\{\mathbf{x},\mathbf{y}\},z)|}{d(\mathbf{x},\mathbf{y})}=0$$

for almost all fugacities z satisfying $0<z<a$.

Proof. In Lemma 5 we take $s=2$ and D any region which contains $z=0$ and whose closure is contained within the set $\{z:|z|\leqq a\}$ where a is any one of the known lower bounds on the radius of convergence of the Mayer fugacity expansion (for Ref. see [3]). Then we can take

$$f(\{\mathbf{x},\mathbf{y}\},z)=u(\{\mathbf{x},\mathbf{y}\},z)/z^2M$$

where M is the upper bound on $u(\{x, y\}, z)/z^2$, independent of x and y, given in [2] or [3]. By continuity at $z=0$, we have from the Mayer series formula

$$f(\{x, y\}, 0) = M^{-1}[\exp(-\beta\phi(x, y)) - 1] .$$

Applying Part (b) of Lemma 4, with $S(X) = d(X)$ and $z_0 = 0$, we complete the proof of Theorem 3.

Like its predecessor, Theorem 3 can be strengthened in the case of ferromagnetic pair interactions.

Theorem 4. *In an Ising ferromagnet for which*

$$\limsup_{d(x,y)\to\infty} \frac{\ln|J(x, y)|}{d(x, y)} = 0$$

the infinite-volume two-point Ursell function $u(\{x, y\}; z)$ *satisfies*

$$\limsup_{d(x,y)\to\infty} \frac{\ln u(\{x, y\}, z)}{d(x, y)} = 0$$

at all positive values of z*: that is, all real values of the magnetic field.*

Proof. The inequalities of Griffiths, Hurst, and Sherman [8] show that the two-point Ursell function at fixed x, y does not decrease as the magnitude of the (real) magnetic field decreases. We have shown in Theorem 3, however, that the rate of exponential decay in this Ursell function is zero for almost all sufficiently small z – i.e. almost all sufficiently large negative h. Therefore the GHS inequalities imply that it is zero at all negative values of h and at 0 too. By symmetry under the reversal of h, it is also zero at all positive values of h. This completes the proof of Theorem 4.

7. Discussion

1. The main result in Lemma 2 is the proof of Part (ii), that $\ln|\lambda_2|$ is subharmonic. This result is not specific to transfer matrices, nor to the second largest eigenvalue λ_2. In fact if $T(z)$ is any compact operator on a Banach space, which depends analytically on z when z lies in a Region D of the complex plane, and if $\lambda_1(z)$, $\lambda_2(z)$, ... are its eigenvalues arranged in decreasing order of magnitude, then for any fixed integer s the function $\ln|\lambda_1(z)\,\lambda_2(z)\ldots\lambda_s(z)|$ is subharmonic for z in D. The proof is similar to that of part (ii) of Lemma 2: if $\lambda_r(z_0)\ldots\lambda_t(z_0)$ are the eigenvalues with modulus equal to $\lambda_s(z_0)$ we show that $f(z) = \lambda_r(z)\,\lambda_{r+1}(z)\ldots\lambda_t(z)$ is an analytic function of z in some neighborhood z_0, hence that $ln|f(z_0)|$ is no greater than the average of $ln|f(z)|$ over a small circle centered at z_0, and the result follows. Parts (i) and (iii) of Lemma 2 also have natural analogues for compact operators.

The resulting generalization of Lemma 2 is practically the same as the one used by Guerra, Rosen, and Simon [21] in their proof that for suitable boundary conditions the $P(\varphi)_2$ Euclidean field with $P(\varphi) = a\varphi^4 + b\varphi^2 - h\varphi, a > 0$ and $h > 0$ has a non-zero mass gap. This field is analogous to a two-dimensional Ising ferromagnet with nearest neighbor interactions in a finite external magnetic field. There were however great additional technical difficulties in the field theory case since there is no simple proof, analogous to that given in section 4, for the existence of a gap at small fugacities (high magnetic fields) independent of the boundary conditions on the transfer matrix.

2. In our results involving the transfer matrix in Sections 3–5 we restricted ourselves to periodic boundary conditions. In general the transfer matrix method is applicable to a cubodial domain Λ with arbitrary boundary conditions b_1 on the faces of Λ perpendicular to the direction of transfer (the planes $x_1 = 0$ and $x_1 = L_1 - 1$ in our notation) and boundary conditions b_W, which are invariant under τ, on the other faces. Writing such boundary conditions as $b_\Lambda = (b_1, b_W)$ it seems essential for the results of Section 3 (subharmonicity of $\ln|\lambda_2/\lambda_1|$ in the Region E) to use periodic boundary conditions for b_1; for only then does Eq. (3.1), $\Xi_\Lambda = \sum_i \lambda_i^m$, hold. The eigenvalues λ_i will, of course, depend on W and b_W as well as on z. The theorems proven in Sections 3–5 then hold (with the appropriate change in the definition of the Region E) for arbitrary b_W. A further relaxation of the conditions is that in order to obtain exponential decay in the x_1-direction we need require in Section 5 that the range of interactions be 1 only in the x_1-direction. To obtain true exponential decay we need of course the restrictions and additional unproven assumptions discussed in Section 5. How to dispense with these remains an open question.

3. Our analysis in Section 6 does not assume any particular boundary conditions, i.e. it holds for all equilibrium states whose Ursell functions satisfy the appropriate analyticity and boundedness conditions. Hence it applies in particular to the nearest-neighbor planar Ising antiferromagnet studied by Brascamp and Kunz [22]. They showed that at low temperatures there is an annulus, centered at the origin of the z plane and containing the zero-magnetic-field point $z = z_0$, within which the infinite-volume Ursell distribution functions obtained using 'alternating boundary conditions' are bounded and unique. This is therefore a further case in which our Lemma 5 could be applied.

4. We have only considered subharmonicity in the fugacity plane here, but very similar results can be obtained for any other parameter in the function βH, for example the inverse temperature β. There are even some generalizations using spaces of many complex variables.

5. A further application of subharmonicity has been made by Guerra, Rosen and Simon [21]. They point out that a non-positive function subharmonic in a Region D can approach zero no faster than linearly as the boundary of D is approached. Consequently for an Ising ferromagnet, the gap $\ln|\lambda_2/\lambda_1|$ goes to zero no faster than linearly as the magnetic field goes to zero at constant temperature. As they point out, this result implies some bounds for critical exponents.

6. There appears to be some relation between the arguments used in this paper, especially those in Section 6, and some very general results of Zerner on the asymptotic behavior of functions depending on a complex parameter [23].

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