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analysis shows that, for large $|\underline{q}' - \underline{q}''|$, the integral is approximately constant, and that

$$\langle \underline{q}' \, | \sigma_1 | \, \underline{q}'' \rangle \simeq \text{const. for large} \qquad |\underline{q}' - \underline{q}''| \tag{8}$$

where the constant is independent of N and is positive. It follows, by § 4 of ref. 2 that B. E. condensation is present and that the wave function of the condensed particles is a constant.

To show that the two methods of defining $\underline{v}_s$ are consistent, we consider the effect of multiplying all wave functions by $\exp(im\underline{u}\cdot\sum_j \underline{q}_j/\hbar)$. This transformation increases the momentum of every particle by $m\underline{u}$ and hence changes the velocity of the ground state to $\underline{u}$, so that Landau's definition of $\underline{v}_s$ now gives $\underline{v}_s = \underline{u}$. The wave function of the condensed particles is transformed to $\Psi = \text{const.}\exp(im\underline{u}\cdot\underline{q}/\hbar)$. According to the ideas of London & Tisza, one should define $\underline{v}_s$ as $(\hbar/im)\,\nabla\,\mathcal{I}m\,\log\,\Psi$. Clearly this definition also gives $\underline{v}_s = \underline{u}$. Thus, under the special simplifying assumptions used here, the two definitions of $\underline{v}_s$ are equivalent.

I wish to thank Professor Lars Onsager for valuable help in planning this calculation.

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LOW TEMPERATURE FLUCTUATIONS*

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It is well known that the fluctuations in a thermodynamic system are intimately related to both the equilibrium and transport properties of such a system. Thus it was shown by Blatt, Butler and Schafroth (1) that the angular momentum of a fluid in a slowly rotating bucket is proportional to the fluctuations of the angular momentum in a stationary bucket. In turn they tried to relate these fluctuations to those of the linear momentum.

It is the purpose of this note to point out that these fluctuations may depend on the topological characteristics of the boundaries; i. e. on whether periodic or rigid box boundary conditions are used. In order to evaluate $\langle P^2 \rangle$ for an N-particle system in a periodicity box we make use of the density matrix

$$\sigma(T,V) \equiv [Z(T,V)]^{-1} e^{-\beta[H-VP]};\ H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{ij} V(r_{ij});\ P = \sum_{i=1}^{N} \vec{p}_i \,.$$

This matrix describes the Gibbs-ensemble representing our system when in contact with a temperature bath moving with velocity v. Since H commutes with P we have

$$\langle (P_x - \langle P_x \rangle)^2 \rangle = \beta^{-1} \frac{d\langle P_x \rangle}{dV} \ : \quad \langle P_x \rangle \equiv tr\left[P_x \sigma(T,V)\right] = \beta^{-1} \frac{d \ln Z}{dV} .$$

Let $\psi_{(F,n)}$ be the eigenfunctions of H which are also eigenfunctions of P and are periodic

$$H\psi_{(F,n)} = E_{(F,n)} \psi_{(F,n)} \ : \ P\psi_{(F,n)} = F\psi_{(F,n)} \ : \ \psi_{(F,n)}(\vec{r}_1, \ldots \vec{r} + L\vec{K}, r) = \psi_{(F,n)}$$

To each $\psi_{(F,n)}$ there is congruent a whole set of states

$$\psi_{(F+N\hbar K/Lm)} = e^{i(2\pi \vec{K}/L \cdot \Sigma \vec{r}_i)} \psi_{(F,n)} \ : \quad K = 0, \pm 1, \pm 2, \ \ldots\ldots$$

these states correspond to uniform motion with velocity (hK/mL). The difference in the energies $E(F + \frac{Nh}{L}K, n)$ and $E(F, n)$ goes as $O(N^{1/3})$ for large N and fixed K.

We now consider the case where the energy of the system is the sum of the energies of individual excitations of momentum $\vec{f}$,

$$E_{(F,n)} = \sum_f n(f)\,\mathcal{E}(f) \ : \ F = \sum_f n(f)\,\vec{f}$$

measured relative to the ground state energy. The partition function $Z(T,V)$ can now be written in the form $Z(T,V) = \sum Z_k(T,V)$

$$Z_K(T,V) = e^{-\frac{\beta N m}{2}(u_K^2 - 2u_K \cdot V)} \left\{ \sum_{n(f)} e^{-\beta \sum_f [n(f)][\mathcal{E}(f) - (V - u_K) \cdot f]} \right\}$$

$$= e^{-\frac{\beta N m}{2}(u_K^2 - 2u_K \cdot V)} \prod_f \left[1 - e^{-\beta[\mathcal{E}(f) - (\vec{V} - \vec{u}_K) \cdot \vec{f}]} \right]^{-1} : u_K = \left(\frac{hK}{mL}\right).$$

For large N, $Z(T,V) \sim Z_{K'}(T,V)$ where K' is an integer satisfying the inequality $|u_{K'} - V| < \frac{1}{2}(h/mL)$. Hence

$$\langle P(T,V) \rangle = Nm[\vec{u}_{K'} + (\rho_n/\rho)(\vec{V} - \vec{u}_{K'}) + O[(V - u_{K'})^2]]$$

and

$$\langle (P_x - \langle P_x \rangle)^2 \rangle = (\rho_n/\rho)\, N m k T + O(N)$$

where $\rho_n = (\beta/3L^3) \sum_f \left\{ f^2 e^{\beta \mathcal{E}(f)} / \left[e^{\beta \mathcal{E}(f)} - 1\right]^2 \right\}$

is the density of the normal fluid, as defined by Landau (2).

The derivation applies to an ideal Bose gas, where the normal mass consists of all the particles with nonzero momentum.

When the N-particle system is confined to a real box by appropriate interactions with the walls, the total momentum no longer commutes with the hamiltonian, it can assume a continuous range of values, and it has a canonical conjugate which locates the center of mass. While $\sigma(T,V)$ is now simply related to $\sigma(T,0)$,

$$\sigma(T,V) = \left\{ e^{\frac{im\vec{V}}{\hbar} \cdot \sum_{j=1}^{N} (\vec{r}_j \cdot \vec{r}_j')} \right\} \sigma_N(T,0)$$

and $\langle P_x(T,V) \rangle = NmV_x$ there is now no longer any simple relation between $\langle P \rangle$ and $\langle P^2 \rangle$. However both the total momentum and its canonical conjugate can be expressed in terms of the phonon amplitudes by way of the current and density operators, respectively:

$$L^3 \rho(\vec{r}) = Nm + 2 \sum_n \alpha_n \rho_n \cos k_n X + \ldots$$

$$L^3 J_x(\vec{r}) = 2 \sum_{n=0}^{\infty} \alpha_n J_n \sin k_n X + \ldots \qquad \dot{\rho}_n = -k_n J_n$$

If we assume rigid walls and constant density up to the wall we have simply

$$k_n = (\pi n/L),\ \alpha_n = 1,\ 0 \leqslant x, y, z \leqslant L.$$

$$P_x = \int J_x d^3 \vec{r} = -(4/\pi) \sum_{n=0}^{\infty} J_{(2n+1)} / (2n+1)$$

and obtain the classical result

$$d \langle P_x^2 \rangle / d\, T = Nmk + O(N)$$

but paradoxically the divergent sum $\frac{4\hbar c}{\pi} \rho_0 L^2 \sum \frac{1}{(2n+1)}$

for $\langle P^2(T = 0) \rangle$. This artifact can be traced to the assumption of constant density, which conflicts with the quantum-mechanical requirement $\psi = 0$ at a rigid wall. It might be interesting to seek a more realistic approximation but we found it easier to introduce elastic walls. The appropriate modifications of the above analysis proved quite tractable and we obtained

$$\langle P_x^2 \rangle = N m k T + \text{const.}\ N^{2/3} \ln N + O(N^{2/3})$$

a result which is probably correct for walls of any reasonable construction.

Landau emphasized the differentiation between gradient and vortex motion in a superfluid. We recognize that the circulation variabics which complete the description of the flow field in multiply connected containers belong essentially to the vortex field. In the Born-Karman box the total momentum is a circulation variable.

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** National Science Foundation Postdoctoral Fellow.

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FURTHER DEVELOPMENT OF AN ELEMENTARY THEORY OF LIQUID HELIUM*

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When exchange effects are important it is convenient to consider atoms in pairs, and it was suggested (1, 2, 3) that N atoms of helium be treated as N/2 pairs. Temperley used a partition function of the form

$$(\text{P. F.}) = [\, 1 + m' \exp(-\epsilon_0'/kT) + m'' \exp(-\epsilon_0''/kT)]^{N/2}$$

This gives, incorrectly, a strong maximum in the specific heat, C, at low temperatures. But various interactions will broaden the energy