ecial cases, the results for the perfect chain and
s. Moreover, for the case where $n_1 = n_2 = n_3$,
single-mass defect with a new $\gamma_T = \gamma_{n_1} + \gamma_{n_2}$
n_2 or $n_2 = n_3$ the double-defects results are

what follows we make the assumption that
$(p = n_1, n_2, \ldots, n_n)$ are so small that higher
may be neglected. Under this assumption and
licative factor γ_{n_i} in the right-hand side of (31)
n dictates that any dependence of the quotients
neglected. Thus, (43a, b) yield, for a given

i, (47a)

n. (47b)

ds the simple result

$+ S_{n,n+1} X^{-n_i}]$, (48)

e of the multiple-mass defects is equal to the
ive mass as if it were the only defective mass
fects).

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SOME REMARKS ON ISING-SPIN SYSTEMS*

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Synopsis

We show that: (a) the free energy and correlation functions of the two-dimensional Ising-spin system with nearest-neighbour ferromagnetic interactions, remain infinitely differentiable with respect to β and h as $h \to 0^{\pm}$ for $\beta > \beta_c$ (where β_c is the reciprocal of the critical temperature) and, (b) the equilibrium equations for the correlation functions of Ising-spin systems may admit a non-physical solution even in the region, $\beta < \beta_c$, where they are known to have a unique physical solution.

1. *Proof of (a)*. Consider an Ising-spin system with ferromagnetic pair interactions in a domain $\Lambda \subset Z^{\nu}$. We shall denote by '+' the boundary condition in which all spins in $Z^{\nu}\backslash\Lambda$ are +1. Let $u_2(x, y; \beta, h, \Lambda, +)$ be the pair correlation: $\langle\sigma_x\sigma_y\rangle - \langle\sigma_x\rangle\langle\sigma_y\rangle$ for this system, $x, y \in \Lambda$. The argument used in ref. 1 (employing the Griffiths, Hurst and Sherman inequality[2]), then shows that when the magnetic field h is in the up direction then

$$u_2(x, y; \beta, h, \Lambda, +) \le u_2(x, y; \beta, h, +) \le u_2(x, y; \beta, h = 0, +), \tag{1}$$

where $u_2(x, y; \beta, h, +) = \lim_{\Lambda\to\infty} u_2(x, y; \beta, h, \Lambda, +)$, the limit being approached monotonically.

We now observe that for the two-dimensional system, $\nu = 2$, with nearest-neighbour attractive interactions, it was shown in ref. 4 that in the infinite-volume limit $\langle\sigma_x\sigma_y\rangle_+(\beta) = \langle\sigma_x\sigma_y\rangle_p(\beta)$; here p indicates periodic (or cylindrical) boundary conditions, and the equality holds for all β even when $h = 0$. (For $h \neq 0$ or $\beta < \beta_c$, the result was already known before[3].) Furthermore in ref. 4 it is also shown that $\lim_{|x-y|\to\infty} \langle\sigma_x\sigma_y\rangle_p = \langle\sigma_x\rangle_+^2$. It follows then from the explicit compu-

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tation of Wu[5]) that the right side of (1) has an exponential decay[1]):

$$u_2(x, y; \beta, h = 0, +) \le \text{const.} \exp -K|x - y|$$

with $K > 0$ for $\beta > \beta_c$. This in turn implies infinite differentiability by the arg ments given in ref. 1. (We note here that Martin-Löf obtained the bound (2) direct computation and communicated it to us prior to our result.)

Actually in ref. 5 the author deals with the case when x and y are on the sa horizontal or vertical line; the general case follows from a careful examination the spectrum of the transfer matrix and it is particularly easy to obtain if one content with a weak estimation of the form

$$|\langle\sigma_x\sigma_y\rangle - \langle\sigma_x\rangle_+^2| \le \text{const.} \exp -\tfrac{1}{2}K|x - y|,$$

where K is the horizontal or vertical correlation length.

2. *Proof of (b).* To prove (b) we consider a one-dimensional system wi nearest-neighbour interaction, $\Lambda = [-L, L]$, and 'open' boundary conditio corresponding to no interactions with spins outside Λ. The hamiltonian of th system, for $h = 0$, then is $H_0(\sigma) = -\sum_{i=-L}^{L-1} \sigma_i\sigma_{i+1}$. Let $H_i(\sigma) = H_0(\sigma) - (i\pi/2$ $\times (\sigma_{-L} + \sigma_L)$. We shall denote with a subscript L, 0 or L, i the average obtain by using $e^{-\beta H_0}$ or $e^{-\beta H_i}$ as weights and by a subscript 0 or i we shall mean th limit as $L \to \infty$, of the corresponding quantities with subscript L, 0 or L, i.

If $x_1 < x_2 < \cdots < x_m$ a simple computation leads to the following result

$$\langle\sigma_{x_1}\cdots\sigma_{x_{2n+1}}\rangle_i = 0 = \langle\sigma_{x_1}\cdots\sigma_{x_{2n+1}}\rangle_0,$$

$$\langle\sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle_i = \prod_{j=1}^{2n-1}\langle\sigma_{x_j}\sigma_{x_{j+1}}\rangle_i,$$

$$\langle\sigma_{x_1}\cdots\sigma_{x_{2n}}\rangle_0 = \prod_{j=1}^{2n-1}\langle\sigma_{x_j}\sigma_{x_{j+1}}\rangle_0.$$

Furthermore it is easy to check that:

$$\langle\sigma_x\sigma_y\rangle_i \equiv \lim_{L\to\infty}\frac{\langle\sigma_x\sigma_y\sigma_{-L}\sigma_L\rangle_{0,L}}{\langle\sigma_{-L}\sigma_L\rangle_{0,L}} = \frac{1}{\langle\sigma_x\sigma_y\rangle_0}.$$

hence $\langle\sigma_x\sigma_y\rangle_i > 1$ and therefore $\langle\sigma_x\sigma_y\rangle_i$ cannot correspond to a physically accep able state. It is, however, easy to see from the definition

$$\langle\sigma_x\sigma_y\cdots\rangle_i = \lim_{L\to\infty}\frac{\sum_\sigma(\sigma_x\sigma_y\cdots)\exp -\beta H_i(\sigma)}{\sum_\sigma \exp -\beta H_i(\sigma)}$$

that the $\langle\sigma_X\rangle_i$ define a family of local distributions $f_\Lambda(X)$* which verify the equilibrium equations (6) as well as the compatibility and normalization conditions (7) that they would have to satisfy if they came from a probability measure on the space of the spin configurations.

Notice that, since $\langle\sigma_x\sigma_y\rangle_0(\beta) = (\text{th}\,\beta)^{|x-y|}$ the functions $\langle\sigma_x\sigma_y\rangle_i(\beta)$ are singular around $\beta = 0$ which explains why they cannot be obtained by the usual perturbative expansions around $\beta = 0$.

It could be directly checked that the Kirkwood–Salsburg equations in zero field are, in general, invariant under the transformation $J_{ij} \to J_{ij} + i\pi/2\beta$ (this is because only $\exp(-4\beta J_{ij})$ enters into the KS equations) and this remark, applied to our case, could be used to provide a simple direct proof that the correlation functions $\langle\sigma_X\rangle_i$ are a solution to the KS equations [one merely notices that $\text{th}(\beta + \frac{1}{2}i\pi) = 1/\text{th}\,\beta$].

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* If $X = (x_1, x_2, \ldots, x_p)$ the functions $f_\Lambda(X)$ are the „probabilities" for finding, inside Λ, spins up in the points $x_1 \ldots x_p$ and spins down in the remaning points; *i.e.*,

$$f_\Lambda(X) = \left\langle \prod_{\xi\in X}\left(\frac{\sigma+1}{2}\right) \prod_{\xi\in\Lambda/x}\left(\frac{1-\sigma_\xi}{2}\right)\right\rangle_i. \quad \text{Also } \langle\sigma_X\rangle = \left\langle \prod_{i=1}^{p} \sigma_{x_i}\right\rangle.$$