STATISTICAL MECHANICS OF CONTINUOUS SPIN SYSTEMS

by

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Abstract: We present results relating to the existence and uniqueness of the free energy and equilibrium states for classical continuous spin systems with superstable interactions.

We consider the lattice $\mathbb{Z}^\nu$ at each site of which there is a vector spin variable S_x, $x \in \mathbb{Z}^\nu$, $S_x \in \mathbb{R}^d$. We denote by $\underline{S} \in \{S\}$ a spin configuration on $\mathbb{Z}^\nu$. Each S_x has associated with it an intrinsic positive measure $\mu(dS)$, the same for all sites, such that $\int \mu(dS)\, e^{-\alpha S^2} < \infty$ for $\alpha > 0$. The energy of a given spin configuration S_Λ in $\Lambda \subset \mathbb{Z}^\nu$ consists of both pair and self interactions and satisfies the following conditions:

a) Superstability There exists $A > 0$, $C \in R$ such that

$$U(S_\Lambda) \geq \sum_{x \in \Lambda} [A\, S_x^2 - C] \tag{1}$$

where S_Λ is a configuration in Λ.

b) Regularity If Λ_1, Λ_2 are disjoint then their interaction energy $W(S_{\Lambda_1}|S_{\Lambda_2}) = U(S_{\Lambda_1} \cup S_{\Lambda_2}) - U(S_{\Lambda_1}) - U(S_{\Lambda_2})$ has the bound

$$|W(S_{\Lambda_1}|S_{\Lambda_2})| \leq \tfrac{1}{2} K \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} |S_x|\,|S_y|\,|x-y|^{-\nu-\epsilon} \tag{2}$$

where $|x| = \max_{1 \leq i \leq \nu} |x^i|$, $|S| = [\sum_{i=1}^{d} (S^i)^2]^{\frac{1}{2}}$.

For Λ bounded in $\mathbb{Z}^\nu$ we consider the restriction, S_{Λ^c} of $\underline{S}$ to Λ^c and define the partition function $Z(\Lambda|S_{\Lambda^c})$ and free energy per site $F(\Lambda|S_{\Lambda^c})$ with 'boundary conditions' (b.c.) determined by $\underline{S}$ as

$$Z(\Lambda|S_{\Lambda^c}) = \int \mu_\Lambda(dS_\Lambda) \exp[-U(S_\Lambda) - W(S_\Lambda|S_{\Lambda^c})]$$

$$F(\Lambda|S_{\Lambda^c}) = |\Lambda|^{-1}\, \ell n\, Z(\Lambda|S_{\Lambda^c})$$

where $\mu_\Lambda(dS_\Lambda) = \prod_{x \in \Lambda} \mu(dS_x)$, $|\Lambda|$ = # of sites in Λ. (The dependence on temperature and magnetic field is included in U and W; it will be made explicit when necessary.)

__Theorem 1.__ Let (1) and (2) hold and let $\underline{S} \in \mathcal{H}_a$:

$\mathcal{H}_a = \{\underline{S} | S_y^2 \leq a \ln |y| \text{ for } |y| > 1\}$. Let $\{\Lambda_n\}$ be a sequence of increasing domains tending to Z^ν in the sense of Van Hove [1] then $\lim_{n\to\infty} F(\Lambda_n | S_{\Lambda^c}) = F$ exists and is independent of the sequence $\{\Lambda_n\}$ and of the b.c. S_{Λ^c}.

Remark: "Zero" b.c. correspond to $S_x = 0$ for $x \in \Lambda^c$. The thermodynamic limit of the "periodic" b.c. free energy can also be shown to exist and be equal to F.

A probability measure ν on the configuration space $\{S\}$ is said to be regular if it satisfies the following condition: There exists $\gamma > 0$, $\delta \geq 0$, such that for every Δ bounded in Z^ν and $N^2 > 0$ the following holds:

$$\nu[B(N^2|\Delta)] \leq \exp[-|\Delta| (\gamma N^2 - \delta)]$$

where

$$B(N^2|\Delta) = \{\underline{S} \mid \sum_{x\in\Delta} S_x^2 \geq N^2 |\Delta|\}$$

For Λ bounded in Z^ν we denote by $F(\Lambda|\nu)$ the free energy in Λ for boundary conditions specified by the measure ν as

$$F(\Lambda|\nu) = \int \nu(d\underline{S})\, F(\Lambda|S_{\Lambda^c})$$

__Theorem 2.__ Let (1) and (2) hold and let Λ_n be as in theorem 1 then for ν regular $\lim_{n\to\infty} F(\Lambda_n|\nu) = F$.

The finite volume equilibrium measure with boundary conditions S_{Λ^c}, $\nu_\Lambda(dS_\Lambda | S_{\Lambda^c})$ is given by

$$\nu_\Lambda(dS_\Lambda|S_{\Lambda^c}) = Z^{-1}(\Lambda|S_{\Lambda^c})\, \mu(dS_\Lambda)\, \exp[-U(S_\Lambda) - W(S_\Lambda|S_{\Lambda^c})] \qquad (3)$$

a measure ν on $\{S\}$ is said to be an equilibrium measure (for our system) if its conditional probabilities $\nu(dS_\Lambda|S_{\Lambda^c})$ satisfy the Dobrushin, Lanford and Ruelle (DLR) [2] equations, i.e. eq. (3).

__Theorem 3.__ Let the conditions of Theorem 1 be satisfied and let $\nu_{\Lambda_n}(dS_{\Lambda_n}|S_{\Lambda^c_n})$ be finite volume equilibrium states then it is always possible to choose subsequences $\{n_i\}$ (which may depend on the b.c.) such that $\nu_{\Lambda_{n_i}}(dS_{\Lambda_{n_i}}|S_{\Lambda^c_{n_i}}) \to \nu$ a __regular__ equilibrium measure on $\{S\}$, [3].

The one component spin system, $S_x \in R$, will be called ferromagnetic with translation invariant interactions if

$$U(S_\Lambda) = -\tfrac{1}{2} \sum_{x \neq y \in\Lambda} J(x-y)\, S_x S_y - \sum_{x\in\Lambda} \Phi(S_x) - h \sum_{x\in\Lambda} S_x;\ J(x) \geq 0.$$

__Theorem 4.__ Let ν be a regular equilibrium measure of a ferromagnetic system in an external field h whose interactions satisfy (1) and (2) then ν is unique (and hence translation invariant) whenever the infinite volume free energy F(h) is differentiable with respect to h [4].

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References

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