Stationary States of the One-dimensional Facilitated Asymmetric Exclusion Process

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Abstract

We describe the translation invariant stationary states (TIS) of the one-dimensional facilitated asymmetric exclusion process in continuous time, in which a particle at site $i \in \mathbb{Z}$ jumps to site $i+1$ (respectively $i-1$) with rate p (resp. $1-p$), provided that site $i-1$ (resp. $i+1$) is occupied and site $i+1$ (resp. $i-1$) is empty. All TIS states with density $\rho \leq 1/2$ are supported on trapped configurations in which no two adjacent sites are occupied; we prove that if in this case the initial state is Bernoulli then the final state is independent of p. This independence also holds for the system on a finite ring. For $\rho > 1/2$ there is only one TIS. It is the infinite volume limit of the probability distribution that gives uniform weight to all configurations in which no two holes are adjacent, and is isomorphic to the Gibbs measure for hard core particles with nearest neighbor exclusion.

1 Introduction

The *facilitated exclusion process* is a model of particles moving on a lattice, which we take to be $\mathbb{Z}^d$. Our primary interest is in the one-dimensional

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version in which particles hop only to nearest neighbor sites, but for completeness we first describe the general model. A configuration of the model is an arrangement of particles on $\mathbb{Z}^d$, with each site either empty or occupied by a single particle. If site i is occupied, *and one of its neighboring sites is also*, then the particle at site i attempts, at rate 1, to jump to another site j, succeeding only if the target is unoccupied. The target site j is chosen with probability $\pi(j-i)$, where $\pi : \mathbb{Z}^d \to \mathbb{R}_+$ is some probability distribution: $\pi \geq 0$ and $\sum_{j\in\mathbb{Z}^d} \pi(j) = 1$.

We will generally consider states of the system—probability measures on the set of configurations—which have a well-defined density ρ, in the sense that, with probability one, a fraction ρ of the sites in each configuration are occupied. (Here, and throughout unless stated otherwise, by "measure" we mean "probability measure.") Since particles are neither created nor destroyed, the density is a conserved quantity. If ρ is not too large there will exist *frozen* configurations in which no two adjacent sites are occupied and hence no particle can move; the maximum density of such a frozen configuration is clearly 1/2.

Most studies of the model with $d \geq 2$ consider the case in which the target sites are uniformly distributed over the nearest neighbors of the jumping particle. For $d = 2$, simulations [12, 18, 19] suggest a somewhat surprising property of the model (which presumably holds for $d \geq 2$): there is a critical density $\rho_c < 1/2$ such that, if the initial state of the models is a Bernoulli distribution with density ρ, then with probability 1, (i) for $\rho < \rho_c$ the model eventually reaches a frozen configuration, while (ii) for $\rho > \rho_c$ the configuration remains active—that is, particles continue to jump—for all time. Note that when $\rho_c < \rho \leq 1/2$ there exist frozen configurations with density ρ; these are traps for the dynamics, but with probability 1 they are avoided. To obtain such a result rigorously, or indeed any interesting rigorous results, seems very challenging (but see [20]). Indeed, we are not able to prove what seems to be self evident: that the configurations eventually freeze for sufficiently small ρ, say $\rho < 10^{-23}$.

In the remainder of this paper we consider only the case $d = 1$, with probabilities p and $1-p$ of jumps to the right and left, respectively (that is, we take $\pi(1) = p$, $\pi(-1) = 1-p$, and $\pi(j) = 0$ for $j \neq \pm 1$). This model is the *Facilitated Asymmetric Simple Exclusion Process* (F-ASEP), with special cases $p = 0, 1$ (*Totally Asymmetric*, the F-TASEP) [1, 2, 4, 6, 7] and $p = 1/2$ (*Symmetric*, the F-SSEP) [3, 5]. A discrete time version of the F-TASEP was studied in [9, 10]. We write $X = \{0, 1\}^{\mathbb{Z}}$ for the configuration space; the condition that configuration η have density ρ is now

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\eta_i=\lim_{N\to\infty}\frac{1}{N}\sum_{i=-N}^{-1}\eta_i=\rho, \tag{1.1}$$

and we let X_ρ denote the set of such configurations. We let $F\subset X$ denote the set of frozen configurations.

We will study the translation invariant (TI) measures on X which are stationary for the dynamics (TIS measures); for this purpose it suffices to consider extremal TIS measures, that is, the set of measures such that every TIS measure is a convex combination of these, and none of these is a proper convex combination of others. As we show below, each extremal measure will be supported on X_ρ for some value of ρ. The stationary measures for the symmetric case were discussed, for finite volume, in [5]; the results given there carry over smoothly to infinite volume. The current paper contains two main results, one each for low density ($0<\rho<1/2$) and high density ($1/2<\rho<1$), discussed in Sections 3 and 4, respectively.

For $0<\rho<1/2$, the TIS states are simple: all such states are frozen, that is, are supported on F, and all TI measures on F are TIS measures. In this case we pose and answer the question: if the initial state is Bernoulli, what is the final state? Our main result that this state is independent of the asymmetry parameter p, and is in fact the state which was earlier shown to answer the same question in the F-TASEP when the continuous time used to define the dynamics here is replaced by discrete time [9, 10]. However, the independence of p does not hold for the discrete-time model [11].

For $1/2<\rho<1$ we prove that there is a unique TIS state for each value of ρ. This state may in fact be identified as the Gibbs measure for a statistical mechanical system in which the only interaction is an exclusion rule that forbids two adjacent empty sites. Results identifying stationary states of particle systems with Gibbs measures have been established earlier [8, 24], but under the assumption that the transition rates for the particle system satisfy a detailed balance condition, which those of the F-ASEP do not (unless $p=1/2$). Our proof of the uniqueness relies on a coupling with the well-studied Asymmetric Simple Exclusion Model (ASEP).

2 The model

In this section we give various definitions and simple results which will be needed later. As indicated above, the configuration space of the F-ASEP model is $X := \{0,1\}^{\mathbb{Z}}$. (In Section 3.2 we will consider also the same dynamics on a ring of L sites with periodic boundary conditions; any notation specific to that case will be introduced as needed). $F \subset X$ denotes the set of frozen configurations in which no two adjacent sites are occupied, and similarly $G \subset X$ denotes the set of configurations with no two adjacent sites empty. We write $\eta = (\eta(i))_{i\in\mathbb{Z}}$ for a typical configuration, and for $j,k \in \mathbb{Z}$ with $j \le k$ we let $\eta(j\,{:}\,k) = (\eta(i))_{j\le i\le k}$ denote the portion of the configuration lying between sites j and k (inclusive). We will occasionally use string notation for configurations or partial configurations, writing for example $\eta(0\,{:}\,4) = \eta(0)\cdots\eta(4) = 10100 = (10)^2 0$.

The expected value of a random variable $f : X \to \mathbb{R}$ with respect to the measure μ will be written as $\mu(f) = \int_X f(\eta)\,d\mu$. We denote by τ the translation operator: if $\eta \in X$ then $(\tau\eta)(i) = \eta(i-1)$, if f is any function on X then $\tau f(\eta) = f(\tau^{-1}\eta)$, and if μ is a (Borel) measure on X then τ acts on μ as $\mu \mapsto \tau_*\mu$, where as usual if $h : A \to B$ and λ is a measure on A then $h_*\lambda$ is the measure on B with $(h_*\lambda)(C) = \lambda(h^{-1}(C))$. We let $\mathcal{L}(X)$ denote the space of functions $f : X \to \mathbb{R}$ for which $f(\eta)$ depends on the values $\eta(k)$ for only finitely many sites k, $\mathcal{C}(X)$ the space of real-valued continuous functions on X, and $\mathcal{M} = \mathcal{M}(X)$ denote the space of translation invariant probability measures on X.

We now turn to a formal specification of the system. The dynamics is controlled by *Site Associated Poisson Processes* (SAPPs); two of these, controlling rightward and leftward jumps, respectively, are associated with each site $i \in \mathbb{Z}$. Specifically, given a TI measure $\mu \in \mathcal{M}$ which specifies the initial distribution of the system, we consider the probability space $(\Omega, \mathbf{P}_p^{\mu})$:

$$\begin{aligned} \Omega &= X \times \Omega_0, \quad \text{with} \quad \Omega_0 = \prod_{i\in\mathbb{Z}} \bigl(\mathcal{T}^{(i,r)} \times \mathcal{T}^{(i,l)}\bigr), \\ \mathbf{P}_p^{\mu} &= \mu \times \mathbf{P}_p, \quad \text{with} \quad \mathbf{P}_p = \prod_{i\in\mathbb{Z}} \bigl(\lambda_p^{(i,r)} \times \lambda_p^{(i,l)}\bigr). \end{aligned} \tag{2.1}$$

Here, for $i \in \mathbb{Z}$ and $\# = l$ or r,

$$\mathcal{T}^{(i,\#)} = \Bigl\{ \bigl((i, t_j^{(i,\#)})\bigr)_{j=1,2,\ldots} \Bigm| 0 < t_1^{(i,\#)} < t_2^{(i,\#)} \cdots, \lim_{j\to\infty} t_j^{(i,\#)} = \infty \Bigr\} \tag{2.2}$$

and under $\lambda_p^{(i,r)}$ (respectively $\lambda_p^{(i,l)}$) the points of $\mathcal{T}^{(i,r)}$ (respectively $\mathcal{T}^{(i,l)}$) are distributed as a Poisson process of rate p (respectively $1-p$.)

The state now evolves as follows: at each time $t = t_j^{(i,r)}$ a particle jumps from site i to site $i+1$ if $\eta_{t-}(i-1) = \eta_{t-}(i) = 1 - \eta_{t-}(i+1) = 1$, and at each time $t = t_j^{(i,l)}$ a particle jumps from i to $i-1$ if $\eta_{t-}(i+1) = \eta_{t-}(i) = 1 - \eta_{t-}(i-1) = 1$. This so-called *graphical* construction leads [23] to a process η_t, well-defined on Ω, with generator L which acts on $\mathcal{L}(X)$ via

$$Lf(\eta) = \sum_{i\in\mathbb{Z}} c(i,\eta)[f(\eta^{i,i+1}) - f(\eta)]. \tag{2.3}$$

Here $\eta^{i,j}$ denotes the configuration η with the values of $\eta(i)$ and $\eta(j)$ exchanged. The rates $c(i,\eta)$ are given by

$$c(i,\eta) = \begin{cases} p, & \text{if } \eta(i-1) = \eta(i) = 1 \text{ and } \eta(i+1) = 0, \\ 1-p, & \text{if } \eta(i) = 0 \text{ and } \eta(i+1) = \eta(i+2) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

L is the generator of a Markov semigroup $S(t) = e^{Lt}$ on $\mathcal{C}(X)$ and so on $\mathcal{M}(X)$ via $\mu \to \mu S(t)$, where $(\mu S(t))(f) = \mu(S(t)f)$ or equivalently $(\mu S(t))(A) = \int_X Q_t(\eta, A)\, d\mu$, with $Q_t(\eta, A) = (S(t)\mathbf{1}_A)(\eta)$ the transition kernel of the Markov process. We will assume that this process, and others to be considered later, have right-continuous sample paths.

Remark 2.1. Since the set of all Poisson times for different sites will a.s. be dense in $(0,\infty)$, one cannot perform all the particle jumps in temporal order, and some care is needed to show that the construction is well defined. Details are given in [23]. In Section 4 we carry out such a construction, for a dynamics governed by *Particle Associated Point Processes*, by a somewhat different method.

If μ is a TI measure on X then, by the ergodic theorem, μ-almost every configuration η has a density, i.e.,

$$r(\eta) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \eta(i) \tag{2.5}$$

exists almost surely. (2.5) defines a map $r : X \to [0,1]$; we will say that a TI measure μ *has density* ρ if $r(\eta) = \rho$ μ-a.s. Note that if μ has density ρ then

$\mu(\eta(i)) = \rho$ for any i, but that the former is a stronger statement, ruling out, for example, the possibility that μ is a superposition of measures of different densities. The next lemma shows that in seeking to describe the set of all stationary TI measures μ it suffices to consider those for which $\mu(F)$ is 0 or 1 and for which $r(\eta)$ is μ-a.s. constant.

Lemma 2.2. *Every TIS measure μ on X is a convex combination of TIS measures for which r is a.s. constant and F has measure 0 or 1.*

Proof. Let $\nu = r_*\mu$; ν is a measure on $[0,1]$ which gives the distribution of the density under μ. Then (see for example [15]) there exists a regular conditional probability distribution for μ, that is, a family $\{\mu_\rho \mid \rho \in [0,1]\}$ of probability measures on X such that μ_ρ has density ρ and for any measurable $A \subset X$,

$$\mu(A) = \int_{0\le\rho\le1} \mu_\rho(A) d\nu(\rho). \tag{2.6}$$

Moreover, $\{\mu_\rho\}$ is unique in the sense that for any other such family $\{\mu'_\rho\}$, $\mu_\rho = \mu'_\rho$ ν-a.s. If we further write $\mu_\rho = \mu_\rho\big|_F + \mu_\rho\big|_{X\setminus F}$ we obtain, after normalization, the desired representation. It remains to verify that these normalized measures, $\mu_\rho\big|_F/\big(\mu_\rho\big|_F(X)\big)$ and $\mu_\rho\big|_{X\setminus F}/\big(\mu_\rho\big|_{X\setminus F}(X)\big)$, are TIS measures.

Now

$$\mu(A) = \tau_*\mu(A) = \int_{0\le\rho\le1} \tau_*\mu_\rho(A) d\nu(\rho), \tag{2.7}$$

and from (2.5) it follows that $\tau_*\mu_\rho$ has density ρ, so that the uniqueness of the conditional probability distribution implies that $\tau_*\mu_\rho = \mu_\rho$ ν-a.s. Stationarity of μ_ρ ν-a.s. follows similarly from the fact that neither dynamics destroys or creates particles. Finally, translation invariance and stationarity of $\mu_\rho\big|_F$ and $\mu_\rho\big|_{X\setminus F}$ follows from the fact that F is translation invariant and invariant under the dynamics. ■

The key idea in the next lemma appears in [5] in the context of a system on a ring.

Lemma 2.3. *If μ is a TIS measure on X then $\mu(F \cup G) = 1$.*

Proof. The argument we give requires that p be strictly positive; by the symmetry of the model under simultaneous spatial reflection and the replacement $p \to 1 - p$, we may assume that this condition holds. Suppose that $\mu(F \cup G) < 1$. If $\eta \notin F \cup G$ then η contains two adjacent zeros and two adjacent ones; let $2k$ be the minimum number of sites by which a double zero follows a double one—that is, for which the string $11(01)^k00$ occurs in a configuration—with nonzero probability. We first note that $k \geq 1$ almost surely. For, using (2.4) and the translation invariance of μ, we have

$$\begin{aligned}\frac{d}{dt}\mu\bigl(\eta(0{:}1) = 11\bigr) &= -p\,\mu\bigl(\eta(0{:}2) = 110\bigr) - (1-p)\,\mu\bigl(\eta(-1{:}1) = 011\bigr)\\ &\qquad + p\,\mu\bigl(\eta(-2{:}1) = 1101\bigr) + (1-p)\,\mu\bigl(\eta(0{:}3) = 1011\bigr)\\ &= -p\,\mu\bigl(\eta(0{:}3) = 1100\bigr) - (1-p)\,\mu\bigl(\eta(-2{:}1) = 0011\bigr).\end{aligned}$$

This quantity must vanish, since μ is stationary, and since p is nonzero the probability of 1100 occurring is zero.

Now by the choice of k, $\mu\bigl(\eta(0{:}k+3) = 11(01)^k00\bigr) > 0$. Then a simple calculation as above, using repeatedly the fact that for any i and for $j < k$, $\mu\bigl(\eta(i{:}i+j+3) = 11(01)^j00\bigr) = 0$, shows that

$$\frac{d}{dt}\mu\bigl(\eta(2{:}k+3) = 11(01)^{(k-1)}00\bigr) = p\,\mu\bigl(\eta(0{:}k+3) = 11(01)^k00\bigr) > 0, \quad (2.8)$$

contradicting stationarity. ∎

Remark 2.4. Let $\eta^* \in X$ be the period-two configuration defined by $\eta^*(i) = i \bmod 2$. η^* and its translate $\tau\eta^*$ consist of alternating 1's and 0's, and the measure $\mu^* = (\delta_{\eta^*} + \delta_{\tau\eta^*})/2$ is a TIS measure with density $\rho = 1/2$. Note that $\mu^*(F) = \mu^*(G) = 1$.

Theorem 2.5. *Let μ be a TIS measure on X with density ρ. Then:*
(a) If $\rho < 1/2$ then $\mu(F) = 1$, i.e., μ is supported on F.
(b) If $\rho = 1/2$ then $\mu = \mu^*$ (see Remark 2.4).
(c) If $\rho > 1/2$ then $\mu(G) = 1$, i.e., μ is supported on G.

Proof. We know that μ is supported on $F \cup G$. Suppose that $\eta \in \operatorname{supp}\mu$; then we may assume that $\eta \in F \cup G$ and $r(\eta) = \rho$. If $\rho = r(\eta) < 1/2$ then, by (2.5), η must contain a positive density of double zeros and so lie in $F \setminus G$, verifying (a); similarly, if $\rho > 1/2$ then $\eta \in G \setminus F$, verifying (c). If $\rho = 1/2$ then (2.5) with $\eta \in F \cup G$ implies that η does not have a positive density of either double ones or double zeros, and hence almost surely has no double ones or double zeros at all, verifying (b). ∎

3 The low density region

In this section we consider TIS states on X_ρ with $0 < \rho < 1/2$; by Theorem 2.5 these are necessarily supported on F. In fact, any TI measure supported on F is clearly a TIS state; see Remark 4.2 for an observation on obtaining such states. Here we address the following question:

Question 3.1. If the system is given an initial measure $\mu^{(\rho)}$, the Bernoulli measure with density $0 < \rho < 1/2$, what is the final measure?

3.1 The totally asymmetric model

In this section we address Question 3.1 for the totally asymmetric model (F-TASEP); we take $p = 1$ but the discussion for $p = 0$ would be similar. The answer is given in [9, 10] for the discrete-time F-TASEP, and the analysis there applies almost unchanged in the continuous-time case, so we content ourselves with a brief summary.

First, it is convenient to enlarge the state space of the process from X to $\widehat{X} := X \times \mathbb{Z}$, writing the state of the system at time t as (η_t, J_t). In this new version of the model, J_t is the signed count of the number of particles passing between sites 0 and 1 up to time t. The new version is defined on the same probability space $(\Omega, \mathbf{P}_p^\mu)$ as the original one (see (2.1), (2.2)), with J_t incremented or decremented by 1 at those times $t_j^{(0,r)}$ or $t_j^{(0,l)}$, respectively, at which jumps actually occur; it is straightforward to verify, as in [23], that this leads to a well-defined process. We always assume that $J_0 = 0$.

The variable J_t allows us to introduce the *height profile* $h_t : \mathbb{Z} \to \mathbb{Z}$ associated with the pair $(\eta_t, J_t) \in \widehat{X}$ (see, e.g., [14]), defined by the requirements that $h_t(i) - h_t(i-1) = 1 - 2\eta_t(i) = (-1)^{\eta_t(i)}$ for all $i \in \mathbb{Z}$ and $h_t(0) = 2J_t$, or more explicitly by

$$h_t(i) = \begin{cases} 2J_t + \displaystyle\sum_{j=1}^{i} (-1)^{\eta_t(j)}, & \text{if } i \geq 0, \\ 2J_t - \displaystyle\sum_{j=i+1}^{0} (-1)^{\eta_t(j)}, & \text{if } i < 0. \end{cases} \tag{3.1}$$

Then, since $0 < \rho < 1/2$, $\lim_{i \to \pm\infty} h_t(i) = \pm\infty$. Moreover, as a function of t, h_t is monotonically increasing; in particular, $h_t(i)$ increases by 2 when a particle jumps from site i to site $i+1$, and such an increase can occur only if $h_t(i-1) > h_t(i)$.

We can now determine the fate of an arbitrary initial configuration $\eta_0 \in X_\rho$. Define $Q = Q(\eta_t) \subset \mathbb{Z}$ by $Q := \{q \in \mathbb{Z} \mid h_t(q) > \sup_{i<q} h_t(i)\}$. From the observation above on how h_t can increase it follows that $Q(\eta_t)$ is in fact independent of t and that for $q \in Q$, $h_t(q)$ is constant. If q and q' are consecutive elements of Q and $i \in \mathbb{Z}$ satisfies $q \le i < q'$, then $h_t(i) < h_t(q') = h_0(q')$, so that $\lim_{t\to\infty} h_t(i)$, and hence also $\eta_\infty(i) = \lim_{t\to\infty} \eta_t(i)$, exist; one sees also that

$$\eta_\infty(q+1\!:\!q') = 1\,0\,1\,0 \cdots 1\,0\,0 = (1\,0)^{(q'-q-1)/2}0. \tag{3.2}$$

This completes the determination of the limiting configuration η_∞.

We now suppose that η_0 is distributed according to the Bernoulli distribution $\mu^{(\rho)}$, and write $\mathbf{P}_p^{(\rho)}$ rather than $\mathbf{P}_p^{\mu^{(\rho)}}$. We ask for the distribution $\mu_\infty^{(\rho)} = \eta_{\infty *}\mathbf{P}_p^{(\rho)}$ of η_∞, where here we think of η_∞ as a map $\eta_\infty : \Omega \to X_\rho$. Let $V = \{\eta \in X \mid 0 \in Q(\eta)\}$; we first describe the conditional measure $\mu_\infty^{(\rho)}(\cdot \mid V)$, then obtain $\mu_\infty^{(\rho)}$ as the (unique) TI measure with this conditional measure. For $\eta_0 \in V$ we may index $Q = Q(\eta_0)$ as $Q = \{q_k\}_{k\in\mathbb{Z}}$, taking $q_0 = 0$ and requiring that the q_k be increasing in k; then we may specify $\mu_\infty^{(\rho)}(\cdot \mid V)$ by giving the joint distribution of the variables n_k defined by $q_{k+1} - q_k = 2n_k + 1$.

It is easy to see that the n_k are i.i.d. To describe the distribution of a single n_k we recall the *Catalan numbers* [22]

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, \dots ; \tag{3.3}$$

c_n counts the number of strings of n 0's and n 1's in which the number of 0's in any initial segment does not exceed the number of 1's. If $q \in Q$ and $l = q+2n+1$ then $q' = l$ if and only if $h_0(l) = h_0(q)+1$ and $h_0(i) \le h_0(q)$ for $q < i < l$, and there are c_n strings $\eta(q+1\!:\!l-1)$ satisfying this condition and hence yielding $q' = l$. Since each such string has probability $\rho^n(1-\rho)^{n+1}$ we have sketched a proof of the next theorem (recall that τ denotes translation):

Theorem 3.2. *(a) The random variables n_k of the F-TASEP, defined on V as above, are i.i.d. under $\mu_\infty^{(\rho)}(\cdot \mid V)$, with distribution*

$$\mu_\infty^{(\rho)}(\{n_k = n\} \mid V) = c_n \rho^n (1-\rho)^{n+1}, \quad n = 0, 1, 2, \dots . \tag{3.4}$$

(b) The measure $\mu_\infty^{(\rho)}$ is given by

$$\mu_\infty^{(\rho)} = \sum_{m\ge 0}\sum_{i=0}^{2m} \tau_*^{-i}\bigl(\mu_\infty^{(\rho)}\big|_{V_m}\bigr), \tag{3.5}$$

where $V_m = \{\eta_0 \in V \mid q_1 - q_0 = 2m+1\}$.

3.2 The model in finite volume

In this section we address Question 3.1, or rather an appropriately modified version of it, for the F-ASEP on a periodic ring of L sites. We first discuss the totally asymmetric model and describe the result corresponding to Theorem 3.2, then show that the limiting measure is in fact independent of the asymmetry parameter p. The ring is denoted $\mathbb{Z}_L = \{0,1,\ldots,L-1\}$; we consider a system of N particles on these sites, governed by the obvious modification of the F-ASEP dynamics defined in Section 2. For a configuration $\eta \in \{0,1\}^{\mathbb{Z}_L}$ we let $|\eta| := \sum_{i=1}^{L}\eta(i)$ denote the number of particles in η; the configuration space of our model is then $X^{(N)} := \{\eta \in \{0,1\}^{\mathbb{Z}_L} \mid |\eta| = N\}$. We will be interested in the fate of an initial measure $\mu^{(N)}$ which is uniform on $X^{(N)}$; the probability space is then $(\Omega^{(N)}, \mathbf{P}_p^{(N)})$:

$$\begin{aligned} \Omega^{(N)} &= X^{(N)} \times \prod_{i\in\mathbb{Z}_L}\bigl(\mathcal{T}^{(i,r)} \times \mathcal{T}^{(i,l)}\bigr),\\ \mathbf{P}_p^{(N)} &= \mu^{(N)} \times \prod_{i\in\mathbb{Z}_L}\bigl(\lambda_p^{(i,r)} \times \lambda_p^{(i,l)}\bigr). \end{aligned} \tag{3.6}$$

Here $\mathcal{T}$ and λ are as in (2.1). (In view of our earlier use of $\mu^{(\rho)}$ and $\mathbf{P}_p^{(\rho)}$, writing $\mu^{(N)}$ and $\mathbf{P}_p^{(N)}$ is admittedly an abuse of notation, but we believe that this will not give rise to confusion.) The system size L will be constant during our analysis and we typically suppress L-dependence. The construction of the dynamics is parallel to the construction in infinite volume, but is technically simpler because the considerations of Remark 2.1 do not apply; we omit details. The auxiliary variable J_t is not needed here.

Now consider the F-TASEP, taking $p = 1$ above. Given an initial configuration $\eta_0 \in \{0,1\}^{\mathbb{Z}_L}$, with $|\eta_0| < L/2$, we extend η_0 to an L-periodic configuration η_0^* on $\mathbb{Z}$, apply the above construction to obtain $Q(\eta_0^*)$, and let $Q(\eta_0) := Q(\eta_0^*) \cap \{0,1,\ldots,L-1\}$; $Q(\eta_0)$ will contain $L - 2|\eta_0|$ sites. An argument as in infinite volume shows that the limiting configuration η_∞ exists and satisfies (3.2) for q, q' consecutive (in cyclic order) elements of $Q(\eta_0)$ (with the expression $q' - q - 1$ in the exponent of (3.2) interpreted mod L).

Now fix $N < L/2$; we will determine the distribution $\mu_\infty^{(N)} = \eta_{\infty *}\mathbf{P}_p^{(N)}$ of η_∞ when η_0 is distributed according to $\mu^{(N)}$ (this is the modified version of Question 3.1 referred to above). Let $V^{(N)} := \{\eta \in X^{(N)} \mid 0 \in Q(\eta)\}$ and note that $|V^{(N)}| = (L-2N)\binom{L}{N}/L$, since if one partitions $X^{(N)}$ into equivalence classes under translation then each class contains a fraction $(L-2N)/L$ of elements belonging to $V^{(N)}$. We can determine the conditional measure $\mu_\infty^{(N)}(\cdot \mid V^{(N)})$ by simple counting: given $0 = q_0 < q_1 < \ldots < q_{L-2N-1} \le L-1$, with $q_{i+1} - q_i \equiv 2n_i + 1 \pmod L$ for $0 \le n_i < L/2$, there are $\prod_{i=0}^{L-2N-1} c_{n_i}$ initial configurations $\eta_0 \in X^{(N)}$ with $Q(\eta_0) = \{q_0, \ldots, q_{L-2N-1}\}$, all leading to $\eta_\infty = \eta^{(q_0,\ldots,q_{L-2N-1})}$, where

$$\eta^{(q_0,\ldots,q_{L-2N-1})} := (10)^{q_1-q_0}0(10)^{q_2-q_1}0\cdots0(10)^{q_0-q_{L-2N-1}+L}0. \tag{3.7}$$

Thus we have

Theorem 3.3. *(a) The possible limiting configurations of the F-TASEP model on $V^{(N)}$ are the $\eta^{(q_0,\ldots,q_{L-2N-1})}$, and*

$$\mu_\infty^{(N)}\big(\{\eta^{(q_0,\ldots,q_{L-2N-1})}\} \mid V^{(N)}\big) = \frac{L\prod_{i=0}^{L-2N-1} c_{n_i}}{(L-2N)\binom{L}{N}}. \tag{3.8}$$

(b) $\mu_\infty^{(N)} = \dfrac{1}{L}\displaystyle\sum_{i=0}^{L-1} \tau_*^i \mu_\infty^{(N)}(\cdot \mid V^{(N)})$.

We now consider the general F-ASEP model on $\mathbb{Z}_L$, with partially asymmetric dynamics governed by the asymmetry parameter p. Since from any initial configuration η_0 there is a sequence of possible transitions leading to a frozen configuration, $\eta_\infty = \lim_{t\to\infty}\eta_t$ exists almost surely, for any η_0, and is frozen. The distribution $\mu_\infty^{(N)} = \eta_{\infty *}\mathbf{P}_p^{(N)}$ of the limiting configurations is then well defined; our goal is to show that this distribution is independent of p (as our notation indicates).

A simple but important observation, which we will use repeatedly, is that if two adjacent sites are empty in η_∞ then they must also be empty in all η_t, $t \ge 0$. Because of this it is convenient to decompose configurations into *components*—strings of 1's and 0's within which no two adjacent sites are empty but which are separated from each other by (at least) two adjacent empty sites. (Formally a *component* of a configuration η is the restriction $\eta|_I$ of η to an interval $I = \{i, i+1, \ldots, j\}$ for which $\eta(i) = \eta(j) = 1$, $\eta(i-2) = \eta(i-1) = \eta(j+1) = \eta(j+2) = 0$, and there is no site k in I such that $\eta(k) = \eta(k+1) = 0$.) We let $c(\eta)$ denote the number of components in η, and write $\mathbf{P}_p^{(N,n)}$ for the measure $\mathbf{P}_p^{(N)}$ conditioned on the event $c(\eta_0) = n$.

Theorem 3.4. *For all L and N, the measure $\mu_\infty^{(N)}$ is independent of p, and so is given by Theorem 3.3(b).*

Proof. We will prove by induction on n, $n = 1, 2, \ldots$, that for all L and N with $L/2 > N \ge n$ the distribution of η_∞ under $\mathbf{P}_p^{(N,n)}$ is independent of p. The theorem then follows from $\mathbf{P}_p^{(N)}(\cdot) = \sum_{n=1}^N \mathbf{P}_p^{(N,n)}(\cdot)\mathbf{P}_p^{(N)}(c(\eta_0) = n)$, since the distribution $\mu^{(N)}$ of η_0 is independent of p. The case $n = 1$ of the induction is trivial: if the initial configuration has a single component then so does the final one, and for any p this component is just $101\cdots01$ (with N 1's) and its position will be uniformly distributed over the ring, by translation invariance.

We now assume inductively that n is such that for all $k \le n$ and all L, N with $L/2 > N \ge n$, the distribution of η_∞ under $\mathbf{P}_p^{(N,k)}$ is independent of p. We then fix a configuration $\zeta \in X^{(N)}$ and show that $\mathbf{P}_p^{(N,n+1)}(\eta_\infty = \zeta)$ is independent of p; we may assume that no two consecutive sites are occupied in ζ, since otherwise this probability is 0. Consider first the case $c(\zeta) > 1$; then if necessary we may rotate ζ (which does not affect the conclusion) to an orientation in which there exists a site i, with $3 \le i \le L-3$, such that $\zeta(0) = \zeta(1) = 0$, $\zeta(i) = \zeta(i+1) = 0$, and $\zeta(j) = 1$ for at least one j in the set $\{2, \ldots, i-1\}$ and at least one j in $\{i+2, \ldots, L-1\}$. Define maps $\pi_1, \pi_2 : X^{(N)} \to \bigcup_{0 \le N' \le N} X^{(N')}$ by

$$(\pi_1\eta)(j) = \begin{cases} \eta(j), & \text{if } 2 \le j \le i-1, \\ 0, & \text{if } 0 \le j \le 1 \text{ or } i \le j \le L-1, \end{cases} \tag{3.9}$$

$$(\pi_2\eta)(j) = \begin{cases} \eta(j), & \text{if } i+2 \le j \le L-1, \\ 0, & \text{if } 0 \le j \le i+1, \end{cases} \tag{3.10}$$

and for integers k_1, k_2 with $1 \le k_m \le |\pi_m\zeta|$, satisfying $k_1 + k_2 = n+1$, define the events

$$E_{k_1,k_2} := \{\eta_0(0) = \eta_0(1) = \eta_0(i) = \eta_0(i+1) = 0, \\ |\pi_m\eta_0| = |\pi_m\zeta|, \text{ and } c(\pi_m\eta_0) = k_m,\ m = 1,2\},$$

and for $m = 1, 2$,

$$E^{(m)}_{k_m} := \{\eta_0 = \pi_m\eta_0,\ |\eta_0| = |\pi_m\zeta|, \text{ and } c(\eta_0) = k_m\}.$$

We will assume in what follows that k_1, k_2 are chosen so that $\mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta \mid E_{k_1,k_2}) \neq 0$. Since

$$\mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta) = \sum_{k_1,k_2} \mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta \mid E_{k_1,k_2})\mathbf{P}^{(N,n+1)}_p(E_{k_1,k_2}) \tag{3.11}$$

and E_{k_1,k_2} depends only on the initial configuration η_0, it suffices to show that for all such k_1, k_2, $\mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta \mid E_{k_1,k_2})$ is independent of p.

Now observe that

$$\mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta \mid E_{k_1,k_2}) = \prod_{m=1,2} \mathbf{P}^{(|\pi_m\zeta|,k_m)}_p(\eta_\infty = \pi_m\zeta \mid E^{(m)}_{k_m}). \tag{3.12}$$

This is because (i) once we have conditioned on E_{k_1,k_2}, $\pi_1\eta_0$ and $\pi_2\eta_0$ are independent, and (ii) given η_0, the probability on the left is determined by the SAPP times $t^{(i',\#)}_j$ for $2 \leq i' \leq i-1$ and $i+2 \leq i' \leq L-1$, and these are independent. But also we have

$$\mathbf{P}^{(|\pi_m\zeta|,k_m)}_p(\eta_\infty = \pi_m\zeta \mid E^{(m)}_{k_m}) = \frac{\mathbf{P}^{(|\pi_m\zeta|,k_m)}_p(\eta_\infty = \pi_m\zeta)}{\mathbf{P}^{(|\pi_m\zeta|,k_m)}_p(E^{(m)}_{k_m})}, \tag{3.13}$$

since, with probability 1 with respect to $\mathbf{P}^{(|\pi_m\zeta|,k_m)}_p$, $\eta_\infty = \pi_m\zeta$ is possible only if $E^{(m)}_{k_m}$ occurs. Both the numerator and denominator on the right hand side of (3.13) are independent of p, the numerator by the inductive assumption and the denominator since it involves only the initial condition. Thus $\mathbf{P}^{(|\pi_m\zeta|,k_m)}_p(\eta_\infty = \pi_m\zeta \mid E^{(m)}_{k_m})$ and hence, by (3.12), $\mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta \mid E_{k_1,k_2})$, are independent of p.

This completes the proof of the p-independence of $\mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta)$ in the case $c(\zeta) > 1$. From the validity of this result for all such ζ it follows that

$$\mathbf{P}^{(N,n+1)}_p(c(\eta_\infty) = 1) = 1 - \sum_{l=2}^{N} \sum_{\{\zeta \mid c(\zeta)=l\}} \mathbf{P}^{(N,n+1)}_p(\eta_\infty = \zeta) \tag{3.14}$$

is also independent of p. Then by translation invariance, $\mathbf{P}_p^{(N,n+1)}(\eta_\infty = \zeta) = \mathbf{P}_p^{(N,n+1)}(c(\eta_\infty) = 1)/L$ if $c(\zeta) = 1$. This completes the proof. ■

3.3 The partially asymmetric model in infinite volume

In this section we return to the (p-dependent) F-ASEP dynamics on $\mathbb{Z}$; we assume that η_0 is distributed as the Bernoulli measure $\mu^{(\rho)}$, $0 < \rho < 1/2$, and show that the distribution of the limiting configuration η_∞ is independent of p. As in Section 3.1 we write $\mathbf{P}_p^{(\rho)}$ for the measure $\mathbf{P}_p^{\mu^{(\rho)}}$ on the sample space Ω of (2.1). We begin with two preliminary results; the first is standard.

Lemma 3.5. *If $f : \Omega \to X$ commutes with translations then $f_*\mathbf{P}_p^{(\rho)}$ is mixing under translations.*

Proof. Note that Ω is in fact a product space, $\Omega = \prod_{i\in\mathbb{Z}}\bigl(\{0,1\} \times \mathcal{T}^{(i,r)} \times \mathcal{T}^{(i,l)}\bigr)$, and that $\mathbf{P}_p^{(\rho)}$ is a product measure. Thus $\mathbf{P}_p^{(\rho)}$ is certainly mixing under translations. Then for any measurable sets $A, B \subset X$,

$$\begin{aligned}
\lim_{n\to\infty} f_*\mathbf{P}_p^{(\rho)}(A \cap \tau^n B) &= \lim_{n\to\infty} \mathbf{P}_p^{(\rho)}(f^{-1}(A \cap \tau^n B)) \\
&= \lim_{n\to\infty} \mathbf{P}_p^{(\rho)}(f^{-1}(A) \cap \tau^n f^{-1}(B)) \\
&= \mathbf{P}_p^{(\rho)}(f^{-1}(A))\mathbf{P}_p^{(\rho)}(f^{-1}(B)) \\
&= f_*\mathbf{P}_p^{(\rho)}(A) f_*\mathbf{P}_p^{(\rho)}(B).
\end{aligned}$$

■

Lemma 3.6. *$\eta_\infty := \lim_{t\to\infty} \eta_t$ exists, and is frozen, $\mathbf{P}_p^{(\rho)}$-almost surely.*

Proof. The cases $p = 1$ and $p = 0$ follow from the discussion of Section 3.1, so we may suppose that $0 < p < 1$. For $t \in \mathbb{Z}_+$ define $\theta_t : \Omega \to X$ by $\theta_t(k) = 1$ if $\eta_t(k) = \eta_t(k+1) = 0$, $\theta_t(k) = 0$ otherwise. The sequence $(\theta_t)_{t\in\mathbb{Z}_+}$ is pointwise decreasing and so $\theta_\infty = \lim_{t\to\infty} \theta_t$ exists; $(\theta_\infty)_*\mathbf{P}_p^{(\rho)}$ is mixing by Lemma 3.5 and moreover, since $\mathbf{P}_p^{(\rho)}\bigl(\theta_t(k) = 1\bigr) \geq 1 - 2\rho$ for all t, $\mathbf{P}_p^{(\rho)}(\theta_\infty(k) = 1) \geq 1 - 2\rho$ by the Monotone Convergence Theorem. Thus if for $k, l > 0$ we define $\Omega_{k,l} = \{\omega \in \Omega \mid \theta_\infty(-k-2) = \theta_\infty(l+1) = 1\}$ then for any $L > 0$, $\mathbf{P}_p^{(\rho)}$-a.e. $\omega \in \Omega$ will lie in $\Omega_{k,l}$ for some $k, l > L$.

We claim that, $\mathbf{P}_p^{(\rho)}$-almost surely on $\Omega_{k,l}$, $\lim_{t\to\infty} \eta_t\big|_{[-k,l]}$ exists and is frozen; by the previous paragraph this suffices for the result. Now conditioning on $\Omega_{k,l}$ simply implies, for the behavior of η_t in $[-k, l]$, that $\eta_0\big|_{[-k,l]}$

has at most $\lfloor (k+l+1)/2 \rfloor$ particles and that no transitions occur across the bonds $\langle -k-1, -k \rangle$ and $\langle l, l+1 \rangle$, and with these restrictions there is, from any initial configuration, a sequence of possible transitions leading to a frozen configuration. ■

Now set $\mu_\infty := \eta_{\infty *}\mathbf{P}_p^{(\rho)}$; μ_∞ is mixing by Lemma 3.5. Our main result, Theorem 3.7 below, is that μ_∞ does not depend on p.

Theorem 3.7. *For all ρ, with $0 < \rho < 1/2$, the distribution of η_∞ is independent of p, and so is given by Theorem 3.2(b).*

Proof. Let $I \subset \mathbb{Z}$ be an interval of integers, let $\zeta \in \{0,1\}^I$ be a configuration on I, and let E be the event that $\eta_\infty\big|_I = \zeta$. We will show that $\mathbf{P}_p^{(\rho)}(E)$ is independent of p, proving the result. We may assume that ζ contains no pair of adjacent occupied sites, since otherwise $\mathbf{P}_p^{(\rho)}(E) = 0$.

Choose $l \in \mathbb{N}$ so large that $I \subset [-l, l]$. Then, since μ_∞ is mixing and hence ergodic, and in μ_∞ there is a strictly positive density $1-2\rho$ of pairs of adjacent empty sites, there must $\mathbf{P}_p^{(\rho)}$-almost surely exist sites i and j, with $i < -l$ and $j > l$, such that $\eta_\infty(i-1) = \eta_\infty(i) = \eta_\infty(j) = \eta_\infty(j+1) = 0$. Focusing on the maximal such i and minimal such j leads to the representation

$$E = \bigcup_{i<-l<l<j} F_i \cap E \cap F'_j, \tag{3.15}$$

where for some $m, m' \geq 0$, $F_i := \{\eta_\infty(i-1 \colon -l-1) = 00(10)^m \text{ or } 00(10)^m 1\}$ and $F'_j := \{\eta_\infty(l+1 \colon j+1) = (01)^{m'}00 \text{ or } 1(01)^{m'}00\}$. Since (3.15) is a disjoint union,

$$\begin{aligned} \mathbf{P}_p^{(\rho)}(E) &= \sum_{i<-l<l<j} \mathbf{P}_p^{(\rho)}(F_i \cap E \cap F'_j) \\ &= \sum_{i<-l<l<j} \alpha(i,j,l)\mathbf{P}_p^{(\rho)}(G_i \cap G'_j). \end{aligned} \tag{3.16}$$

where $G_i := \{\eta_\infty(i) = \eta_\infty(i-1) = 0\}$, $G'_j := \{\eta_\infty(j) = \eta_\infty(j+1) = 0\}$, and $\alpha(i,j,l) := \mathbf{P}_p^{(\rho)}(F_i \cap E \cap F'_j \mid G_i \cap G'_j)$. We show below that, as the notation indicates, $\alpha(i,j,l)$ is independent of p. Now suppose that $\epsilon > 0$; since $\mathbf{P}_p^{(\rho)}(G_i) = \mathbf{P}_p^{(\rho)}(G'_j) = 1-2\rho$, Lemma 3.5 implies that there is an l_p^* such that

$$|\mathbf{P}_p^{(\rho)}(G_i \cap G'_j) - (1-2\rho)^2| < \epsilon \text{ for } l \geq l_p^*. \tag{3.17}$$

Now (3.16) and (3.17) imply that if $\epsilon < (1-2\rho)^2$,

$$\sum_{i<-l<l<j} \alpha(i,j,l) < \frac{1}{(1-2\rho)^2-\epsilon}. \tag{3.18}$$

But then for any p, p' with $0 \le p, p' \le 1$ we have for $l > \max(l_p^*, l_{p'}^*)$,

$$|\mathbf{P}_p^{(\rho)}(E) - \mathbf{P}_{p'}^{(\rho)}(E)| < 2\epsilon \sum_{i<-l<l<j} \alpha(i,j,l) \le \frac{2\epsilon}{(1-2\rho)^2-\epsilon}. \tag{3.19}$$

Since ϵ is arbitrary, $\mathbf{P}_p^{(\rho)}(E) = \mathbf{P}_{p'}^{(\rho)}(E)$.

To see that $\mathbf{P}_p^{(\rho)}(F_i \cap E \cap F_j' \mid G_i \cap G_j')$ is independent of p we appeal to Theorem 3.4. Let $L = j+1-i$, let N denote the number of particles in η_0 which lie in the interval $[i-1, j+1]$, and let $H_n = \{N = n\}$. If G_i and G_j occur then necessarily $N < L/2$, and we may write

$$\begin{aligned}\mathbf{P}_p^{(\rho)}&(F_i \cap E \cap F_j' \mid G_i \cap G_j')\\ &= \sum_{n<L/2} \mathbf{P}_p^{(\rho)}(F_i \cap E \quad \cap F_j' \mid G_i \cap G_j' \cap H_n)\mathbf{P}_p^{(\rho)}(H_n).\end{aligned} \tag{3.20}$$

Now consider a system with n particles on a ring of L sites, which for convenience we label as $i, i+1, \ldots j$; the state space is $X^{(n)} \subset \{0,1\}^{[i,j]}$ and there is a natural map $\pi_n : H_n \to X^{(n)}$ given by restriction. Further,

$$\mathbf{P}_p^{(\rho)}(F_i \cap E \cap F_j' \mid G_i \cap G_j' \cap H_n) = \mathbf{P}_p^{(n)}(\pi(F_i \cap E \cap F_j') \mid \pi(G_i \cap G_j')), \tag{3.21}$$

since, under the conditioning on $G_i \cap G_j'$ and $\pi(G_i \cap G_j')$, respectively, if some initial condition and sequence of particle jumps in the system on $\mathbb{Z}$ produces $G_\theta \cap E \cap G_\sigma'$ then the corresponding initial condition and sequence of jumps in the finite system will produce $\pi(G_\theta \cap E \cap G_\sigma')$. Since the right hand side of (3.21) is independent of p by Theorem 3.4, so is $\mathbf{P}_p^{(\rho)}(F_i \cap E \cap F_j' \mid G_i \cap G_j')$, by (3.20). ■

4 The high density region

We now turn to consideration of the TIS measures for the F-ASEP with density $\rho > 1/2$. By Theorem 2.5 such measures are supported on the set $G \subset X$ of configurations with no two adjacent holes, so in this section we will regard the F-ASEP as a Markov process on G, and write $\mathcal{M}(G)$ for the space of TI probability measures on G. We will prove:

Theorem 4.1. *For each $\rho > 1/2$ there is a unique TIS measure with density ρ for the F-ASEP.*

Some context for this result arises from a familiar equilibrium statistical mechanical system of particles on a one-dimensional lattice, sometimes referred to as the *nearest-neighbor hard core* model, in which the only interaction is an infinitely strong repulsion between particles on adjacent sites, so that the possible configurations are those with no two particles adjacent. When this system is considered on a ring all configurations satisfying this restriction are equally likely, and in the thermodynamic limit there is a unique (for given density ρ) Gibbs measure. If we exchange the roles of particles and holes we obtain from this a measure $\bar{\mu}^{(\rho)}$ supported on G, and this measure is a TIS state for the F-ASEP, whatever the asymmetry—which must, of course, be the unique such state identified in Theorem 4.1.

Results identifying all stationary states of particle systems as canonical Gibbs measures have been established in fairly general contexts in [8] and [24] (see also [21] for a review of the situation). These results, however, require that the rates satisfy a detailed balance condition, which ours do not unless $p = 1/2$, and even in this symmetric case certain non-degeneracy hypotheses on the rates exclude the F-SSEP. Rather than attempting to extend or modify the arguments of these papers we give an independent proof of the theorem for the F-ASEP, based on a coupling with the Asymmetric Simple Exclusion Model (ASEP). Recall [17] that this model has configuration space $Y\,(= X) = \{0,1\}^{\mathbb{Z}}$; we will write a typical ASEP configuration as $\zeta = \bigl(\zeta(i)\bigr)_{i\in\mathbb{Z}}$ and write $\mathcal{M}(Y)$ for the space of TI probability measures on Y. The dynamics is defined in parallel with that of the F-ASEP (see Section 2), using the same SAPPs $\bigl((i,t_j^{(i,r)})\bigr)_{j=1,2,\ldots}$ and $\bigl((i,t_j^{(i,l)})\bigr)_{j=1,2,\ldots}$ and requiring that a particle jump from i to $i+1$ at $t = t_j^{(i,r)}$ if $\eta_{t-}(i) = 1 - \eta_{t-}(i+1) = 1$ and from i to $i-1$ at $t = t_j^{(i,l)}$ if $\eta_{t-}(i) = 1 - \eta_{t-}(i-1) = 1$. It is known [17] that for *0* $\le \hat{\rho} \le 1$ the Bernoulli measure is the unique TIS state of density $\hat{\rho}$ for the ASEP.

To define the coupling we first introduce the map $\phi : Y \to G$ defined by the substitutions $1 \to 1$, $0 \to 10$; more specifically, for $\zeta \in Y$,

$$\phi(\zeta) = \cdots \psi(\zeta(-1))\psi(\zeta(0))\psi(\zeta(1))\psi(\zeta(2)) \cdots, \tag{4.1}$$

where $\psi(1) = 1$, $\psi(0) = 10$, and the substitution is made so that $\psi(\zeta(1))$ begins at site 1. Further, we define $\gamma_\zeta : \mathbb{Z} \to \mathbb{Z}$ so that $\gamma_\zeta(i)$ is the initial site of the string $\psi(\zeta(i))$ which is substituted for $\zeta(i)$ under ϕ. For example, if $\zeta(-1{:}4) = 0\,1\,1\,0\,0\,1$ then $\phi(\zeta)(-2{:}6) = 1\,0\,1\,1\,1\,0\,1\,0\,1$ and $\gamma_\zeta(-1) = -2$, $\gamma_\zeta(0) = 0$, $\gamma_\zeta(1) = 1$, $\gamma_\zeta(2) = 2$, $\gamma_\zeta(3) = 4$, and $\gamma_\zeta(4) = 6$. ϕ is clearly a bijection of Y with G_1, where G_σ denotes the set of configurations $\eta \in G$ with $\eta(1) = \sigma$; we write $\phi^{-1} : G_1 \to Y$ for the inverse of this bijection.

Suppose now that $\hat\mu \in \mathcal{M}(Y)$ and that $\hat\mu$ has density $\hat\rho$. $\phi_*\hat\mu$ cannot be TI, since it is supported on G_1. However, ϕ does give rise to a map $\Phi : \mathcal{M}(Y) \to \mathcal{M}(G)$, obtained as follows. Write $G_1 = G_{10} \cup G_{11}$, where $G_{\sigma\sigma'} := \{\eta \in G \mid \eta(1) = \sigma, \eta(2) = \sigma'\}$. $G_0 = G \setminus G_1$ is just the translate $\tau^{-1}G_{10}$, and for $\hat\mu \in \mathcal{M}(Y)$, $\Phi(\hat\mu)$ on G_0 should be just the translate of $\phi_*\hat\mu\big|_{G_{10}}$. This leads us to define

$$\Phi(\hat\mu) := \rho\Big(\phi_*\hat\mu + \tau_*^{-1}\phi_*\hat\mu\big|_{G_{10}}\Big), \tag{4.2}$$

where $\rho = 1/(2-\hat\rho)$ is a normalizing factor. $\Phi(\hat\mu)$ is a TI probability measure of density ρ and Φ is a bijection with inverse $\Phi^{-1}(\mu) = \mu(G_1)^{-1}\phi_*^{-1}\big(\mu\big|_{G_1}\big)$. Moreover, Φ preserves convex combinations and this, with the invertibility of Φ, implies that $\hat\mu$ is ergodic (i.e., extremal) if and only if $\Phi(\hat\mu)$ is. Finally, as we shall see in Theorem 4.5 below, Φ also commutes with the time evolutions in the two systems.

Remark 4.2. As noted in Section 3, the TIS states of the F-ASEP at low density, $0 < \rho < 1/2$, are precisely the TI measures supported on F. Such measures may be obtained by defining a map from X to F via the substitutions $0 \to 0$, $1 \to 01$, in parallel with (4.1); then as with (4.2) we obtain a bijective correspondence between the set of all TI measures on X of density $\bar\rho$ and the set of TI measures on F of density $\bar\rho/(1+\bar\rho)$.

For $\zeta \in Y$ we let $K_\zeta = \zeta^{-1}(1)$ be the set of particle locations in ζ, and let $(k_i)_{i\in\mathbb{Z}}$ be an ordered enumeration of K_ζ ($k_i < k_{i'}$ if $i < i'$). If $\eta \in G$ is an F-ASEP configuration then we will refer to certain particles in η as *true* particles; the true particles are those which are immediately followed by another particle. The mapping γ_ζ, when restricted to K_ζ, then gives a bijective correspondence between the particles in the ASEP configuration ζ and the true particles in $\phi(\zeta)$. Note that if $\eta \in G$ is any F-ASEP configuration and there exists an ordered enumeration $(k'_i)_{i\in\mathbb{Z}}$ of the sites of the true particles in η satisfying

$$k'_{i+1} - k'_i = \gamma_\zeta(k_{i+1}) - \gamma_\zeta(k_i) \big(= 2(k_{i+1} - k_i) - 1\big) \tag{4.3}$$

for all i, then η is a translate of $\phi(\zeta)$.

The idea behind the coupling is to establish the correspondence between particles in the ASEP and true particles in the F-ASEP at time 0, through γ_{ζ_0}, and then to maintain this correspondence as the configurations evolve. As a preliminary we introduce a minor modification of the F-ASEP dynamics: we keep the SAPPs $\big((i,t_j^{(i,r)})\big)_{j=1,2,\ldots}$ and $\big((i,t_j^{(i,l)})\big)_{j=1,2,\ldots}$ introduced in Section 2 through (2.2), but replace the exchanges which they trigger by exchanges corresponding to those in the ASEP. Thus at a time $t=t_j^{(i,r)}$ an exchange occurs only if $\eta_{t-}(i\,{:}\,i+2)=1\,1\,0$, and then the true particle at i exchanges with the pair $1\,0$ to its right, yielding $\eta_t(i\,{:}\,i+2)=1\,0\,1$. Similarly for $t=t_j^{(i,l)}$: if $\eta_{t-}(i-2\,{:}\,i+1)=1\,0\,1\,1$ then the true particle at i exchanges with the pair to its left, yielding $\eta_t(i-2\,{:}\,i+1)=1\,1\,0\,1$. It is clear that these are the same exchanges which took place in the earlier formulation of the dynamics, although triggered by Poisson times associated with different sites, so that the process defined in this way is the same as the F-ASEP process defined earlier. A formal proof of this is easily given.

Theorem 4.3. *For any $\zeta_0\in Y$ there exists a process (ζ_t,η_t), with state space $Y\times G$, such that ζ_t is the ASEP process with initial configuration ζ_0 and η_t is the F-ASEP process with initial configuration $\eta_0=\phi(\zeta_0)$. Moreover, for all t, η_t is a translation of $\phi(\zeta_t)$.*

Proof. We regard $K:=K_{\zeta_0}$, the set of initial ASEP particle positions, as a set of labels for the ASEP particles, and keep these labels as the particles move to different sites. K also labels the true particles in the F-ASEP through the map γ_{ζ_0}. We will obtain the coupled dynamics from a set of *Particle Associated Poisson Processes* (PAPPs), defined on the probability space $(\overline{\Omega},\overline{\mathbf{P}}_p^{\hat\mu})$ (compare (2.1)–(2.2)):

$$\begin{aligned}
\overline{\Omega}&=Y\times\overline{\Omega}_0,\quad\text{with}\quad \overline{\Omega}_0=\prod_{k\in K}\big(\overline{\mathcal{T}}^{(k,r)}\times\overline{\mathcal{T}}^{(k,l)}\big),\\
\overline{\mathbf{P}}_p^{\hat\mu}&=\hat\mu\times\overline{\mathbf{P}}_p,\quad\text{with}\quad \overline{\mathbf{P}}_p=\prod_{k\in K}\big(\overline{\lambda}_p^{(k,r)}\times\overline{\lambda}_p^{(k,l)}\big),\\
\overline{\mathcal{T}}^{(k,\#)}&=\Big\{\big((k,\bar t_j^{(k,\#)})\big)_{j=1,2,\ldots}\,\Big|\,0<\bar t_1^{(k,\#)}<\bar t_2^{(k,\#)}\cdots,\ \lim_{j\to\infty}\bar t_j^{(k,\#)}=\infty\Big\},
\end{aligned}\tag{4.4}$$

with $\overline{\lambda}_p^{k,r}$ and $\overline{\lambda}_p^{k,l}$ Poisson processes with rates p and $1-p$, respectively. We take the initial measure $\hat\mu$ to be δ_{ζ_0} and write $\overline{\mathbf{P}}_p^{(\zeta_0)}$ rather than $\overline{\mathbf{P}}_p^{\delta_{\zeta_0}}$.

Given a specific realization of the PAPPs we define the corresponding space-time configuration $(\zeta,\eta) = \big((\zeta_t(i),\eta_t(i))\big)_{t\in[0,\infty),i\in\mathbb{Z}}$ as follows (see Remark 2.1). For each $N\in\mathbb{N}$ we let $I^{(N)} := [k_1^{(N)},k_2^{(N)}]$ be the minimal interval for which (i) $k_1^{(N)},k_2^{(N)}\in K$, (ii) $k_1^{(N)}\le 0<k_2^{(N)}$, and (iii) no Poisson events occur for particles $k_1^{(N)}$ and $k_2^{(N)}$ during the time interval $[0,N]$. Formally, the particles $k_1^{(N)}$ and $k_2^{(N)}$ are inactive during the time interval $[0,N]$ and thus insulate the sites in $I^{(N)}$ from outside influence during this time interval. Note that $I^{(N)}$ exists a.s. and that clearly $I^{(N)}\subset I^{(N+1)}$ and $I_N\nearrow\mathbb{Z}$ a.s.

The next step is to define the space-time configuration $(\zeta^{(N)},\eta^{(N)})$ on $[0,N]\times I^{(N)}$: $(\zeta^{(N)},\eta^{(N)}) = \big((\zeta_t^{(N)}(i),\eta_t^{(N)}(i))\big)_{t\in[0,N],i\in I^{(N)}}$. The definition is such that $\zeta_0^{(N)} = \zeta_0\big|_{I^{(N)}}$ and that the restriction of $(\zeta^{(N+1)},\eta^{(N+1)})$ to $[0,N]\times I^{(N)}$ is $\zeta^{(N)}$. Once this is done we define $(\zeta,\eta)=\lim_{N\to\infty}(\zeta^{(N)},\eta^{(N)})$, where of course the limit exists trivially.

To define $\zeta^{(N)}$ we first specify that particles with labels $k_1^{(N)}$ and $k_2^{(N)}$ (in either the ASEP or F-ASEP) stay at their initial position through time N: $\zeta_t^{(N)}(k_1^{(N)})=\zeta_t^{(N)}(k_2^{(N)})=1$ and $\eta_t(\gamma_{\zeta_0}(k_1^{(N)}))=\eta_t(\gamma_{\zeta_0}(k_2^{(N)}))=1$ for $0\le t\le N$. Next, note that the set of Poisson events $(k,\overline{t}_j^{(k.\#)})$ with $k_1^{(N)}<k<k_2^{(N)}$, $\overline{t}_j^{(k,\#)}\in[0,N]$, and $\#=r$ or l, is a.s. finite. Taking these events in their time order, we specify that the particles move as follows:

- for an event $(k,\overline{t}_j^{(k.r)})$: if at time $\overline{t}_j^{(k.r)}-$ the site to the right of the ASEP particle with label k is empty, then that particle moves to its right, and the F-ASEP particle with label k exchanges with the 1 0 pair on its right (as in the modified F-ASEP dynamics above), and
- for an event $(k,\overline{t}_j^{(k.l)})$: if at time $\overline{t}_j^{(k.l)}-$ the site to the left of the ASEP particle with label k is empty, then that ASEP particle moves to its left and the F-ASEP particle with label k exchanges with the 1 0 pair to its left.

It is clear intuitively that the first and second components of this process are respectively the ASEP and F-ASEP as defined earlier using the SAPPs; we give a formal proof of this in Appendix A. Moreover, one checks easily that if $K_{\zeta_t}=(k_i)_{i\in\mathbb{Z}}$ and the set of true particles in η_t is $(k_i')_{i\in\mathbb{Z}}$ then (4.3) is satisfied, so that η_t is a translate of $\phi(\zeta_t)$. ■

For the next main result we need a lemma. Let $\hat{S}(t)$ and $S(t)$ be the evolution operators for the ASEP and F-ASEP, respectively (see Section 2).

Lemma 4.4. *$S(t)$ and $\hat{S}(t)$ preserve ergodicity.*

Proof. We prove the lemma for $S(t)$; the proof for $\hat{S}(t)$ is the same. The lemma follows from two elementary observations: (i) a covariant image of an ergodic measure is ergodic (for our purposes here, a covariant image of a measure P is a measure f_*P, where f commutes with translations); and (ii) the product of an ergodic dynamical system with one that is weakly mixing is ergodic. (i) is trivial; (ii) is a well-known fact that the reader can easily verify (or find in [13]).

The lemma then follows from the observation that for any measure μ on X we have that $\mu S(t) = \eta_{t*}\mathbf{P}_p^\mu$ (see (2.1)). (Here it is irrelevant whether η_t is defined on Ω using the original jump rule or the modified one.) Since $\mathbf{P}_p^\mu = \mu \times \mathbf{P}_p$ with the product measure $\mathbf{P}_p$ mixing and hence weakly mixing, (i) and (ii) imply that $\mu S(t)$ is ergodic if μ is. ∎

The idea of the proof of the following result is taken from [10]:

Theorem 4.5. *(a) For any TI measure $\hat{\mu}$ on Y, $\Phi(\hat{\mu})S(t) = \Phi\bigl(\hat{\mu}\hat{S}(t)\bigr)$.*
(b) Φ is a bijection of the TIS measures for the ASEP and F-ASEP systems.

Proof. (b) is an immediate consequence of (a) and the remark above that Φ is a bijection of TI measures, and clearly it suffices to verify (a) for $\hat{\mu}$ ergodic. Let us write $\nu_t := \Phi(\hat{\mu})S(t)$ and $\widetilde{\nu}_t := \Phi\bigl(\hat{\mu}\hat{S}(t)\bigr)$. Since $S(t)$ and $\hat{S}(t)$ preserve ergodicity, as does Φ, ν_t and $\widetilde{\nu}_t$ are ergodic, so that these two measures are either equal or mutually singular. Hence to prove their equality it suffices to find a nonzero TI measure λ_t with $\lambda_t \le \nu_t$ and $\lambda_t \le \widetilde{\nu}_t$, where for measures α, β we write $\alpha \le \beta$ if $\alpha(C) \le \beta(C)$ for every measurable set C.

Let π_Y and π_G be the projections of $Y \times G$ onto its first and second components, respectively, and let κ_t be the measure on $Y \times G$ giving the distribution of (ζ_t, η_t); then $\pi_{Y*}\kappa_t = \hat{\mu}\hat{S}(t)$ and $\pi_{G*}\kappa_t = \nu_t$. From Theorem 4.3 there is an $m \in \mathbb{Z}$ such that $\kappa_t(B_m) > 0$, where $B_m = \{(\zeta, \eta) \in Y \times G \mid \eta = \tau^m \phi(\zeta)\}$. Let $\widetilde{\lambda}_t = \rho\phi_*\pi_{Y*}(\chi_{B_m}\kappa_t)$ and $\lambda_t = \rho\pi_{G*}(\chi_{B_m}\kappa_t)$, with χ_{B_m} the characteristic function of B_m. Then we have

$$\widetilde{\lambda}_t \le \rho\phi_*\pi_{Y*}\kappa_t = \rho\phi_*(\hat{\mu}\hat{S}(t)) \le \Phi(\hat{\mu}\hat{S}(t)) = \widetilde{\nu}_t \tag{4.5}$$

where we have used (4.2), and

$$\lambda_t \le \rho\pi_{G*}\kappa_t = \rho\nu_t \le \nu_t. \tag{4.6}$$

But by the definition of B_m, (4.5), and the translation invariance of $\widetilde{\nu}_t$,

$$\lambda_t = \tau_*^m \widetilde{\lambda}_t \le \tau_*^m \widetilde{\nu}_t = \widetilde{\nu}_t. \tag{4.7}$$

Since λ_t is clearly not zero, by the choice of m, the result follows. ∎

Now we can prove our main result:

Proof of Theorem 4.1. The Theorem is an immediate consequence of Theorem 4.5(b) and the fact [17] that there is a unique TIS (the Bernoulli measure) for the ASEP. ∎

Remark 4.6. The ASEP system also has non-TI stationary states as long as $p \neq 1/2$, that is, as long as there is a true asymmetry [17]. We conjecture that this is also true for the F-ASEP, but we do not have a proof.

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A An equivalence result

The lemma proved in this appendix is essentially a completion of the proof of Theorem 4.3, and we will adopt the notation of that proof; in particular, we let ζ_0 be the initial ASEP configuration and write $K := K_{\zeta_0}$.

Remark A.1. It will be convenient to use a representation of the sample points for the SAPP and PAPP which is different from that of (2.1)–(2.2) and (4.4). If $\omega \in \Omega_0$ then, from (2.2), $\omega = \bigl((\omega^{(i,r)}, \omega^{(i,l)})\bigr)_{i\in\mathbb{Z}}$, with $\omega^{(i,\#)}$ a sequence $\bigl((i, t_j^{(i,\#)})\bigr)_{j\in\mathbb{N}}$, and we may identify ω with a set of labeled (by r or l) *Poisson points* in $\mathbb{Z}\times\mathbb{R}_+\times\{r,l\}$: $\omega \sim \{(i, t_j^{(i,\#)}, \#)\}$. For given ω we may and do assume that the Poisson times $t_j^{(i,\#)}$ are all different and all nonintegral, since this is true $\mathbf{P}_p$-a.s. (this avoids discussion of irrelevant special cases). A similar representation $\overline{\omega} \sim \{(k, \overline{t}_j^{(k,\#)}, \#)\}$, with $k \in K$, holds for $\overline{\omega} \in \overline{\Omega}_0$. We say that the PAPP point $(k, \overline{t}, \#)$ is *located at* $(i, \overline{t})$ if particle k is at site i at time $\overline{t}-$.

Lemma A.2. *The SAPP and PAPP definitions of the ASEP and F-ASEP processes are equivalent.*

Proof. The proofs for the two processes are completely parallel; to be definite we will consider the ASEP. We let ζ_t and $\bar{\zeta}_t$ be respectively the SAPP and PAPP ASEP processes, each with initial configuration ζ_0. The probability spaces for these processes, $(\Omega_0, \mathbf{P}_p)$ and $(\overline{\Omega}_0, \overline{\mathbf{P}}_p)$, are defined in (2.1), (2.2), and (4.4); see also Remark A.1. We define a map $\Psi : \Omega_0 \to \overline{\Omega}_0$ as follows: with the point $\omega \in \Omega_0$ is associated a well-defined space-time history $\zeta(\omega) = \bigl(\zeta_t(\omega, i)\bigr)_{(i,t)\in\mathbb{Z}\times\mathbb{R}_+}$, and hence, for each particle $k \in K$, a well defined space-time trajectory. We define $\Psi(\omega)$ by the condition that $(k, t, \#)$ is a PAPP point of $\Psi(\omega)$ iff $(i, t, \#)$ is a SAPP point of ω which lies on the closure of this trajectory. Then clearly $\zeta_t = \bar{\zeta}_t \circ \Psi$ for all t, and the lemma will follow once we show that Ψ has the correct distribution, that is, that $\Psi_*\mathbf{P}_p = \overline{\mathbf{P}}_p$.

We verify this by defining, for each $N \in \mathbb{N}$, a certain approximation $\Psi^{(N)}$ of Ψ. As a preliminary, let $J^{(N)} = [j_1^{(N)}, j_2^{(N)}]$ denote the interval defined in parallel with the construction of $I^{(N)}$ in the proof of Theorem 4.3, but using the SAPP rather than the PAPP and ensuring that $J^{(N)} \supset [-N, N]$: $J^{(N)}$ is the minimal interval for which $j_1^{(N)}, j_2^{(N)} \in K$, $j_1^{(N)} \le -N < N \le j_2^{(N)}$, and no Poisson events occur in the SAPP process for sites $j_1^{(N)}$ and $j_2^{(N)}$ during the time interval $[0, N]$. Let $A^{(N)}$ be the set of SAPP points $(i, t, \#)$ of ω with $(i, t) \in J^{(N)} \times [0, N]$. $A^{(N)}$ is a.s. finite; we let $m^{(N)}$ denote the number of points in $A^{(N)}$, and index these points as $(i_n, t_n, \#_n)_{n=1,\ldots,m^{(N)}}$ with $0 < t_1 < \cdots < t_{m^{(N)}} < N$. By convention we take $t_0 = 0$.

We now construct recursively a sequence $\bigl(\psi^{(N,n)}\bigr)_{n=0,1,\ldots}$ of maps $\psi^{(N,n)} : \Omega_0 \to \overline{\Omega}_0$; $\psi^{(N,n)}(\omega)$ will be independent of n for $n \ge m^{(N)}(\omega)$ and $\Psi^{(N)}$ will then be defined by $\Psi^{(N)} := \psi^{(N,m^{(N)})}$. We first take $\psi^{(N,0)}(\omega)$ to be such that, for each particle $k \in K$, $(k, \bar{t}, \#)$ is a PAPP point of $\psi^{(N,0)}(\omega)$ if and only if it is a SAPP point of ω. Suppose then that we have defined $\psi^{(N,n-1)}(\omega)$. To define $\psi^{(N,n)}(\omega)$ we suppose first that $n \le m^{(N)}(\omega)$, consider the SAPP point $(i_n, t_n, \#_n)$, and let i'_n denote the target site to which a particle at site i_n *might* jump at time t_n: $i'_n = i_n + 1$ if $\#_n = r$ and $i'_n = i_n - 1$ if $\#_n = l$. If (in the SAPP process) either there is no particle at site i_n at time t_n-, or the target site i'_n is occupied at time t_n-, then we define $\psi^{(N,n)}(\omega) = \psi^{(N,n-1)}(\omega)$. Otherwise, the particle in the SAPP process at site i_n at t_n-, say particle k, jumps to site i'_n in the SAPP process, and $\psi^{(N,n)}(\omega)$ is defined to have the same Poisson points as $\psi^{(N,n-1)}(\omega)$, *except* that we replace the (labeled) times of the PAPP points for particle k which lie in the future of t_n with the times of the SAPP points for site i'_n which lie in the future of t_n: for $\overline{t} > t_n$, $(k, \overline{t}, \#)$ is a PAPP point of $\psi^{(N,n)}(\omega)$ if and only if $(i'_n, \overline{t}, \#)$ is a SAPP point of ω. Continuing in this way we define $\psi^{(N,0)}(\omega), \ldots, \psi^{(N,m^{(N)})}(\omega) =: \Psi^{(N)}(\omega)$. Finally, if $n \geq m^{(N)}(\omega)$ we take $\psi^{(N,n)}(\omega) = \psi^{(N,m^{(N)})}(\omega)$.

We next show that the $\psi^{(N,n)}$ satisfy (P1)–(P3) below:

(P1) For $\omega \in \Omega_0$, $N \in \mathbb{N}$, and $0 \leq n \leq m^{(N)}$, the locations of the set of PAPP points $(k, \overline{t}, \#)$ of $\psi^{(N,n)}(\omega)$ with $(k, \overline{t}) \in J^{(N)} \times [0, t_n]$ coincide with the PAPP points of $\Psi(\omega)$ satisfying the same restrictions.

(P2) For $\omega \in \Omega_0$, $N \in \mathbb{N}$, $0 \leq n \leq m^{(N)}$, and particle $k \in J^{(N)}$, if k is located at site i at time t_n then for $\# = l, r$ the locations of the set of PAPP points $(k, \overline{t}, \#)$ of $\psi^{(N,n)}(\omega)$ with $\overline{t} > t_n$ coincide with the set of SAPP points $(i, t, \#)$ of ω with $t > t_n$.

(P3) $\psi^{(N,n)}_* \mathbf{P}_p = \overline{\mathbf{P}}_p$ for all n.

(P1)–(P3) are trivially satisfied for $n = 0$; to verify them for general n we argue recursively.

First, (P1) for $\psi^{(N,n-1)}$ implies that (P1) holds for $\psi^{(N,n)}$, except possibly for PAPP points in $J^{(N)} \times (t_{n-1}, t_n]$. Since there are no SAPP points for ω in $J^{(N)} \times (t_{n-1}, t_n)$, and hence no PAPP points for either $\psi^{(N,n)}(\omega)$ or $\Psi(\omega)$ located in this region, it remains to show that either (i) no PAPP point for either $\psi^{(N,n)}(\omega)$ or $\Psi(\omega)$ is located at $(i_n, t_n, \#_n)$, or (ii) a PAPP point $(k, t_n, \#_n)$ for both is located there. It is clear that (i) holds if no particle is located at (i_n, t_n-). On the other hand, if particle k is located at (i_n, t_n-), then certainly $(k, t_n, \#_n)$ is a PAPP point of $\Psi(\omega)$; moreover, k must also be located at (i_n, t_{n-1}), so that $(k, t_n, \#_n)$ is a PAPP point of $\psi(N, n-1)(\omega)$ from (P2) for $\psi^{(N,n-1)}(\omega)$, and so also of $\psi^{(N,n)}(\omega)$, since a jump at time t_n only changes the PAPP points in the future of t_n.

Second, (P2) for $\psi^{(N,n)}(\omega)$ follows from (P2) for $\psi^{(N,n-1)}(\omega)$ and the observation that if particle k jumps at time t_n then the change in the PAPP points of this particle which takes place in passing from $\psi^{(N,n-1)}(\omega)$ to $\psi^{(N,n)}(\omega)$ is precisely what is needed to maintain (P2).

Finally, we verify (P3) for $\psi^{(N,n)}$, assuming (P3) for $\psi^{(N,n-1)}$. We will consider conditional measures $\mathbf{P}_p(\cdot \mid \mathcal{Q})$, where $\mathcal{Q}$ is specified by certain events and/or values of certain random quantities, and the family of all such

$\mathcal{Q}$'s forms a partition of Ω_0. The $\mathcal{Q}$'s which we use will be specified during the course of the proof. For each $\mathcal{Q}$ which arises we will show that

$$\psi_*^{(N,n)}\mathbf{P}_p(\cdot \mid \mathcal{Q}) = \psi_*^{(N,n-1)}\mathbf{P}_p(\cdot \mid \mathcal{Q}). \tag{A.1}$$

Integrating (A.1) against the marginal $\mathbf{P}_p(d\mathcal{Q})$ yields $\psi_*^{(N,n)}\mathbf{P}_p = \psi_*^{(N,n-1)}\mathbf{P}_p$, which with (P3) for $\psi^{(N,n-1)}$ yields (P3) for $\psi^{(N,n)}$. As a first step, let $\mathcal{Q}_0$ be the event that $m^{(N)} \geq n$. On the complimentary event $\mathcal{Q}_0^c$, $\psi^{(N,n)} = \psi^{(N,n-1)}\big(= \psi^{(N,m^{(N)})}\big)$, so that (A.1) holds trivially with $\mathcal{Q} = \mathcal{Q}_0^c$.

Next, we let $\mathcal{Q}_1$ be defined by specifying, in addition to $\mathcal{Q}_0$, values of the interval $J^{(N)}$, of the time t_n (which is well-defined on $\mathcal{Q}_0$), and of the entire past of t_n, including in particular the values of all $(i_{n'}, t_{n'}, \#_{n'})$ with $n' \leq n$. For $i \in \mathbb{Z}$ let S_i be the set of Poisson points $(i, t, \#)$ at site i in the future of t_n. We can describe the joint distribution of the sets S_i under $\mathbf{P}_p(\cdot \mid \mathcal{Q}_1)$ in terms of the measure $\kappa_u^{(i)}$ defined, for $u \geq 0$, to be the translate by u of the measure $\lambda^{(i,r)} \times \lambda^{(i,l)}$ (see (2.1)): (i) the S_i, $i \in \mathbb{Z}$, are independent; (ii) S_i is distributed as $\kappa_{t_n}^{(i)}$ if either (ii.a) $i \notin J^{(N)}$, (ii.b) $i \in [-N, N]$, (ii.c) $i \notin K$, or (ii.d) $i = i_{n'}$ for some $(i_{n'}, t_{n'}, \#_{n'})$ with $n' \leq n$; (iii) S_i has no points in $(t_n, N]$ and on (N, ∞) is distributed as $\kappa_N^{(i)}$, if $i = j_1^{(N)}$ or $i = j_2^{(N)}$; (iv) S_i is distributed as the conditional distribution of $\kappa_{t_n}^{(i)}$, given that there is at least one point in (t_n, N), otherwise.

Now conditioning on $\mathcal{Q}_1$ determines whether or not a jump takes place at time t_n; let $\mathcal{Q}_1'$ and $\mathcal{Q}_1''$ be $\mathcal{Q}_1$ with the additional restriction that the jump respectively does or does not take place. Under $\mathcal{Q}_1''$, $\psi^{(N,n)} = \psi^{(N,n-1)}$, so that (A.1) holds with $\mathcal{Q} = \mathcal{Q}_1''$. On the other hand, under $\mathcal{Q}_1'$, some particle k will jump from site i_n to i_n'; let $\mathcal{Q}_2$ be obtained by specifying $\mathcal{Q}_1'$ together with values of all the sets S_i for $i \neq i_n, i_n'$. Consider then (A.1) with $\mathcal{Q} = \mathcal{Q}_2$; the left side of this equation is obtained from the right by the replacement of the (labeled) times of the PAPP points for particle k which lie in the future of t_n—and, by (P2), these are just the times of S_{i_n}—with the times of $S_{i_n'}$ lying in that same future. But S_{i_n} and $S_{i_n'}$ have distributions $\kappa_{t_n}^{(i_n)}$ and $\kappa_{t_n}^{(i_n')}$ under $\mathbf{P}_p(\cdot \mid \mathcal{Q}_1)$ and hence, by the independence noted in (i) above, under $\mathbf{P}_p(\cdot \mid \mathcal{Q}_2)$; this is because i_n falls under case (ii.d), and i_n' under either case (ii.c) or case (ii.d), of the previous paragraph. Since $\kappa_{t_n}^{(i_n)}$ and $\kappa_{t_n}^{(i_n')}$ agree, this verifies (A.1) for $\mathcal{Q} = \mathcal{Q}_2$ and completes the verification of (P1)–(P3) for $\psi^{(N,n)}$.

To complete the proof of the lemma, observe that (P1), together with the fact that there are no SAPP points of ω in $J^{(N)} \times (t_{m^{(N)}}, N]$ and hence no PAPP points of either $\Psi^{(N)}(\omega) = \psi^{(N,m^{(N)})}(\omega)$ or $\Psi(\omega)$ located there, implies that the the set of PAPP points $(k, \bar{t}, \#)$ of $\Psi^{(N)}(\omega)$ which satisfy $-N \leq k \leq N$ and $0 \leq \bar{t} \leq N$ coincides with the corresponding set of PAPP points of $\Psi(\omega)$. By (P3), then, the marginal distribution of PAPP points of Ψ in this region is distributed as the marginal of $\overline{\mathbf{P}}_p$. Since N is arbitrary, we can conclude that $\Psi_*\mathbf{P}_p = \overline{\mathbf{P}}_p$. ■

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