CHAPTER 3

Deterministic and Stochastic Hydrodynamic Equations Arising From Simple Microscopic Model Systems

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1. Introduction

Nature has a hierarchical structure with strongly separated levels. This allows us, in many cases, to treat the higher levels (almost) independently of the lower ones. For example, the hydrodynamic laws which describe, with a high degree of accuracy, the large scale behavior of fluids were discovered before and are to a large extent independent of the detailed microscopic structure of matter.

These laws generally take the form of autonomous nonlinear partial differential equations,

$$\frac{\partial}{\partial t}\mathbf{M}(\mathbf{r},t) = \mathbf{F}(\mathbf{M})(\mathbf{r},t), \tag{1.1}$$

where $\mathbf{M}(\mathbf{r},t)$ denotes an appropriate set of macroscopic variables depending on space and time and the structure of $\mathbf{F}$ in (1.1) depends in general only on the phenomena considered and not on the nature of the microscopic constituents of the macroscopic objects studied. Thus, Fourier's law of heat conduction has the same form for solid gold and liquid water and the same Navier–Stokes equations describe the flow of air and the flow of water: the details of microscopic structure generally enter in $\mathbf{F}$ only through some parameters, e.g. heat conductivity, viscosity.

The origin of such generic hydrodynamic laws is a consequence of the existence of very large separation between the spatial and temporal scales of microscopic and macroscopic phenomena and some very general features of the microscopic dynamics. Chief among these are the approximate locality and additivity of the microscopic interactions and the resulting law of large numbers for the macroscopic fields. Consequently, our microscopic models can be rather crude, even blatantly wrong, and still give rise to correct macroscopic equations. All that is necessary is that the models contain the essential features responsible for the phenomena of interest.

The apparent robustness or universality of certain equations of type (1.1) is really quite remarkable, extending far beyond the description of *simple* macroscopic systems. Reaction–Diffusion equations of similar form can describe both chemically reacting molecular mixtures and the propagation of genetic traits in biological populations. The main aim of the present work is to discuss stochastic improvements to the deterministic description provided by (1.1).

The possibility and (occasional) need for improvement of this description comes from the fact that the different hierarchical levels are of course not completely isolated – there is no sharp demarcation line between an atomic beam and the jet stream. In fact, one of the basic dogmas of science is that the behavior at any level can be deduced, at least in principle, entirely from the dynamics of the level below it, i.e. there are no new physical laws, only new phenomena, as one goes from atoms to fluids to galaxies. The existence of the lower level will manifest itself in the apparently random fluctuations in the macroscopic fields about the solutions of (1.1): *the quantities fluctuate*, as one says (see [**73**], p.344). While these fluctuations are generally very small on the macroscopic scale in cases where the solutions of (1.1) are smooth and stable, they may become macroscopically relevant in cases where these solutions are unstable, singular, or non unique. There is thus a practical motivation for improving the deterministic equations in situations where the hydrodynamic laws fail, partially or entirely, to describe fully the phenomena of interest: the conceptual and mathematical motivations are of course not limited to such cases.

An *improved* version of (1.1) is often written in the form

$$\frac{\partial}{\partial t}\mathbf{M}(\mathbf{r},t) = \mathbf{F}^{\epsilon}(\mathbf{M})(\mathbf{r},t), \tag{1.2}$$

in which $\mathbf{F}^{\epsilon}$ is dependent on a small positive parameter ϵ, representing the contribution of the finer scale, and $\mathbf{F}^{\epsilon}$ approaches $\mathbf{F}$ as ϵ goes to zero. A frequent choice for $\mathbf{F}^{\epsilon}$, based mostly on heuristic considerations, is $\mathbf{F}^{\epsilon} = \mathbf{F} + \epsilon\omega(\mathbf{r},t)$, in which $\omega(\mathbf{r},t)$ is of white noise type (both in space and time) or the gradient, in the spatial variable, of a white noise. Examples (in which there is in fact no explicit small parameter ϵ) include the *fluctuating hydrodynamic theory* of Landau and Lifshitz (see [**73**], p.524) and the fluctuating Boltzmann equation, [**95**]. In these examples the covariances of the fluctuating terms are chosen on the basis of equilibrium statistical mechanical considerations. These equations are quite successful in predicting the general pattern of behavior occurring near the onset of instabilities or bifurcations. Most of these considerations follow however by considering linearized versions of the original PDE's. This is just as well, since generally these type of noise terms make the nonlinear problem mathematically ill–posed.

To help give mathematical meaning to such SPDE's we shall here review their origin in simple models in which the transition from microscopic to macroscopic scales and ipso facto the corrections to (1.1) can be investigated with some mathematical rigor. To be more precise we will consider lattice systems with stochastic microscopic dynamics, the so called Interacting Particle Systems (IPS) **[30]**,**[81]**,**[95]**, where the configurations are updated at random (Poisson) times according to some local rules. The importance of models is well established for equilibrium behavior, where the simple *Ising model*, has played a fundamental role in elucidating the nature of *phase transitions*, both from the physical and the mathematical viewpoint. There is unfortunately no comparable model for the more complex non equilibrium phenomena. This is not surprising since even the solutions of the macroscopic hydrodynamic equations are far from being understood in many interesting cases. Nevertheless, computer simulations and even some rigorous analysis have shown that very simple IPS models can capture the essence of large scale hydrodynamic behavior and so we shall assume that their behavior is of physical interest.

The IPS dynamics are divided into two main groups: the Glauber (or *spin–flip*) dynamics, which does not conserve the sum of the occupation variables (or any other analogous quantity), and the Kawasaki (or *exchange*) dynamics, which has at least one (particle) conservation law. For both classes the derivation of macroscopic behavior from microscopic model systems via hydrodynamical scaling limits is currently an active field of research. There are reviews of this work to which we refer the reader for background **[30]**,**[95]**.

1.1. The Micro–Macro Connection. The hydrodynamic limit (or hydrodynamic scaling limit: HSL) gives a reduced description of the collective behavior of some particle systems which involves only the slowly varying fields, e.g. those that satisfy local conservation laws: generally the same ones that describe the thermodynamic equilibria of the system. The hydrodynamic equations are obtained mathematically by suitable space-time scaling limits; we have a scaling parameter ϵ such that as $\epsilon \to 0^+$ the evolution converges, in a sense to be specified, to some autonomous deterministic limit laws of the form (1.1). Corrections may then be introduced to describe *physically important* effects present in the system when $\epsilon > 0$ which are lost after the limit and thus no longer present in the PDE. We shall always take ϵ to be the ratio of microscopic to macroscopic length scales ($\epsilon \sim 10^{-10} - 10^{-5}$ in many typical physical situations). The ratio of the macroscopic time scale to the microscopic one will be $\epsilon^{-\alpha}$, with α between 1 and 4 in the problems considered in this chapter; $\alpha = 1, 2$ are generally referred to as respectively the Euler and the diffusive scale. The HSL leaves out many interesting features of the IPS. We enumerate some of these here.

1). Small fluctuations in the hydrodynamical variables about their deterministic values provided by the solutions of (1.1). These will always be present since, by definition, the description given by the HSL is only valid when $\epsilon \to 0$, while in physical situations ϵ is always finite. These fluctuations have been studied most extensively for systems in equilibrium, when the right hand side of (1.1) vanishes. In the case of systems with unique equilibrium states (away from the critical point), the situation is analogous to the behavior of sums of (approximately) independent random variables. The deterministic behavior corresponds to the law of large numbers while the fluctuations around stable evolutions obey the central limit theorem.

2). Deterministic corrections to (1.1) which change the qualitative behavior of the evolution on time scales larger than those on which (1.1) holds. A very striking example of such a situation are the expected corrections, on the time scale ϵ^{-2}, to the Euler equations of hydrodynamics, which give the correct behavior on the time scale ϵ^{-1} **[38]**,**[24]**. These correction terms, containing the viscosity and heat conductivity, lead to the Navier-Stokes equations **[73]**, whose solutions may have a behavior which is very different from that of the Euler equations. While the time evolution described by the Euler equations is, in the absence of shocks, entirely reversible, that described by the Navier-Stokes equations is time asymmetric, leading, in the absence of external forcing, to a final equilibrium independent of the initial conditions. We shall generally refer to such deterministic corrections to (1.1), leading to deterministic equations of form (1.2), as *Navier-Stokes corrections.* The rigorous derivation of such corrections for IPS has been achieved so far only in some very special cases, corresponding to the incompressible Navier–Stokes case (**[38]**,**[39]**,**[74]**). We shall discuss some such examples later.

3). Stochastic corrections to (1.1) which go beyond that of small fluctuations discussed in 1). These concern mostly cases in which the solutions of (1.1) are not linearly stable. They correspond in many ways to phase transitions and critical phenomena in equilibrium statistical mechanics. These corrections lead to the most difficult and also most interesting questions regarding the utility of SPDE's for improving the HSL. Unfortunately all we shall be able to do in most cases is raise questions rather than give answers.

1.2. Outline of the Chapter. The outline of the rest of the chapter is as follows. Part I (Sections 2 to 5) deals with asymmetric non-reversible models. In Section 2 we will introduce the HSL by starting from a *microscopic model* which is itself a deterministic PDE for a scalar field, i.e. the Burgers equation (2.1). The *coarse graining* leads to a hydrodynamic equation (1.1) which in this case is the same Burgers equation but without a viscosity term, (2.2). The corrected equation (1.2) will simply be the viscid Burgers equation (2.1) we started from.

In Section 3 we introduce two IPS, the asymmetric simple exclusion process (ASEP) and, for comparison, the independent particle process (IPP). The hydrodynamic equations for these systems are respectively the inviscid Burgers equation (2.1) and the free streaming equation (3.9). The first question we examine there concerns the Navier–Stokes correction, namely the analysis of effects in the particle system that require the addition of a viscosity term. Following this we discuss the behavior of the shocks, which, for the inviscid Burgers equation, are discontinuities of the density profile. Thus, at the shock, the basic assumption for hydrodynamic behavior, i.e. the slowly varying condition, fails. We will see that at a finer level of description (than the hydrodynamic one) stochastic effects become dominant. These may be explained still using the Burgers equation: the stochasticity will only involve the initial datum, namely the initial random fluctuations cannot be neglected, but those arising from the evolution are, at this order, negligible.

In Section 4 we study the ASEP in the case of weak asymmetry, the so called *weakly asymmetric simple exclusion process* (WASEP). For this system we can obtain, in suitable scalings, corrections that are captured by the addition of a stochastic forcing term to the Burgers equation. Some of the results for the WASEP are translated in Section 5 into the language of growing interfaces [**72**],[**67**]. Such interfaces often display anomalous (nonlinear) fluctuations: the particular model considered here is the *restricted solid on solid* model. The microscopic analogy between the ASEP and an interface model is reflected by the macroscopic analogy between the stochastic Burgers equation and the Kardar–Parisi–Zhang equation [**67**].

In Part II (Sections 6 to 9) we consider IPS with *long range potentials*. These give rise to non trivial translation invariant states of the dynamics, which is now not necessarily conservative. We discuss first, in Section 6, the hydrodynamics, the small fluctuations and the relation between Ornstein-Uhlenbeck processes and *fluctuating hydrodynamic theory* for these systems. We then describe, in Section 7, the critical fluctuations (mostly for $d = 1$), i.e. fluctuations at the critical point, leading to one dimensional Reaction–Diffusion [**93**] and Cahn–Hilliard [**13**] equations with noise. In Section 8 we consider the fluctuations in unstable situations and the macroscopic effects that they can generate: in the context which we consider, the noise will trigger the origin of (random) interfaces. In Section 9 we look briefly at large deviation phenomena as a possible means for establishing the nature of the stochastic corrections to the HSL.

PART I
NON REVERSIBLE DYNAMICAL SYSTEMS: ASYMMETRIC MODELS WITH SHOCKS

2. The Burgers Equation

To make concrete some of the issues discussed in the Introduction we consider in this section the relatively trivial case of a *microscopic model* which is itself described by a deterministic PDE. This is the Burgers equation

$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho(1-\rho) = \nabla \cdot [D \nabla \rho], \tag{2.1}$$

where $\rho = \rho(r,t) \in [0,1]$, $r \in \mathbb{R}^d$, $t \geq 0$, $v \in \mathbb{R}^d$, $v \neq 0$, and $D \geq 0$ a non negative, constant matrix. The equation with $D = 0$

$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho(1-\rho) = 0, \tag{2.2}$$

is called the inviscid Burgers equation. We recall, see for instance [**93**], that the Cauchy problem for (2.2) in general has no global, classical solution, as there are smooth initial data that develop singularities in a finite time, after which the solution is defined only in a weak sense. Uniqueness, which is lost at this stage, may be recovered by restricting to the class of *entropic solutions*. The equation has therefore a rich and intriguing mathematical structure and is of physical interest in fluid mechanics as a model for the formation and propagation of shocks. In the next section we will see that it is naturally related to an important class of stochastic IPS.

We begin with the HSL: while it is usually considered for IPS, the definition extends to several other dynamical systems. In particular we will next regard any of the Burgers equations (2.1) as defining the evolution of a microscopic system to which we want to apply the HSL procedure. Thus the space and time variables in

(2.1), that we denote by x and t, should be now regarded as microscopic coordinates. We then choose an initial datum that depends on a small, positive parameter ϵ and set $\rho(x,0;\epsilon) = \rho_0(\epsilon x)$, ρ_0 a fixed, smooth function on $\mathbb{R}^d$. Thus when ϵ is small $\rho(x,0;\epsilon)$ is a *slowly varying function* of x. Calling $\rho(x,t;\epsilon)$ the solution of (2.1) which starts from $\rho(x,0;\epsilon)$, we write

$$\rho^{(\epsilon)}(r,\tau) := \rho(\epsilon^{-1}r, \epsilon^{-1}\tau; \epsilon), \tag{2.3}$$

With this change of variables in general $\rho^{(\epsilon)}(r,\tau)$ will no longer be a slowly varying function of its arguments, it is then natural to regard $r = \epsilon x$ and $\tau = \epsilon t$ as the macroscopic space and time coordinates and ϵ as the ratio between microscopic and macroscopic units. $\rho^{(\epsilon)}(r,\tau)$ and $\rho(x,t;\epsilon)$ are just the same density profile, but expressed in macroscopic and microscopic coordinates, respectively.

The HSL is, by definition, the limit behavior of $\rho^{(\epsilon)}(r,\tau)$ as $\epsilon \to 0$. The existence of a limit proves that the macroscopic description of the profile becomes less and less sensitive to the actual value of the ratio between micro and macro scales. Thus the limit $\epsilon \to 0$ and the HSL should be regarded as a macroscopic limit. If the evolution is given by (2.1) we have for any test function ϕ and any $\tau \geq 0$

$$\lim_{\epsilon \to 0+} \int_{\mathbb{R}^d} dr\, \rho^{(\epsilon)}(r,\tau)\phi(r) = \int_{\mathbb{R}^d} dr\, \rho(r,\tau)\phi(r), \tag{2.4}$$

where $\rho(r,\tau)$ is the entropic solution of (2.2) starting from $\rho_0(r)$, [**93**].

Notice that the limit (2.4) holds independently of which D is used in (2.1) to define $\rho(x,t;\epsilon)$, namely all the equations (2.1) have the same hydrodynamic limit described by the same inviscid Burgers equation (2.2) (with the prescription of taking its entropic solution, a rule thus justified by this procedure). One of the main questions about hydrodynamic limits is therefore to determine domains of attraction to given hydrodynamic equations: as we will see, the domain of attraction of (2.2) is much larger than the class of evolutions defined by (2.1). Observe that, in particular, the hydrodynamic limit of (2.2) is (2.2) itself. In general the *pure hydrodynamic equations* are those identical to their hydrodynamic limit. They are thus fixed points of the scaling transformation (2.3) and they are at the origin of some scaling phenomena observed in macroscopic systems.

It is clear that for the *microscopic model* we have considered, namely (2.1), there are only Navier-Stokes corrections to the hydrodynamic equation (1.1), which is just (2.2). These corrections are in fact trivial: the analog of (1.2) is here just the same as (2.1) written in macroscopic coordinates. In fact after the substitution (2.3), (2.1) becomes

$$\frac{\partial \rho^{(\epsilon)}}{\partial \tau} + v \cdot \nabla \rho^{(\epsilon)}(1-\rho^{(\epsilon)}) = \epsilon \nabla \cdot [D \nabla \rho^{(\epsilon)}]. \tag{2.5}$$

The HSL procedure has been applied to a large variety of deterministic equations, including cases where they are *far* from their hydrodynamic limits: the analysis is then much more intricate than in the simple case we have considered above. We just mention, very briefly, two of the most interesting examples. In the first one the Boltzmann equation plays the role of the microscopic model and the *true* Euler and Navier Stokes equations, [**14**],[**23**], are its hydrodynamic limits, respectively in the Euler and in the *diffusive–incompressible* scaling limits. The second example is rather different since the microscopic model, in contrast to the previous cases, has no conservation laws. The evolution is defined by the

Allen-Cahn equation

$$\frac{\partial u}{\partial t} = \Delta u - U'(u) \tag{2.6}$$

in $\mathbb{R}^d$, $d \geq 2$ with $U(\cdot)$ a double well, symmetric potential: to be definite let us take $U(u) = u^4/4 - u^2/2$. The HSL procedure, object of many studies in the recent years, see for instance **[41]** and references therein, is applied to initial data that are equal (or approximately equal) to either of the two phases ($u = \pm 1$, the minimizers of $U(u)$) in macroscopically large regions. The diffusive scaling limit gives rise to a motion by mean curvature which describes the macroscopic forces (due to the surface tension) between the two phases at the interface. In the second part of this chapter we will come back to some problems connected to (2.6).

3. The Asymmetric Simple Exclusion and the Independent Particle System

A single random walk is the process on $\mathbb{Z}^d$ where the particle (say at x) waits for an exponential time of mean 1 and then jumps to a site y chosen with probability $P(x,y)$, x, y in $\mathbb{Z}^d$. After that it starts anew from y, independently of the past. We suppose that $P(x,y)$ is translation invariant, $P(x,y) = P(0,y-x)$, nearest neighbor, $P(x,y) = 0$ unless $|x-y| = 1$, and asymmetric:

$$\sum_y P(0,y)y = v \neq 0. \tag{3.1}$$

The ASEP is a Markov process which describes the evolution of random walks on $\mathbb{Z}^d$ which interact only by exclusion: the exclusion rule (hard core interaction) prohibits the occupation of any lattice site by more than one particle. Starting from any configuration satisfying the exclusion rule we stipulate that if the site chosen by a particle for jumping on is already occupied by another particle then that jump is suppressed and the particle stays where it was, waiting for the next attempt. The ASEP is therefore a process on $\{0,1\}^{\mathbb{Z}^d}$; its elements, denoted by $\eta = \{\eta(x),\ x \in \mathbb{Z}^d\}$, are particle configurations, $\eta(x) = 1$ meaning that x is occupied by a particle (particles are here indistinguishable). Let L be the (pre)generator of the process, which is defined by its action over the cylindrical functions $f(\eta)$, i.e. functions which depend only on the values of $\eta(x)$, for x in a finite set:

$$Lf(\eta) = \sum_{x,y} P(x,y)\eta(x)\big(1-\eta(y)\big)[f(\eta^{x,y}) - f(\eta)], \tag{3.2}$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation numbers at x and y, that is

$$\eta^x(y) = \begin{cases} \eta(y) & \text{if } y \neq x \\ -\eta(y) & \text{if } y = x. \end{cases} \tag{3.3}$$

The closeness of the ASEP to the Burgers equation can be readily understood with the help of the following heuristic argument. Let $f(\eta) = \eta(x)$ in (3.2), then

$$L\eta(x) = -\sum_{i=1}^d \big[j_i(x, x+e_i) - j_i(x-e_i, x)\big], \tag{3.4}$$

where e_i is the unit vector along the i-th direction and $j_i(x, x+e_i)$ is the (expected) current through the bond $(x, x+e_i)$:

(3.5)
$$j_i(x, x+e_i) = P(x, x+e_i)\eta(x)\big(1-\eta(x+e_i)\big) - P(x+e_i, x)\eta(x+e_i)\big(1-\eta(x)\big).$$

Under a propagation of chaos ansatz, supported by the fact that the Bernoulli measures (a Bernoulli measure on a product space is a product measure) ν_ρ on $\mathbb{Z}^d$ (with density $\rho \in [0,1]$) are invariant for the ASEP, we get

$$\begin{aligned} E\big(j_i(x, x+e_i, t)\big) &\approx P(x, x+e_i)E\big(\eta(x,t)\big)\big[1-E\big(\eta(x,t)\big)\big] \\ &\quad -P(x+e_i, x)E\big(\eta(x+e_i,t)\big)\big[1-E\big(\eta(x,t)\big)\big]. \end{aligned} \tag{3.6}$$

Assuming that $E\big(\eta(x,t)\big)$ is a slowly varying function of x and keeping only the leading orders we get from (3.4) and (3.6) the inviscid Burgers equation (2.2) with v as in (3.1).

The statement is made precise using the HSL procedure described in the previous section. Let then ρ_0 be a smooth function on $\mathbb{R}^d$ and let the ASEP start from the product measure $\mu^{(\epsilon)}$ on $\{0,1\}^{\mathbb{Z}^d}$ with averages

$$E_{\mu^{(\epsilon)}}\big(\eta(x)\big) = \rho_0(\epsilon x) \qquad x \in \mathbb{Z}^d \tag{3.7}$$

This choice makes the local equilibrium condition (reasonably well) satisfied at time 0, since the Bernoulli measures ν_ρ with constant average ρ are the true equilibria of the ASEP. Let $P_{\mu^{(\epsilon)}}$ be the law of the ASEP starting from $\mu^{(\epsilon)}$, then, recalling (2.3) and (2.4),

THEOREM 3.1. *For any $\delta > 0$, $\tau \geq 0$ and any $\phi \in C_0^\infty(\mathbb{R}^d)$*

$$\lim_{\epsilon\to 0} P_{\mu^{(\epsilon)}}\Big(\Big|\epsilon^d \sum_{x\in\mathbb{Z}^d} \phi(\epsilon x)\eta(x, \epsilon^{-1}\tau) - \int_{\mathbb{R}^d} dr\, \phi(r)\rho(r,\tau)\Big| < \delta\Big) = 1, \tag{3.8}$$

where $\rho(r,\tau)$ is the entropic solution of (2.2) *with v as in* (3.1) *and initial datum ρ_0.*

Theorem 3.1 is proved in **[91]** where more general initial data are also allowed. The proof follows previous works where particular cases had been studied in great detail, see references in **[91]**.

The most remarkable fact about Theorem 3.1 is its validity even after the formation of shocks when the condition of slow variations in the previous heuristic argument fails. The use of the test functions ϕ in the limit (3.8) allows for a coarse graining analysis that avoids a finer examination of the behavior of the system at the shock and leaves open the question of why the system still follows (2.1) even when the slow variation condition is not satisfied. A possible and instructive guess is that the shock is such only at the macroscopic level and that on a finer scale, intermediate between micro and macro, it is resolved into a smooth function. This turns out to be wrong for the ASEP but right for the independent particle process (IPP) where an analogue of Theorem 3.1 holds, as we shall see next.

3.1. The Independent Particle Process: Hydrodynamics and Navier-Stokes Corrections. The IPP is the Markov process on $\mathbb{N}^{\mathbb{Z}^d}$ defined by random walks on $\mathbb{Z}^d$ which evolve independently of each other. The extremal, invariant measures (still denoted by) ν_ρ, $\rho \in \mathbb{R}_+$, are the product measures where each $\eta(x)$ has the Poisson distribution of mean ρ. The initial condition $\mu^{(\epsilon)}$ is then chosen as the product of Poisson distributions with averages $\rho_0(\epsilon x)$. Theorem 3.1 holds with $\rho(r,\tau)$ in (3.8) satisfying

$$\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0. \tag{3.9}$$

No shocks are present here since (3.9) is a linear equation, but we can always simulate a shock by choosing ρ_0 equal to

$$\chi_{\rho_-,\rho_+}(r) = \rho_- \mathbf{1}_{\{r_1 \le 0\}}(r) + \rho_+ \mathbf{1}_{\{r_1 > 0\}}(r), \qquad r_1 = r \cdot e_1,\ \ 0 \le \rho_- < \rho_+. \tag{3.10}$$

We suppose $v \cdot e_1 \neq 0$. The solution of (3.9) with initial datum χ_{ρ_-,ρ_+} is

$$\chi_{\rho_-,\rho_+}(r,t) = \chi_{\rho_-,\rho_+}(r - vt) \tag{3.11}$$

which is the same step function moving rigidly with velocity $v \cdot e_1$.

To determine a (possible) smoothening of the shock, which would be incompatible with (3.9), we need a criterion finer than that used in (3.8). To this end it is convenient to extend the test functions ϕ to include characteristic functions of suitable sets. Let $C^{(\ell)}(x)$ be the cube in $\mathbb{R}^d$ of side ℓ and center x, $|C^{(\ell)}(x)|$ its volume, $\rho(y)$ a function on $\mathbb{R}^d$. We define the *distance* between a configuration η and a density $\rho(y)$, $y \in \mathbb{R}^d$, as

$$d_{\ell,x}(\eta,\rho) = \frac{1}{|C^{(\ell)}(x)|} \left| \sum_{y \in C^{(\ell)}(x) \cap \mathbb{Z}^d} \eta(y) - \int_{C^{(\ell)}(x)} dy\, \rho(y) \right|. \tag{3.12}$$

We use the distance (3.12) to compare the random particle configurations $\eta(\cdot,t)$ and the density $\rho(y,t;\epsilon) = \rho(\epsilon y, \epsilon t)$, where $\rho(r,\tau)$ is the solution of (3.9) starting from ρ_0, i.e. $\rho(y,t;\epsilon)$ is the same $\rho(r,\tau)$, but in microscopic coordinates (recall the discussion in Section 2). It can be proved that for any $\tau > 0$, $\ell > 0$ and $R > 0$

$$\lim_{\epsilon \to 0} \sup_{|x| \le \epsilon^{-1} R} P_{\mu^{(\epsilon)}}\Big(d_{\epsilon^{-1}\ell,x}\big(\eta(\cdot,\epsilon^{-1}\tau), \rho(\cdot,\epsilon^{-1}\tau;\epsilon)\big) < \delta \Big) = 1. \tag{3.13}$$

Without the supremum, taking $x = \epsilon^{-1}\bar r$, $\bar r$ a fixed point in $\mathbb{R}^d$, (3.13) becomes (3.8) with $\phi(r) = \mathbf{1}_{\{|r - \bar r| \le \ell\}}$, see for instance Ch.2 of **[30]**, to which we also refer for the statements below that concern the IPP. We also mention that (3.13) holds as well with the supremum inside the probability.

Let us next go back to the shocks, taking as the initial datum the function χ_{ρ_-,ρ_+} in (3.10) and improving our *accuracy* by replacing in (3.13) $\epsilon^{-1}\ell$ with $\epsilon^{-\alpha}$, $\alpha \in (0,1)$. The result changes drastically: if $R > |v|\tau$, $\alpha \le 1/2$ and $\delta > 0$ is small enough, then

$$\lim_{\epsilon \to 0} \sup_{|x| \le \epsilon^{-1} R} P_{\mu^{(\epsilon)}}\Big(d_{\epsilon^{-\alpha},x}\big(\eta(\cdot,\epsilon^{-1}\tau), \chi_{\rho_-,\rho_+}(\cdot,\epsilon^{-1}\tau;\epsilon)\big) < \delta \Big) = 0 \tag{3.14}$$

At this level of accuracy, therefore, (3.9) does not describe anymore the behavior of the IPP.

Consider the new equation

$$\frac{\partial \rho^\star}{\partial t} + v\nabla(\rho^\star) = \nabla \cdot [D\nabla \rho^\star], \tag{3.15}$$

with D the constant matrix

$$D_{i,j} = \frac{1}{2}\delta_{i,j}[P(0,e_i) + P(0,-e_i)]. \tag{3.16}$$

Call $\rho^\star(x,t)$ the solution of (3.15) starting from χ_{ρ_-,ρ_+}, then for any $R > 0$, $\tau \geq 0$ and $\alpha > 0$,

$$\lim_{\epsilon\to 0} \sup_{|x|\leq \epsilon^{-1}R} P_{\mu^{(\epsilon)}}\Big(d_{\epsilon^{-\alpha},x}(\eta(\cdot,\epsilon^{-1}\tau),\rho^\star(\cdot,\epsilon^{-1}\tau)) < \delta\Big) = 1. \tag{3.17}$$

By comparing (3.14) and (3.17) we conclude that (3.15) gives a better description of the IPP than (3.9), while in the pure hydrodynamic limit (3.13) they are indistinguishable.

An alternative, maybe more convenient way to select (3.15) is to look at longer times while keeping the same coarse grained accuracy as in (3.13). We have for any $\tau \geq 0$, $\ell > 0$ and $R > 0$:

$$\lim_{\epsilon\to 0} \sup_{|x|\leq \epsilon^{-2}R} P_{\mu^{(\epsilon)}}\Big(d_{\epsilon^{-1}\ell,x}(\eta(\cdot,\epsilon^{-2}\tau),\rho^\star(\cdot,\epsilon^{-2}\tau)) < \delta\Big) = 1, \tag{3.18}$$

while the limit is 0 if $\rho^\star$ is replaced by $\chi_{\rho_-,\rho_+}(\cdot,\epsilon^{-2}\tau)$, see (3.11) for notation.

This formulation allows for a scaling limit procedure similar to that in Theorem 3.1. To be more precise let $\rho^\star(x,0;\epsilon) = \rho_0(\epsilon x)$ and set

$$\rho^{(\epsilon)}(r,\tau) = \rho^\star(\epsilon^{-1}r + \epsilon^{-2}v\tau, \epsilon^{-2}\tau;\epsilon), \tag{3.19}$$

then $\rho^{(\epsilon)}$ is actually independent of ϵ, $\rho^{(\epsilon)} = \tilde\rho$ where

$$\frac{\partial \tilde\rho}{\partial \tau} = \nabla \cdot D\nabla\tilde\rho \tag{3.20}$$

with $\tilde\rho(r,0) = \rho_0(r)$. We then have that, for a general class of initial data ρ_0,

$$\lim_{\epsilon\to 0} P_{\mu^{(\epsilon)}}\Big(\Big|\epsilon^d \sum_{x\in\mathbb{Z}^d} \phi(\epsilon[x - v\epsilon^{-2}\tau])\eta(x,\epsilon^{-2}\tau) - \int_{\mathbb{R}^d} dr\, \phi(r)\tilde\rho(r,\tau)\Big| < \delta\Big) = 1. \tag{3.21}$$

Thus a diffusive scaling (with the introduction of *Lagrangian coordinates*) gives rise to the heat equation (3.20), while the Euler scaling with space and time scaled by the same factor gives the hyperbolic equation (3.9): longer times in the IPP yield a non zero Navier Stokes correction.

3.2. Random Fluctuations of the Shock in the ASEP. It is natural at this point to ask whether similar Navier–Stokes corrections arise also in the ASEP, the answer, as we will see, is negative. Let $d = 1$ and, going back to (3.2), set

$$p = P(0,1) > 1/2, \quad q = P(0,-1), \quad p + q = 1. \tag{3.22}$$

The shocks in the inviscid Burgers equation have also the form (3.10)-(3.11) with $0 \leq \rho_- < \rho_+ \leq 1$ and

$$\chi_{\rho_-,\rho_+}(x,t) = \chi_{\rho_-,\rho_+}(x - ct), \qquad c = (p-q)(1 - \rho_- - \rho_+). \tag{3.23}$$

Since $\chi_{\rho_-,\rho_+}(\epsilon x)$ is independent of ϵ, the family of initial measures $\mu^{(\epsilon)}$ is independent of ϵ, $\mu^{(\epsilon)} \equiv \mu$ being the product measure with averages $\chi_{\rho_-,\rho_+}(x)$. (3.14) holds also for the ASEP, but, contrary to what happens in the IPP, it cannot be fixed, as in (3.18), by replacing $\chi_{\rho_-,\rho_+}(x,t)$ by the solution of the equation with an additional viscosity. In fact let $\rho(x,t)$ be the solution of (2.1) with some $D>0$ (in $d=1$, D is a scalar) starting from χ_{ρ_-,ρ_+}: it is then possible to show that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\chi_{\rho_-,\rho_+;K}(x,t)-\rho(x,t)\right|=0, \tag{3.24}$$

where the functions $\chi_{\rho_-,\rho_+;K}(x,t)$ are the traveling waves

(3.25)

$$\chi_{\rho_-,\rho_+;K}(x,t)=\frac{\rho_-+\rho_+}{2}+\frac{\rho_+-\rho_-}{2}\tanh\left\{\frac{x-ct}{K}\right\},\qquad K=\frac{(\rho_+-\rho_-)D}{4(p-q)}$$

(ρ_-, ρ_+ and c as before).

The discontinuity at $x=0$ of $\chi_{\rho_-,\rho_+}(x)$ disappears instantaneously in (2.1) with $D>0$ just as in (3.15), but, contrary to what happens in the latter, the shock width does not diverge as $t\to\infty$, instead it approaches that in (3.25). $\chi_{\rho_-,\rho_+,K}(\cdot,t)$ behaves just like $\chi_{\rho_-,\rho_+}(\cdot,t)$ as far as the criterion (3.14) is concerned and since the inviscid Burgers equation does not approximate correctly the ASEP, in the sense of (3.14), then none of the Burgers equations, no matter what is $D>0$, gives a correct approximation either. The *shocks* (3.25) have an essentially finite width (of the order of K) and they could only be distinguished from the shock χ_{ρ_-,ρ_+} by a finite coarse graining, i.e. if $\epsilon^{-1/2}\ell$ is substituted by ℓ. But then the statistical fluctuations intrinsic to the microscopic description would be no longer negligible.

The failure of the Burgers equation to describe the ASEP is not at all due to a problem of Navier-Stokes corrections, but rather to a completely new phenomenon of stochastic nature:

THEOREM 3.2. *Given any ρ_- and ρ_+ as in* (3.23) *there is a probability ν on $\{0,1\}^{\mathbb{Z}}$ and a random process ξ_t, $t\geq 0$, $\xi_0=0$, with values in $\mathbb{Z}$, adapted to $\eta(\cdot,t)$, $t\geq 0$, such that for any $\delta>0$, $\alpha>0$, $\tau\geq 0$ and $R>0$*

$$\lim_{\epsilon\to 0}\sup_{|x|\leq\epsilon^{-1}R}P_\nu\Big(d_{\epsilon^{-\alpha},x}\big(\eta(\cdot,\epsilon^{-1}\tau),\chi_{\rho_-,\rho_+}(\cdot+\xi_{\epsilon^{-1}\tau},0)\big)<\delta\Big)=1. \tag{3.26}$$

Moreover

$$\lim_{\epsilon\to 0}E_\nu\big(\epsilon\xi_{\epsilon^{-1}\tau}\big)=c, \tag{3.27}$$

where c is the speed of the shock, given in (3.23).

This beautiful result, proved in **[43]**, shows that in all the typical configurations of the ASEP which starts from ν, there is always, somewhere, a shock. Initially it is at 0 then it moves randomly as ξ_t.

By (3.26) with $\tau=0$ we can say that ν is an acceptable choice for describing the initial macroscopic profile χ_{ρ_-,ρ_+}, just like the traditional product measure μ. (3.26) can then be seen as the analogue of (3.18) in the case of the ASEP. The result proved in **[43]** is even stronger since it states also that the measure ν is invariant when the ASEP is seen from ξ_t. In particular this means that we can take any time, say $\epsilon^{-100}\tau$, and still find the shock somewhere, in all the typical configurations of the process.

There are sharper results about the location of the shock. We restrict ourselves for simplicity to the totally asymmetric ASEP, i.e. $p = 1$ in (3.22), and refer to **[44]** for a more detailed survey on the argument and for references. It is proved that the deviation

$$\epsilon^{1/2}\Big\{\xi_{\epsilon^{-1}\tau} - c\epsilon^{-1}\tau\Big\} \tag{3.28}$$

converges weakly to a Brownian motion. The main contribution to these fluctuations can be ascribed to the randomness of the initial datum and the behavior of the shocks in the ASEP can be described, at the level of accuracy (3.26), by the Burgers equation with any given value D of the viscosity in terms of the solution $\rho = \rho(r,t;\eta)$, $r \in \mathbb{R}$, $t \geq 0$, of the following Cauchy problem

$$\frac{\partial \rho}{\partial t} + v \cdot \frac{\partial}{\partial x}\rho(1-\rho) = D\frac{\partial^2}{\partial x^2}\rho, \tag{3.29}$$

with initial datum:

$$\rho(x,0;\eta) = \eta([x]), \tag{3.30}$$

where $[x]$ denotes the integer part of x. Here the $\eta(x)$'s are independent random variables with average $\chi_{\rho_-,\rho_+,K}(x,0)$ and K is related to D as in (3.25). To be more specific, define

$$x_\pm(t) = \pm(p-q)\big[1 - 2\rho_- - (1-\rho_- - \rho_+)\big]t \tag{3.31}$$

($p - q = 1$ in the actual case we are considering) and

$$\xi_t^\star = \frac{1}{\rho_+ - \rho_-}\sum_{x\in\mathbb{Z}\cap[x_-(t),x_+(t)]}\eta(x). \tag{3.32}$$

The main result is that if ν (the same as in Theorem 3.2) is the distribution of the η's, then

$$\lim_{\epsilon\to 0} E_\nu\Big(\epsilon^{1/2}|\xi_{\epsilon^{-1}\tau} - \xi^\star_{\epsilon^{-1}\tau}|\Big) = 0 \tag{3.33}$$

(the same result holds starting from the product measure with averages χ_{ρ_-,ρ_+}). Moreover for any $\delta > 0$, $\alpha > 0$, $\tau \geq 0$ and $R > 0$, we expect that

$$\lim_{\epsilon\to 0}\sup_{|x|\leq\epsilon^{-1}R} P_\nu\Big(d_{\epsilon^{-\alpha},x}\big(\rho(\cdot,\epsilon^{-1}\tau;\eta),\chi_{\rho_-,\rho_+,K}(\cdot+\xi^\star_{\epsilon^{-1}\tau},0)\big) < \delta\Big) = 1. \tag{3.34}$$

We do not know of a proof of the above statement which should thus be regarded as a conjecture (partial results in this direction can however be found in **[34]**).

According to Theorem 3.2, the random microscopic position of the shock, ξ_t, has a nice interpretation in terms of second class particles, since it is the position of a second class particle added to the system. The dynamics of a second class particle is like that of the other particles in exchanges with vacancies, but it acts like a vacancy in exchanges with regular (i.e. first class) particles. In this way the second class particle does not affect the dynamics of the first class particles. It can be shown (and this is the reason for introducing it here) that the second class particle moves with velocity $1 - 2\rho_-$ (respectively $1 - 2\rho_+$) to the left and right of the shock and hence it is attracted by the shock. The particles around it develop a pattern which is described by the measure ν, whose densities are asymptotically ρ_+ and ρ_- respectively to the right and to the left of the second class particle. A complete characterization of ν, first introduced in **[43]**, has been obtained in **[33]** where it is shown that, looking from the second class particle, the asymptotic convergence to the Bernoulli measures with densities ρ_- and ρ_+ is exponentially fast.

We have seen so far how to localize the shock, we next discuss about its shape, in particular whether the smoothened shock $\chi_{\rho_-,\rho_+,K}$, (3.25) is better than the sharp step χ_{ρ_-,ρ_+}, (3.10). *Better* here means a better approximation to the profiles described by the measure ν. Since we need large averaging blocks $C^{(\ell)}$ to dampen the statistical fluctuations of the empirical density, the shape of the shock is essentially undetectable unless its width is much larger than the side of the averaging block. The Burgers profile $\chi_{\rho_-,\rho_+,K}$ has width proportional to K. K can be made larger by decreasing the asymmetry $p-q$, see (3.25). In **[33]** it is shown that as $p-q \to 0$ the profiles of the typical configurations of ν agree with $\chi_{\rho_-,\rho_+,K}$. The statement is made precise under a proper rescaling where $p-q=\epsilon$ is the micro-macro space ratio. The ASEP in such a case is called the *weakly asymmetric simple exclusion process* (WASEP), a process that will be discussed in the next section.

We conclude this section with a few words on the problem of the Navier-Stokes correction for the ASEP. We have seen that contrary to what happens in the IPP the analysis of the shock does not select any value for the viscosity coefficient. There is however evidence that the correct equation for the ASEP is a Burgers equation with viscosity. This comes from studying the ASEP in $d=3$, starting from an initial datum $\rho^{(\epsilon)}(x) = 1/2 + \epsilon\theta(\epsilon x)$, where θ is independent of ϵ. It has been proved in **[38]** that at times $\epsilon^{-2}\tau$ the correct profile is, to leading orders, $1/2 + \epsilon\theta(\epsilon x, \tau)$ with $\theta(r,\tau)$ satisfying a Burgers equation with a density dependent diffusion coefficient obtained from a Green–Kubo formula **[95]**.

4. The Weakly Asymmetric Simple Exclusion Process: Hydrodynamics and Stochastic Corrections

We have seen in the previous section that there are stochastic effects which survive in the hydrodynamic limit. They appear as small, random displacements of the shock profile caused primarily by the fluctuations of the initial data (and not by the randomness in the dynamics). They are therefore more appropriately treated in the hydrodynamic description by the Burgers equation with random initial condition rather than by adding a random forcing term to the Burgers equation.

In this section we will instead examine a variation of the ASEP, whose behavior appears to be captured by the following one dimensional SPDE

$$\frac{\partial}{\partial t}\rho^{(\epsilon)} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho^{(\epsilon)} - \frac{\partial}{\partial x}\left[\rho^{(\epsilon)}\left(1-\rho^{(\epsilon)}\right) + \sqrt{\epsilon\rho^{(\epsilon)}(1-\rho^{(\epsilon)})}\,\dot{W}_t^{\epsilon}\right], \tag{4.1}$$

with

$$\dot{W}_t^{\epsilon} = \varphi_\epsilon * \dot{W}_t, \tag{4.2}$$

φ_ϵ an approximate δ-function, i.e. $\varphi_\epsilon(\cdot) = \epsilon^{-1}\varphi(\cdot\,\epsilon^{-1})$, φ a non negative, even, compactly supported, smooth function with $\int_{\mathbb{R}}\varphi(r)dr = 1$, and $*$ denotes the convolution. $\dot{W}_t$ is the space–time white noise, that is the Gaussian process with mean zero and covariance

$$E\left(\dot{W}_t(r)\dot{W}_{t'}(r')\right) = \delta(t-t')\delta(r-r'). \tag{4.3}$$

The particle system we consider here is the one dimensional weakly asymmetric simple exclusion process (WASEP) already mentioned in the previous section.

Recall that this is the exclusion process with p and q in (3.22) chosen as

$$p = \frac{1+\epsilon}{2}, \qquad q = \frac{1-\epsilon}{2}. \tag{4.4}$$

This means that the bias v in (3.1) is equal to ϵ and goes to zero in the hydrodynamic limit. However, if we use the diffusive scaling with times proportional to ϵ^{-2}), then the effect of the drift survives in the limit and in fact the hydrodynamic equation for the WASEP is the viscous Burgers equation (2.1). There is also a Central Limit Theorem which describes the small deviations from the hydrodynamic limit as an Ornstein-Uhlenbeck process and thus making rigorous, in this context, the *fluctuating hydrodynamic theory*. We will then explain how a nonlinear SPDE, i.e. the stochastic Burgers equation with viscosity and with additive noise, (4.1), arises as a particular scaling limit of the WASEP. This scaling is obtained by looking at much longer times (proportional to ϵ^{-4}) and very special initial data.

4.1. The Hydrodynamic Limit for WASEP. To understand the scalings let us go back to Section 2 and to (2.1). Let $\rho = \rho(x,t;\epsilon)$, $x \in \mathbb{R}$, $D = 1$ and $v = \epsilon$, then (2.1) becomes

$$\frac{\partial \rho}{\partial t} + \epsilon \cdot \frac{\partial}{\partial x}\rho(1-\rho) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho. \tag{4.5}$$

As in Section 2 we first look at (4.5) as the microscopic system and perform the HSL. We thus set $\rho(x,0;\epsilon) = \rho_0(\epsilon x)$ and, in contrast to (2.3), we scale times diffusively. We define

$$\rho(r,\tau) := \rho(\epsilon^{-1}r, \epsilon^{-2}\tau; \epsilon). \tag{4.6}$$

As the notation suggests, the rescaled density $\rho(r,\tau)$ is independent of ϵ and, moreover, it solves the viscous Burgers equation

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial r}\rho(1-\rho) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho, \tag{4.7}$$

with initial condition $\rho_0(r)$, which is therefore the hydrodynamic equation associated to (4.5).

The same conclusions carry through to the WASEP. We take as usual $\mu^{(\epsilon)}$ to be the product measure on $\{0,1\}^{\mathbb{Z}}$ with $\mu^{(\epsilon)}(\eta(x)) = \rho_0(\epsilon x)$, $\rho_0 \in C^2(\mathbb{R})$, $0 \le \rho_0(r) \le 1$ for all $r \in \mathbb{R}$. We call P_ν the law of the WASEP starting from a probability ν.

THEOREM 4.1. *For any $\tau \ge 0$, any $\phi \in C_0^\infty(\mathbb{R})$ and any $\delta > 0$ we have*

$$\lim_{\epsilon \to 0} P_{\mu^{(\epsilon)}}\left(\left| \epsilon \sum_{x\in\mathbb{Z}} \eta(x, \epsilon^{-2}\tau)\phi(\epsilon x) - \int_{\mathbb{R}} \rho(r,\tau)\phi(r)dr \right| \le \delta \right) = 1, \tag{4.8}$$

where $\rho \in C^{1,2}(\mathbb{R}^+,\mathbb{R})$ is the unique (classical) solution of (4.7) with initial condition ρ_0.

This result says that the hydrodynamic limit of the WASEP, on the proper diffusive time scale, is the Burgers equation with viscosity. For a proof see [**31**] and

[53]. See also [70], where the large deviations from the hydrodynamic limit are established (in macroscopically finite volumes). Theorem 4.1 holds in any dimension.

4.2. The Normal Fluctuations. Let us now look at the fluctuations around the limit behavior given by (4.7). To do this we consider the distribution valued process $X^\epsilon_\cdot \equiv \{X^\epsilon_\tau, \tau \geq 0\}$

$$\langle X^\epsilon_\tau, \phi\rangle \equiv \epsilon^{1/2} \sum_{x\in\mathbb{Z}} \left[\eta(x, \epsilon^{-2}\tau) - \rho(\epsilon x, \tau)\right] \phi(\epsilon x), \tag{4.9}$$

where ϕ is a test function in $\mathcal{S}$. For any $\tau \in \mathbb{R}^+$, X^ϵ_τ is an element in $\mathcal{S}'$ (Schwarz distributions) and $X^\epsilon_\cdot$ is an element in $D([0,\infty); \mathcal{S}')$, the Skorohod space of distribution valued functions: we denote by $\langle \cdot, \cdot \rangle$ the duality between $\mathcal{S}$ and $\mathcal{S}'$.

If we assume that $\mu^{(\epsilon)}$ is the product measure considered in Theorem 4.1, then $X^\epsilon_0 \in \mathcal{S}'$ converges weakly to a $\mathcal{S}'$–valued process X_0, where $\langle X_0, \phi\rangle$ is a centered Gaussian variable with variance $\int \rho_0(r)(1-\rho_0(r))\phi(r)^2 \mathrm{d}r$ (i.e. X_0 is a *spatially modulated* white noise).

We have the following result (see [31],[35])

THEOREM 4.2. *$X^\epsilon_\cdot \in D([0,\infty); \mathcal{S}')$ converges weakly to $X_\cdot \in C([0,\infty); \mathcal{S}')$, where $X_\cdot$ is the unique solution in $C([0,\infty); \mathcal{S}')$ of*

$$\langle X_t, \phi\rangle = \langle X_0, \phi\rangle + \int_0^t \langle Y_s, \left(\frac{1}{2}\Delta + (2\rho_s - 1)\nabla\right) \phi\rangle ds + \langle M_t, \phi\rangle, \tag{4.10}$$

for all $\phi \in \mathcal{S}$ and $t \geq 0$. $\langle M_t, \phi\rangle$ is the continuous martingale with independent increments and quadratic variation given by

$$\int_0^t ds\, \langle \rho(\cdot, s)\big(1 - \rho(\cdot, s)\big), \frac{\partial}{\partial r}\phi\rangle^2. \tag{4.11}$$

In other words, the fluctuations of the particle distribution around its deterministic limit is given by an Ornstein–Uhlenbeck process in which the drift term is the linearization of the Burgers equation: X_t satisfies the linear SPDE

$$\frac{\partial}{\partial t} X_t = \frac{1}{2}\Delta X_t - \nabla[(2\rho(\cdot, t) - 1)X_t] + \nabla\left(\sqrt{\rho(\cdot, t)\big(1 - \rho(\cdot, t)\big)}\, \dot{W}_t\right). \tag{4.12}$$

Theorem 4.2 can be extended to $d > 1$, but some care has to be taken in defining (4.9): the fluctuation field in general should be defined by subtracting the mean of $\eta(x, \epsilon^{-2}\tau)$ and not its hydrodynamic value, which will differ by terms of $O(\epsilon)$ from the mean, since, for example as in the proof of Theorem 4.1, we are replacing a discrete version of (4.7) on a grid of spacing ϵ with its continuum limit. This problem is not seen in $d = 1$, where this difference is masked by the magnitude of the fluctuations which are of $O(\sqrt{\epsilon})$, but, for general d, the fluctuations will be of order $\epsilon^{d/2}$, so the difference between the mean of the process its hydrodynamic limit is important for $d \geq 2$.

The result analogous to Theorem 4.2 for the ϵ–dependent SPDE problem (4.1) is showing that

$$\epsilon^{-1/2}\left[\rho^{(\epsilon)}(\cdot,t)-\rho(\cdot,t)\right]$$

converges weakly to to X_t, a fact that can be easily verified at a formal level.

We want to stress that the appearance of a *linear* SPDE as the law of the fluctuations is no surprise and the structure of Theorem 4.2 is expected to hold in great generality: the field of fluctuations is, in the limit, a diffusion with drift given by the linearization of the hydrodynamic equation. There is however, for the WASEP, a scaling limit in which a *suitable* fluctuation field converges to a *nonlinear* SPDE and we describe this next.

4.3. Nonlinear Fluctuations for WASEP: the Space–Time Scaling. The next results are restricted to $d=1$. Let us start by explaining the limit that we will consider. Theorem 4.1 and Theorem 4.2 show that (4.1) is correct up to first order in ϵ in the hydrodynamic scaling. We will then use (4.1) (beyond the regime where its validity has been proved) for some formal manipulations aimed at understanding what could be a scaling in which both noise and nonlinearity appear together. In particular, for simplicity, we will replace $\dot{W}_t^\epsilon$ by $\dot{W}_t$. $\dot{W}_t$ denotes different processes which coincide in law.

Let $m_t^{(\epsilon)}(r)=\rho^{(\epsilon)}(\epsilon^{-1}r,\epsilon^{-2}t)$, this corresponds to looking at the empirical density on the longer space-time scale $t\sim\epsilon^{-4}$, $x\sim\epsilon^{-2}$: recall that the forcing term is $\sim\epsilon$, so that this new scaling, in spite of being still diffusive, is different from the previous one. Using the scaling properties of white noise, we get from (4.1)

$$\frac{\partial m_t^{(\epsilon)}}{\partial t}=\frac{1}{2}\Delta m_t^{(\epsilon)}-\partial_r\left[\epsilon^{-1}m_t^{(\epsilon)}\left(1-m_t^{(\epsilon)}\right)+\epsilon\sqrt{m_t^{(\epsilon)}\left(1-m_t^{(\epsilon)}\right)}\dot{W}_t\right],\tag{4.13}$$

which gives a diverging current when $\epsilon\to 0$. In order to make this term finite we shall consider what has been called, in the context of the derivation of the Navier-Stokes equation, the *incompressible limit*, i.e. we analyze a small perturbation of the global equilibrium of constant density. We assume that $m^{(\epsilon)}$ has the form

$$m_t^{(\epsilon)}=\frac{1}{2}\left[1+\epsilon\,u_t^{(\epsilon)}\right],\tag{4.14}$$

and that $u_t^{(\epsilon)}$ has a limit as ϵ tends to zero, i.e. we are considering a perturbation of the same order as the noise in (4.13), for this reason we *may be able* to see both the nonlinearity and the random force in the scaling limit. Recall that at the end of Section 3 we have considered initial profiles of the form (4.14): they were used in the ASEP in $d\geq 3$ to study the Navier Stokes correction, **[38]**, in a diffusive scaling limit. In that case in fact we do not see stochastic effects, but an extra viscosity shows up.

With the choice (4.14) the diverging term in (4.13) disappears and, in the limit $\epsilon\to 0$, the equation that we obtain by substituting (4.14) into (4.13) formally converges to

$$\frac{\partial u_t}{\partial t}=\frac{1}{2}\Delta u_t+\frac{1}{2}\nabla(u_t^2)+\nabla\dot{W}_t.\tag{4.15}$$

The stochastic Burgers equation (4.15) should thus describe the first order correction to a global equilibrium profile.

In this heuristic discussion we are always *assuming* that the form (4.14), which can be assumed at $t = 0$, will be stable under the time evolution, but this is to a certain extent the real issue. In other words we are asking what happens to a perturbation of order ϵ (on the space scale ϵ^{-2}) after a time ϵ^{-4} .

At this point, in conformity with the existing literature, we change slightly the definition of the WASEP process. From now on in this section the WASEP process will be generated by

$$(4.16) \qquad L_\epsilon f(\eta) = \sum_{x\in\mathbb{Z}} \left\{ \left(\frac{1}{2}+\epsilon\right) \left[f(\eta^{x-1,x}) - f(\eta)\right] + \frac{1}{2}\left[f(\eta^{x,x+1}) - f(\eta)\right] \right\},$$

where f is, as before, a local function of the configuration η. The difference with the process introduced before, generated by (3.2) with the choices (3.22) and (4.4), is that here the bias is toward the left, i.e. $q > p$, and the parameter ϵ which appears in (4.4) should be replaced by $\epsilon/(1+\epsilon) = \epsilon + O(\epsilon^2)$ and the time should be speeded up of a factor $1+\epsilon$, i.e. the generator (3.22) should be multiplied by $1+\epsilon$. The change of time as well as the small change in the size of the asymmetry do not change the statements of Theorem 4.1 and Theorem 4.2, apart for the different sign in the drift term of the Burgers equation, i.e. the sign of the second term in the right–hand side of (4.7) should be exchanged (and analogous change in Theorem 4.2). Some of the result we are going to present now are more refined and the results would have to be restated in a slightly different way for the process with jump rates given by (4.4), but no substantial change would be needed. The bias toward the left has been introduced because it gives a more direct picture of the *growing interface* process that will be introduced it the next section.

The heuristic considerations we made above suggest that in order to derive the stochastic Burgers equation from the WASEP, we should consider the fluctuation field

$$(4.17) \qquad \langle X_t^\epsilon, \phi\rangle = \epsilon \sum_{x\in\mathbb{Z}} \phi(\epsilon^2 x)\left[2\eta(x,\epsilon^{-4}t) - 1\right],$$

where $\{\eta\}$ is the WASEP. The initial condition $\mu^{(\epsilon)}$ is the product measure on $\{0,1\}^{\mathbb{Z}}$ with marginals

$$(4.18) \qquad \mu^{(\epsilon)}\left(\eta(x)\right) = H\left(\frac{1}{2} + \epsilon^{-1}\left[h(\epsilon^2 x) - h(\epsilon^2(x-1))\right]\right),$$

$$H(r) = \begin{cases} 1 & \text{if } r > 1 \\ r & \text{if } r \in [0,1] \\ 0 & \text{if } r < 0, \end{cases}$$

with $h = h(r)$ a α–Hölder continuous function ($\alpha > 1/2$) on $\mathbb{R}$ which satisfies the condition that there is $a > 0$ such that, for every $r \in \mathbb{R}$, $|h(r)| \leq a(1+|r|)$. In [5] a much larger class of initial data is considered, but here we will restrict ourselves to this case. It is by no means obvious how to give a meaning to (4.15) and, as a matter of fact, we will be selecting one of the possible *regularizations* of (4.15). We therefore first discuss this point.

4.4. The Burgers Equation with Conservative Noise. We consider the space $C^0(\mathbb{R})$ of continuous functions on the real line equipped with the topology of uniform convergence over compact sets. As test functions we use the space $\mathcal{D}$, i.e. the function in $C_0^\infty(\mathbb{R})$ with the inductive limit topology; its strong topological dual $\mathcal{D}'$ is the space of distributions over $\mathbb{R}$. We denote by $\langle\cdot,\cdot\rangle$ the duality between the test functions and the distributions. In this notation the space–time W_t in (4.3) is the distribution valued centered Gaussian process such that

$$E\left(\langle W_t,\phi_1\rangle\langle W_s,\phi_2\rangle\right) = t\wedge s(\phi_1,\phi_2), \tag{4.19}$$

where $\phi_1,\phi_2\in\mathcal{D}$, are test function, $(\cdot,\cdot)$ is the scalar product in $L^2(\mathbb{R})$ and $a\wedge b := \min\{a,b\}$. We denote by $\mathcal{A}_t$ the natural filtration of W_t and denote by P the law of W_t.

For the initial datum of the stochastic Burgers equation we take a random field with trajectories u_0 in $\mathcal{D}'$ of the form that we now explain. We specify a random field $\mathcal{P}_0$ with trajectories $h_0\in C(\mathbb{R})$ which is independent of P and satisfies the condition: for all $p>0$ there exists a such that

$$\sup_{r\in\mathbb{R}} e^{-a|r|}\mathcal{E}_0\left(e^{-ph_0(r)}\right) < \infty. \tag{4.20}$$

For $\phi\in\mathcal{D}$ we then define

$$\langle u_0,\phi\rangle = -\int h_0(r)\phi'(r)\mathrm{d}r \tag{4.21}$$

To summarize, u_0 is the derivative, in distributional sense, of a continuous trajectory $h_0 = h_0(r)$ which has for every $r\in\mathbb{R}$ an exponential moment that grows at most exponentially in r. One can obviously take h_0 to be a deterministic function with at most linear growth.

We will characterize the solution of the stochastic Burgers equation through a limiting procedure. Accordingly we introduce a regularized version of the cylindrical Wiener process which will define a family of approximating problems.

Let $\varphi\in C_0^\infty(\mathbb{R})$ be as in (4.2). Introduce, for $\kappa>0$, the approximate identity $\delta_r^\kappa(r') = \varphi_{1/\kappa}(r-r') = \kappa\varphi(\kappa(r-r'))$ and here we use the notation $W_t^\kappa(r) = \langle W_t,\delta_r^\kappa\rangle$; its covariance is then

$$E\left(W_t^{1/\kappa}(r)W_s^{1/\kappa}(r')\right) = (t\wedge s)C_\kappa(r-r'),\qquad C_\kappa(r) = \int \delta^\kappa(r-r')\delta^\kappa(r')\mathrm{d}r'. \tag{4.22}$$

We then write the stochastic Burgers equation with this regularized Wiener process against test functions as

$$\langle u_t^\kappa,\phi\rangle = \langle u_0,\phi\rangle + \int_0^t \frac{1}{2}\left\{\langle u_s^\kappa,\phi''\rangle + \langle (u_s^\kappa)^2,\phi'\rangle\right\}\mathrm{d}s - \langle W_t^{1/\kappa},\phi'\rangle, \tag{4.23}$$

which is formally obtained from (4.15) by integrating by parts. The limit $\kappa\to\infty$ is now taken according to the following strategy. When the cutoff κ is finite $W_t^{1/\kappa}(r)$ is smooth so that (4.23) makes sense in the space of continuous functions; we thus obtain a processes u^κ in $C((0,T];C^0(\mathbb{R}))$. Since a limiting process will not be continuous in space we have no hope to get a convergent sequence in the topology of $C((0,T];C^0(\mathbb{R}))$; however $\{u^\kappa\}_{\kappa>0}$ does form a weak convergent family as $\kappa\to\infty$ in the topology of $C([0,T];\mathcal{D}')$.

In fact we have the following

PROPOSITION 4.1. *Under the above assumptions on the initial condition* u_0, *for all* $\kappa > 0$ *there exists a unique process* $u_t^\kappa = u_t^\kappa(r)$ *in* $C((0,\infty); C^0(\mathbb{R}))$, *adapted to the filtration* $\mathcal{A}_t$, *which solves almost surely* (4.23) *for all* $\phi \in \mathcal{D}$ *and every* $t > 0$. *Moreover the family* $\{u^\kappa\}_{\kappa>0}$, $u_t^\kappa \in C((0,T]; \mathcal{D}')$, *converges weakly as* $\kappa \to \infty$ *to a limit process denoted by* u.

Although this result characterizes uniquely the solution of the stochastic Burgers equation (4.15) through the approximating problems (4.23) it is not completely satisfactory since it avoids the issue of showing that u satisfies a limit equation. Below we will give an alternative definition of the process u. But let us give first the result linking the WASEP with the stochastic Burgers equation.

4.5. WASEP and the Stochastic Burgers Equation. Let us go back to the particle system and consider (4.17). We regard $X^\epsilon = (X_t^\epsilon)_{t\in[0,T]}$ as a random element in $D([0,T]; \mathcal{D}')$. In the scaling limit $\epsilon \to 0$ the distribution of the fluctuation field (4.17) is determined by the Burgers equation with random noise on the current, namely

THEOREM 4.3. *The family* $\{X^\epsilon\}_{\epsilon>0}$ *is weakly convergent in* $D([0,T]; \mathcal{D}')$ *as* $\epsilon \to 0$. *Furthermore the weak limit is concentrated on* $C([0,T]; \mathcal{D}')$ *and coincides with the process* u *constructed in Proposition 4.1, with* $\langle u_0, \phi\rangle$ *a Gaussian variable with mean* $-\langle m, \phi'\rangle$ *and variance* (ϕ, ϕ), *for any* $\phi \in \mathcal{D}$.

The initial condition is easily derived from (4.18) and it simply says that $u_0 = \nabla m + W$, where W is a white noise. We have thus found a scaling in which the fluctuations are nonlinear. We are now going to say a few words about the derivation of this result.

4.6. The Cole–Hopf Transform and the Stochastic Heat Equation. The proof of Theorem 4.3 follows a strategy parallel to the one in [**53**]. The heuristic observation is simply that if u_t is a solution of the stochastic Burgers equation (4.15) and h_t is a function such that $u_t = \nabla h_t$, then the process $\theta_t := \exp\{-h_t\}$ solves the *linear* SPDE

$$d\theta_t = \frac{1}{2}\Delta\theta_t dt - \theta_t dW_t, \tag{4.24}$$

with the stochastic differential interpreted in the Ito sense. Equation (4.24) is usually referred to as *the stochastic heat equation*, see [**98**],[**84**],[**4**] for existence, uniqueness and other properties of the equation, with the stochastic differential interpreted in the Ito sense. Here we just recall that if we set

$$C_+^0(\mathbb{R}) = \{f \in C^0(\mathbb{R}) : f(r) > 0 \text{ for all } r \in \mathbb{R}\} \tag{4.25}$$

and $\theta_0 \in C_+^0$ a.s., then $\theta \in C([0,\infty); C_+^0(\mathbb{R}))$ a.s. (this has been proved by C. Müller [**84**]). We can then define the logarithm of θ and in [**5**] it is proven that the process u given by Proposition 4.1 is in law the same as $-\nabla \log\theta$, thus providing a better characterization of the limit behavior of the particle system. The choice of the initial condition for (4.24) to match (4.21) is straightforward.

The idea is now to perform a transformation similar to the Cole–Hopf on the particle system, as it is done in [**53**]. The resulting particle model is simpler than the original one and one can prove convergence of the new process to the solution of (4.24). Theorem 4.3 is then obtained by performing the inverse Cole–Hopf transform.

5. Driven Surfaces and Fluctuations

We describe here one particular model out of an extremely wide field, explored by physicists and recently also by mathematicians. There is however, up to now, very little mathematical understanding of the basic physical results, in spite of the very many connections with active fields of probability, like random walks in random environments and polymer problems. An interesting review of the physical ideas is in [**72**].

5.1. Interface Models of Solid on Solid Type. The WASEP can be easily mapped into a one–dimensional interface growth model and we will explain how the results obtained in the context of the stochastic Burgers equation map into results for this interface model.

The microscopic interface model has state space

$$\hat{\Omega} = \left\{\zeta \in \mathbb{Z}^{\mathbb{Z}} : |\zeta(x+1) - \zeta(x)| = 1 \text{ for all } x \in \mathbb{Z}\right\}. \tag{5.1}$$

Its time evolution is governed by the jump Markov process generated by

$$\hat{L}_\epsilon f(\zeta) = \sum_{x\in\mathbb{Z}} \left\{c_\epsilon^+(x,\zeta)\,[f(\zeta+2\delta_x) - f(\zeta)] + c_\epsilon^-(x,\zeta)\,[f(\zeta-2\delta_x) - f(\zeta)]\right\}, \tag{5.2}$$

where f is a cylindrical function on $\hat{\Omega}$, $\delta_x \in \mathbb{Z}^{\mathbb{Z}}$ is defined by $\delta_x(y) = \delta_{x,y}$ (the Kronecker symbol) and

$$c_\epsilon^+(x,\zeta) = \begin{cases} 1/2+\epsilon & \text{if } \zeta(x+1) - 2\zeta(x) + \zeta(x-1) = 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$c_\epsilon^-(x,\zeta) = \begin{cases} 1/2-\epsilon & \text{if } \zeta(x+1) - 2\zeta(x) + \zeta(x-1) = -2 \\ 0 & \text{otherwise.} \end{cases}$$

The dynamics is easily explained: view $\zeta(x)$ as the height of the interface at the site x. There is the constraint that, moving along x, the interface is never flat and changes its height by ±1 at each step. The interface evolves according to a Poisson process and the only changes allowed are the ones which make a local minimum into a local maximum or vice versa. This is performed with a slight bias (ϵ) toward increasing the height, that is more minima become maxima than vice versa. Notice that the dynamics preserves the single step constraint. This model goes under the name of restricted solid on solid process (SOS). We shall consider ζ as a continuous function by linear interpolation on its value on the lattice $\mathbb{Z}$.

To give the relation between the SOS process and the WASEP we put a *tag* on the particle that at time zero is the closest to the origin on the positive half-axis. Let x_t^0 be the position of the tagged particle under the time evolution. It is straightforward to verify that we can construct a version of the SOS process, starting from the WASEP, essentially by summing from the tagged particle

$$\zeta_t(x) = \begin{cases} \sum_{x_t^0 < y \le x} (2\eta_t(y) - 1) - x_t^0 & \text{if } x > x_t^0 \\ -\sum_{x < y \le x_t^0} (2\eta_t(y) - 1) - x_t^0 & \text{if } x < x_t^0 \\ -x_t^0 & \text{if } x = x_t^0 \end{cases} \tag{5.3}$$

where η_t is the WASEP.

In the reverse direction, given ζ_t a realization of the SOS process, we obtain a version of the WASEP by looking at the increments, i.e.

$$\eta_t(x) = \frac{(\zeta_t(x) - \zeta_t(x-1)) + 1}{2}.$$

5.2. Continuum Interfaces. We first translate (briefly and informally) Theorem 4.1 and Theorem 4.2 into the interface language (the precise statements can be easily recovered).

The law of large numbers this time involves $h_t^\epsilon(r) \equiv \epsilon\zeta_{t\epsilon^{-2}}([\epsilon^{-1}r])$, an element in $D(\mathbb{R}^+; C^0(\mathbb{R}))$. It is then an easy corollary of Theorem 4.1 to show that $h_t^\epsilon \in C^0(\mathbb{R})$ converges in probability to $h \in C^{1,2}(\mathbb{R}^+, \mathbb{R})$, the unique classical solution of

$$\frac{\partial h_t}{\partial t} = \frac{1}{2}\Delta h_t + \frac{1}{2}[1 - (\nabla h_t)^2] \tag{5.4}$$

with initial condition $h_0 = h$.

In a similar way the normal fluctuations consist in looking at

$$X_t^\epsilon(r) = \epsilon^{-1/2}\left[\epsilon\zeta_{t\epsilon^{-2}}([\epsilon^{-1}r]) - h_t(r)\right] \tag{5.5}$$

and the weak limit X of $X^\epsilon \in D(\mathbb{R}^+; C^0(\mathbb{R}))$ solves the linear SPDE

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\Delta X_t - \nabla h_t \nabla X_t + \sqrt{1 - (\nabla h_t)^2}\,\dot{W}_t. \tag{5.6}$$

We now pass to the nonlinear fluctuations. These are of particular interest because they give rise to the celebrated Kardar–Parisi–Zhang (KPZ) equation [**67**]. In this case the object of interest turns out to be

$$Z_t^\epsilon(r) = \sqrt{\epsilon}\left(\zeta_{\epsilon^{-4}t}(\epsilon^{-2}r) - v_\epsilon t\right), \tag{5.7}$$

where $v_\epsilon = (1/2)\epsilon^{-3} - (1/24)\epsilon^{-1}$ and we regard Z^ϵ as a random element in $D([0,T]; C(\mathbb{R}))$. The statement is then that the family $\{Z^\epsilon\}_{\epsilon>0}$ is weakly convergent as $\epsilon \to 0$ to the limit $Z = -\log\theta$, where θ is the solution of the stochastic heat equation.

Also in this case we can write a mollified KPZ equation against test functions as

$$\langle h_t^\kappa, \phi\rangle = \langle h_0, \phi\rangle + \int_0^t \frac{1}{2}\left\{\langle h_s^\kappa, \phi''\rangle - \langle\left[(\nabla h_s^\kappa)^2 - C_\kappa(0)\right], \phi\rangle\right\} \mathrm{d}s + W_t^{1/\kappa}(\phi) \tag{5.8}$$

where $\phi \in \mathcal{D}(\mathbb{R})$. The weak limit of h^κ coincides with $-\log\theta$ (this is the analog of Proposition 4.1).

We notice that, to a certain extent, the result on the KPZ equation is more complex than the one for Burgers. First of all there is the term $C_\kappa(0) \sim \kappa^{-1}$, which corresponds to a *renormalization* of the nonlinearity: this term disappeared in the Burgers case, due to the extra space gradient. Moreover to define the fluctuation field (5.7) we had to subtract not only the average speed of the interface, that is the term of order ϵ^{-3} in v_ϵ, but also an extra term of order ϵ^{-1}. While the origin of the first term is easily understood, since the interface has a drift proportional to ϵ acting for a time $O(\epsilon^{-4})$, the fact that there is no $O(\epsilon^{-2})$ term and, in particular, the value of the constant in front of the ϵ^{-1} term (1/24) are highly non obvious. Using the result we just (informally) stated, it has been possible to prove [**6**] that the value of *sample Lyapunov exponent* ([**83**],[**15**]) of the stochastic heat equation (4.24)is 1/24. This result can be found in the physical literature ([**72**],[**66**]), where the *quenched* free energies of the directed polymer and other interface models are computed via a non rigorous *replica computation* : the stochastic heat equation can be interpreted as the partition function of a one dimensional directed polymer in random environment (t in this case becomes the size of the system) and the sample Lyapunov exponent is then the *quenched* free energy of the system.

The KPZ equation is expected to describe the large scale fluctuations of a very general class of driven interfaces. These fluctuations are *anomalous*: in particular they are not of Gaussian type and this phenomenon goes under the name of *kinetic roughening*. This behavior is intimately related to the appearance of superdiffusive behavior of a random walk in random environment (the directed polymer mentioned above: see e.g. [**72**] and [**66**]). The physical phenomena are very dependent on the dimensionality of the space and the one–dimensional case is definitely different from the higher dimensional cases (from the physical viewpoint, the one–dimensional and two–dimensional cases are the most complex ones). The results that we have just reviewed, besides establishing the validity of the physicists' picture in one particular case, provide a rigorous mathematical framework (first of all a precise definition of *the solution to the KPZ equation*) within which one could try to formulate and eventually prove the physical statements.

PART II
Reversible Dynamical Systems: Symmetric Models with Long Range Interactions

6. Ising Models with Kac Potentials: Glauber and Kawasaki Dynamics

The systems considered so far have trivial translation invariant stationary states which are just the Bernoulli measures (indexed by the density parameter $\rho \in [0,1]$) and their linear combinations. We now switch to systems whose stationary states have a much richer structure, in particular we will be concerned with systems which can undergo phase transitions. In the cases we are going to consider it is convenient to replace the particle density $\rho \in [0,1]$ by the magnetization $m = 2\rho - 1 \in [-1,+1]$; the analogous change is also done at the microscopic level, where from now on the occupation value $\eta(x) = 0, 1$ at $x \in \mathbb{Z}^d$ is replaced by the spin variable $\sigma(x) = \pm 1$. We will look both at systems with exchange dynamics, like all the particle models considered in Part I, and at systems with spin–flip dynamics, i.e. where the elementary event is the change of the value (flip) of a single spin. We will briefly consider also evolutions where both mechanisms are present.

The easiest examples of these dynamics are given by the following two *non interacting systems*: the first one is the independent spin–flip process, i.e. a process $\{\sigma_t\}_{t \in \mathbb{R}^+}$, $\sigma_t \in \{-1,+1\}^{\mathbb{Z}^d}$, which is a collection of independent processes

$\{\sigma_t(x)\}_{t\geq 0}$ indexed by $x \in \mathbb{Z}^d$, and for each x the only possible transition, which happens at exponentially distributed random times (with mean 1), is the spin–flip $\sigma(x) \to -\sigma(x)$. It is obvious that this dynamics has only one invariant measure: the Bernoulli measure with average 0. The second one is the Symmetric Simple Exclusion Process, that is the process generated by (3.22) with $p = q = 1/2$ and $\sigma = 2\eta - 1$. For this model, as we have already pointed out, the combinations of Bernoulli measures with average $m \in [-1, 1]$ are invariant (they are actually the only invariant measures).

More complex systems are naturally built by choosing the rate of transition at time t dependent on the configuration at time t^- (indeed the exclusion process does have a dependence, given by the exclusion rule, but one can view it as an independent process on the bonds). We will choose configuration dependent rates, but we will do it in such a way that the *Gibbs measures*, the equilibrium measures according to statistical mechanics, are stationary for our systems. We thus know from the beginning (at least some of) the invariant measures, and this is a real advantage since often the existence and characterization of the stationary states is a very hard problem. A Gibbs measure is usually specified by an *interaction potential*, related to the transition rates of the dynamics.

The structure of the Gibbs states, which is the main object of study in equilibrium statistical mechanics, may indeed be very complex. In the simpler case of the Ising system that we consider here, if the temperature is sufficiently large or the *interaction* weak, there is a unique Gibbs measure (and a unique invariant measure for the spin flip dynamics as well) which is not too different from the Bernoulli measure with zero average. But at low temperatures and 0 magnetic field, i.e. in a *phase transition situation*, this will no longer be true and there will be (at least) two stationary Gibbs measures with non zero average magnetization $\pm m$. Correspondingly the spin flip evolution has two (or more) extremal, invariant measures. In the case of the exchange dynamics, the total magnetization is invariant and consequently, contrary to the case of the spin flip dynamics, any Gibbs measure, irrespectively of the value of the magnetic field, is invariant. Thus, analogously to the case of the symmetric SEP, we have a family of extremal invariant measures (the Gibbs measures corresponding to all the possible values of the magnetic field) which will be indexed by the magnetization m. If there is no phase transition, m takes all the values in the interval $[-1, 1]$, but when there is a phase transition there is an interval $I \subset [-1, +1]$ of values of m that are forbidden, namely for no value of the magnetic field there is an extremal, translation invariant state with magnetization in I. As one can easily guess, this has dramatic consequences on the macroscopic behavior of the system leading to phenomena like *spinodal decomposition*, free boundary and Stefan problems, interface dynamics, formation of mushy regions, metastable behavior and critical fluctuations. Here we will concentrate on the spinodal decomposition and the critical fluctuations.

The spinodal decomposition is a phenomenon that appears when the temperature of a system is suddenly lowered (*quenched*) from above to below its *critical* value (namely the temperature where there is a bifurcation from absence to presence of phase transition). The magnetization, which during the (very rapid) quenching process does not change significantly, is chosen so that it lies, at the final temperature, in the forbidden interval I. In the interesting cases, the state is then stationary, but unstable for the deterministic macroscopic equation that describes the system. The stochastic corrections to this equation have then the fundamental role of inducing the escape from such an unstable stationary state with the formation of the pure phases in a non trivial, random spatial pattern. The critical fluctuations refer to the anomalously large magnetization fluctuations that occur at the critical temperature and they are described by nonlinear SPDE's, like the Cahn–Allen equation with additive white noise.

6.1. Gibbs Measures. We use the following notation: for any $\Lambda \subset \mathbb{Z}^d$, we denote by σ_Λ a configuration in Λ, i.e. an element in $\{-1,+1\}^\Lambda$. When $\Lambda = \mathbb{Z}^d$, we simply write σ. We call $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$ and equip it with the product topology.

Since the analysis of models with phase transitions presents great difficulties in general, we will restrict drastically the class of systems by considering Ising systems with Kac potentials. For any value of the Kac parameter $\gamma > 0$, we define the *Gibbs measure* as a probability measure which can be formally written as

$$\frac{e^{-\beta H_\gamma(\sigma)}}{\sum_{\sigma\in\Omega} e^{-\beta H_\gamma(\sigma)}}, \tag{6.1}$$

$\beta > 0$ is the inverse temperature and

$$H_\gamma(\sigma) = \frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} J_\gamma(x,y)\sigma(x)\sigma(y) + h\sum_{x\in\mathbb{Z}^d}\sigma(x), \tag{6.2}$$

where $h \in \mathbb{R}$ is the external magnetic field and $J_\gamma(x,y)$ the *coupling* strength:

$$J_\gamma(x,y) = \gamma^d J(\gamma(x-y)). \tag{6.3}$$

We suppose that $J \in C^\infty(\mathbb{R}^d)$ is a non negative function supported in the unit ball, with $\int_{\mathbb{R}^d} J(r)dr = 1$. We are therefore looking at a system with interaction range γ^{-1} and interaction strength γ^d, we will take γ small. Clearly neither (6.1) nor (6.2) make sense, since they contain unbounded terms.

There are various ways to define properly *Gibbs measures* (for a large amount of information on this one can look at [**55**],[**37**]) and we will choose to define a Gibbs measure by giving its conditional expectation at each site. After introducing the molecular magnetic field $h_\gamma(x;\sigma_\Delta)$ at $x \notin \Delta$ due to the spin configuration σ_Δ in Δ as

$$h_\gamma(x;\sigma_\Delta) = \sum_{y\in\Delta} J_\gamma(x,y)\sigma(y), \tag{6.4}$$

we define a Gibbs measure $\mu_{\beta,h,\gamma}$ as a probability on the Borel sets of Ω such that for any x the conditional probability that $\sigma(x) = \overline{\sigma} \in \{-1,+1\}$, given the configuration in x^c (short–hand notation for $\{x\}^c$) is

$$\mu_{\beta,h,\gamma}\left(\sigma(x) = \overline{\sigma}|\sigma_{x^c}\right) = \frac{e^{-\beta\overline{\sigma}[h_\gamma(x;\sigma_{x^c})+h]}}{e^{-\beta[h_\gamma(x;\sigma_{x^c})+h]} + e^{\beta[h_\gamma(c;\sigma_{x^c})+h]}} \tag{6.5}$$

almost surely w.r.t. $\mu_{\beta,h,\gamma}(d\sigma_{x^c})$ (which is the marginal of the measure $\mu_{\beta,h,\gamma}$ on $\{-1,1\}^{x^c}$). The conditioning is on the event $\{\sigma' : \sigma'(y) = \sigma_{x^c}(y)$ for all $y \in x^c\}$.

The set of Gibbs measures is non empty, because Ω is compact, and it may contain more than one element, in which case we say that the system undergoes a phase transition. By general theorems on Ising ferromagnetic systems with finite range interactions, for any $\gamma > 0$ there is a unique Gibbs measure, $\mu_{\beta,h,\gamma}$, for any h in $d = 1$ and for $h \neq 0$ in $d \geq 2$. Moreover([**10**],[**18**],[**11**]) in $d \geq 2$ and $h = 0$ there is a unique Gibbs measure if $\beta < 1$ and γ sufficiently small, while if $\beta > 1$, and γ

sufficiently small there are two different, extremal, invariant Gibbs measures, $\mu^{\pm}_{\beta,\gamma}$. They are translationally invariant and any other translationally invariant Gibbs measure is a convex combination of $\mu^{\pm}_{\beta,\gamma}$. For simplicity, even if uniqueness does not hold, we will keep using the notation $\mu_{\beta,h,\gamma}$ to indicate any Gibbs measure with parameters β, h and γ.

6.2. Glauber and Kawasaki Dynamics. We will consider two classes of *reversible* processes: Glauber (spin–flip) dynamics and Kawasaki (spin–exchange) dynamics. The elementary events in the former are spin–flips (i.e. changes of sign of a single spin, (6.7) below) while, in the Kawasaki case, two spins exchange their position, like in the dynamics presented in Part I. In both cases, we build the dynamics from jump rates which will only depend on the difference of energy before and after an elementary transition. To simplify the exposition, from now on a Gibbs measure is always a *translationally invariant* Gibbs measure and an *extremal* Gibbs measure is a measure ergodic with respect to spatial translation.

The dynamics we will be looking at are Markov processes in $D(\mathbb{R}^+;\Omega)$ generated by self–adjoint operators in $L^2(\Omega;\mu)$, where μ is a Gibbs measure at given β, h and γ. Self–adjointness (of the generator of the dynamics in $L^2(\Omega;\mu)$) is sometimes called *reversibility* and the process *reversible*, see [**95**]). Self–adjointness trivially implies that μ is invariant for the dynamics. All the details on the construction of these processes are given in [**81**].

The generator of the particular Glauber dynamics we shall consider here, L^G_γ, acts on the local (cylinder) functions f as

$$L^G_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_\gamma(x;\sigma)\,[f(\sigma^x)-f(\sigma)]\,, \tag{6.6}$$

where

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x, \end{cases} \tag{6.7}$$

$$c_\gamma(x;\sigma) = \frac{e^{-\beta\sigma(x)[h_\gamma(x,\sigma|_{x^c})+h]}}{e^{-\beta[h_\gamma(x,\sigma|_{x^c})+h]}+e^{\beta[h_\gamma(x,\sigma|_{x^c})+h]}}. \tag{6.8}$$

Observe that the *flip rate* $c_\gamma(x;\sigma)$ in (6.8) is a nonnegative function $\Phi(E)$ of the difference of energy E before and after a spin–flip

$$H_{\gamma,\{x\}}(-\sigma|\sigma_{x^c}) - H_{\gamma,\{x\}}(\sigma|\sigma_{x^c}) = 2\sigma(x)\,[h_\gamma(x,\sigma_{x^c})+h]\,. \tag{6.9}$$

The choice (6.8) is not the only possible one: any flip rate defined in terms of a function $\Phi(E)$ (with some regularity conditions) that satisfies the detailed balance condition (see [**95**], page 161, and [**81**], Section 4)

$$\Phi(E) = \exp(-\beta E)\Phi(-E) \qquad \text{for all } E\in\mathbb{R}, \tag{6.10}$$

leads to a reversible process with invariant measure μ. (6.8) is obtained by choosing $\Phi(E) = e^{-\beta E/2}/(e^{\beta E/2}+e^{-\beta E/2})$.

The generator of the Kawasaki dynamics acts on the local functions as

$$L^K_\gamma f(\sigma) = \sum_{x,y\in\mathbb{Z}^d} c_\gamma(x,y;\sigma)[f(\sigma^{x,y})-f(\sigma)]. \tag{6.11}$$

By $\sigma^{x,y} \in \Omega$, as in (3.3), we mean the configuration obtained from σ by exchanging the spins at the sites x and y,

$$\sigma^{x,y}(z) = \begin{cases} \sigma(x) & \text{if } z = y \\ \sigma(y) & \text{if } z = x \\ \sigma(z) & \text{otherwise.} \end{cases} \tag{6.12}$$

We set

$$c_\gamma(x, y; \sigma) = \begin{cases} \Phi(\Delta_{x,y} H_\gamma) & \text{if } |x - y| = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{6.13}$$

with $\Phi \in C^2$ which satisfies (6.10) and

$$\Delta_{x,y} H_\gamma(\sigma) = H_{\gamma,\{x,y\}}\left(\sigma^{x,y} \big| \sigma|_{\{x,y\}^c}\right) - H_{\gamma,\{x,y\}}\left(\sigma \big| \sigma|_{\{x,y\}^c}\right). \tag{6.14}$$

For convenience we set $\Phi(0) = 1$, which by (6.10) implies $\Phi'(0) = -\beta/2$ and, as it will be explained below, at the hydrodynamic level all that can be observed of Φ is its value and the value of its first derivative at 0. The general case is obviously recovered by a time change.

It is easy to realize that the rates in (6.13) are h independent. Due to the condition (6.10), any measure in the family $\{\mu_{\beta,h,\gamma}\}_{h\in\mathbb{R}}$ is invariant. This is the first important difference between the Glauber and the Kawasaki dynamics. In the Glauber dynamics for each different choice of the parameters β, h and γ (we have fixed once for all J) we have different jump rates. Correspondingly only the Gibbs measures $\mu_{\beta,h,\gamma}$ with the same magnetic field as the one used in the definition of the dynamics are invariant (apart for degenerate cases). For the Kawasaki dynamics, instead, the magnetic field h does not appear in the definition of the rates: this is an immediate consequence of the fact that the magnetization is locally conserved in a spin exchange. As a consequence all the Gibbs measures, no matter what is the value of h, are invariant and we thus have the one parameter family of invariant measures $\{\mu_{\beta,h,\gamma}\}_{h\in\mathbb{R}}$ with the magnetic field h that plays for the spin systems the role that the *chemical potential* has for the particle systems.

The presence of a *first order* phase transition at $h = 0$ for $\beta > 1$ and h sufficiently small means that, for the same value of β and γ, the function $m(h) = E_{\beta,h,\gamma}(\sigma(x))$, well defined for $h \neq 0$ (and in fact analytical and, due to the condition $J \geq 0$, monotonic increasing in $\mathbb{R}\setminus\{0\}$), see [**55**],[**37**]), has left limit $-m_{\beta,\gamma}$ and right limit $+m_{\beta,\gamma}$, with $m_{\beta,\gamma} > 0$. Therefore if we call I the interval $(-m_{\beta,\gamma}, +m_{\beta,\gamma})$ (the *forbidden* interval), the equation $m(h) = \overline{m} \in I$ cannot be solved. Any Gibbs measure with average $m \in I$ is a linear combination of the extremal states with magnetization $m \notin I$.

6.3. Macroscopic Limits and Non Local Evolution Equations. We take γ^{-1} as our macroscopic space unit and we are interested in the limit as $\gamma \to 0$. Let $m_0 \in C^\infty(\mathbb{R}^d; [-1, 1])$ be a macroscopic initial profile which at the spin level is represented by the product measures $\mu^{(\gamma)}$ on Ω with averages

$$E_{\mu^{(\gamma)}}\big(\sigma(x)\big) = m_0(\gamma x) \tag{6.15}$$

(just as in Section 3, (3.7) with γ instead of ϵ). Denoting by $P_\nu^{(\gamma,G)}$ and by $P_\nu^{(\gamma,K)}$ the processes starting from ν with generators L_γ^G and, respectively, L_γ^K we have ([26] for Glauber, [78],[58],[59] for Kawasaki[1])

THEOREM 6.1. *For any $\delta > 0$, $\tau \geq 0$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ the following holds.*
(i). Glauber dynamics.

$$\lim_{\gamma\to 0} P_{\mu^{(\gamma)}}^{(\gamma,G)}\left(\left|\gamma^d \sum_{x\in\mathbb{Z}^d} \phi(\gamma x)\sigma(x,\tau) - \int_{\mathbb{R}^d} dr\, \phi(r) m^{(G)}(r,\tau)\right| < \delta\right) = 1, \tag{6.16}$$

where $m^{(G)}(r,\tau)$ is the unique solution of

$$\frac{\partial m^{(G)}}{\partial \tau}(r,\tau) = -m^{(G)}(r,\tau) + \tanh\left\{\beta((J * m^{(G)})(r,\tau) + h)\right\}, \tag{6.17}$$

with initial datum m_0. Here $(J * m^{(G)})(r,\tau)$ denotes the convolution of J and $m^{(G)}(\cdot,\tau)$ evaluated at r.
(ii). Kawasaki dynamics.

$$\lim_{\gamma\to 0} P_{\mu^{(\gamma)}}^{(\gamma,K)}\left(\left|\gamma^d \sum_{x\in\mathbb{Z}^d} \phi(\gamma x)\sigma(x,\gamma^{-2}\tau) - \int_{\mathbb{R}^d} dr\, \phi(r) m^{(K)}(r,\tau)\right| < \delta\right) = 1, \tag{6.18}$$

where $m^{(K)}(r,0) = m_0(r)$ and $m^{(K)}(r,\tau)$ is the unique solution of

$$\frac{\partial m^{(K)}}{\partial t}(r,\tau) = \Delta m^{(K)}(r,\tau) - \beta\nabla\left[(1 - m^{(K)}(r,\tau)^2)(\nabla J * m^{(K)})(r,\tau)\right], \tag{6.19}$$

and $*$ is again the convolution in the spatial variable r.

A more accurate analysis of the process shows that for both dynamics the spins are to leading order (as $\gamma \to 0$) mutually independent. This is a mean field effect due to the fact that for small γ the direct interaction between two spins is negligible. Such a property is also true for the extremal Gibbs measures, but, while in equilibrium the magnetization has constant values, in Theorem 6.1 the value is determined by the initial datum and the evolution equation and may thus be an arbitrary function. This is a short time effect, as it is clear in the Glauber case where times are not scaled at all and each spin undergoes only a finite number of flips in a finite time. Thus Theorem 6.1 describes the early stage of the evolution while the main interest is the long time behavior, in particular if and how the true equilibrium is reached. The natural question is then if such effects are still well described by (6.17) and (6.19). Behind this there is an inversion of limits: we want first to run the process for a long time and then take $\gamma \to 0$, while the opposite is done if we look at the long time behavior of (6.17) and (6.19). It is not obvious at all that the two procedures agree. In fact we will see that there are cases where this is not true and we will concentrate on those where the corrections to (6.17) and (6.19) are of stochastic nature. Before considering the fluctuations from the

[1]The result stated for the case of the Kawasaki dynamics does not appear in the literature. The correlation function techniques in [78],[58] can however be used to establish it. The result in [59] is the one quoted here, but in finite macroscopic volumes. In order to cover the infinite volume statement presented here one needs some extra arguments, like in [47].

deterministic evolutions (6.17) and (6.19), we give a heuristic explanation of why the relevant time scale in the Kawasaki case turns out to be γ^{-2}. This can be understood in the following way: the jump rate can be expanded to obtain

$$\begin{aligned} c_\gamma(x,y;\sigma) &= 1-\frac{\beta}{2}[\sigma(y)-\sigma(x)]\sum_{z\neq x,y}(J_\gamma(z,x)-J_\gamma(z,y))\sigma(z)+O(\gamma^2) \\ &= 1-\frac{\beta\gamma}{2}[\sigma(y)-\sigma(x)]\gamma^d\sum_{z\neq x,y}(y-x)\cdot\nabla J(\gamma(x-z))\sigma(z)+O(\gamma^2), \end{aligned} \tag{6.20}$$

which in particular says that the process is a $O(\gamma)$ correction of the symmetric SEP. It is in fact similar to the WASEP considered in Section 4, but the asymmetry, i.e. the $O(\gamma)$ term in (6.20), is configuration dependent. This dependence upon the configurations is nice, since it is averaged over a length of order γ^{-1}, exactly the scale on which we are looking for the result. Recalling that the hydrodynamic limit of the WASEP process is (2.1), it does not come as a surprise that in this case the hydrodynamic equation is (6.19).

6.4. Small Fluctuations. In agreement with the Fluctuating Hydrodynamic Theory, the small deviations from the macroscopic limit are described by the solution of a linear SPDE obtained by linearizing the macroscopic equation and adding a stochastic noise. Using the notation and assumptions of Theorem 6.1 we introduce the fluctuation fields

$$X_\tau^{(\gamma,G)}(\phi)=\gamma^{d/2}\sum_{x\in\mathbb{Z}^d}\phi(\gamma x)[\sigma(x,\tau)-m^{(G)}(\gamma x,\tau)], \tag{6.21}$$

$$X_\tau^{(\gamma,K)}(\phi)=\gamma^{d/2}\sum_{x\in\mathbb{Z}^d}\phi(\gamma x)[\sigma(x,\gamma^{-2}\tau)-m^{(K)}(\gamma x,\tau)]. \tag{6.22}$$

THEOREM 6.2. *For any $T>0$ the family $\{X^{(\gamma,G)}\}_{\gamma>0}$ is weakly convergent in $D([0,T];\mathcal{S}')$ as $\gamma\to 0$. The limit process $\{\xi^{(G)}(r,t)\}$ is in $C([0,T];\mathcal{S}')$ and satisfies the following SPDE*

$$\begin{aligned} \frac{\partial\xi^{(G)}(r,t)}{\partial t} &= \int_{\mathbb{R}}\mathcal{L}(r,r',t)\xi^{(G)}(r',t)\,dr' \\ &+ \sqrt{2[1-m^{(G)}(r,t)\tanh\{\beta(J*m^{(G)})(r,t)+h\}}\,\dot W(r,t) \end{aligned} \tag{6.23}$$

and

$$\mathcal{L}(r,r',t)=-\delta(r-r')+\frac{\beta J(r-r')}{\cosh^2[\beta(J*m^{(G)})(r,t)+h]}. \tag{6.24}$$

We stress once again that the drift term of the diffusion process $\xi^{(G)}$ is simply given by the linearization of the operator on the right–hand side of (6.17) around the solution $m^{(G)}(r,t)$.

Also in the Kawasaki case the fluctuations are expected to scale to an infinite dimensional diffusion process whose drift is given by the linearization of (6.19) around a solution $m^{(K)}(r,\tau)$ and the martingale part is the gradient of white noise, in space and time, with a spatial modulation. In SPDE terms they should scale to the solution of

$$\begin{aligned}\frac{\partial \xi^{(K)}}{\partial t} = \Delta \xi^{(K)} \quad &- \quad \beta \nabla \left[(1 - [m^{(K)}]^2) \nabla J * \xi^{(K)} \right] \\ &+ \quad 2\beta \nabla \left[\xi^{(K)} \nabla J * m^{(K)} \right] + \nabla \left[\sqrt{1 - [m^{(K)}]^2} \dot{W} \right].\end{aligned} \tag{6.25}$$

This result is not proven, but it should be inside the realm of the known techniques **[24]**,[31],[26],**[19]**.

7. Nonlinear Fluctuations: Stochastic Allen–Cahn and Cahn–Hilliard Equations

In this section we will see that in a neighborhood of the critical temperature both the Glauber and the Kawasaki dynamics considered in the previous section have anomalous fluctuating behaviors. We will show that, for certain initial conditions, a suitable random function on the $d = 1$ Glauber spin flip process converges to the solution of a stochastic Reaction–Diffusion equation with additive white noise. We will also discuss partial results for the $d = 2$ Glauber process and present some heuristics about the Kawasaki dynamics that indicate convergence to a fourth order PDE with *additive conservative white noise.*

7.1. The General Strategy. We will only study deviations from the constant profile $m(r) = 0$, represented at the spin level by the product measure μ with 0 averages, i.e. $E_\mu(\sigma(x)) = 0$ for all x. At the macroscopic level the state $m \equiv 0$ is stationary because $m(r,t) = 0$ is a stationary solution both for (6.17) (with $h = 0$) and (6.19). The first order corrections to the state $m \equiv 0$ are described by the limit behavior of the fluctuation fields, characterized by (6.23) and (6.25). These are linear equations that can be solved quite explicitly. The interesting case for us is when their solutions diverge as $t \to \infty$, so that, according to the linear theory, the deviations from the state $m \equiv 0$ grow indefinitely. This cannot be correct at the spin level because the validity of the linear approximation, which leads to (6.23) and (6.25), fails when the fluctuations become too large. Exactly at this point the non linear effects, that could be neglected so far, become relevant and we may have both non linear and stochastic terms all together, hence non linear SPDE's.

We start from the Glauber dynamics with $h = 0$. Since $m^{(G)}(r,t) = 0$ we get from (6.23) that the limit fluctuation field $\xi^{(G)}(r,t)$, which we simply write here as $\xi(r,t)$, satisfies the equation

$$\frac{\partial \xi(r,t)}{\partial t} = -\xi(r,t) + \beta (J \star \xi)(r,t) + \dot{W}(r,t). \tag{7.1}$$

The covariance is

$$C_t(r,r') = E\big(\xi(r,t)\xi(r',t)\big), \qquad C_0(r,r') = \delta(r - r'). \tag{7.2}$$

Setting $\alpha = 1 - \beta$ and recalling that the integral of J is equal to 1 (we look here at the case in which J is *isotropic*, i.e. it depends on r only via $|r|$), we have

$$C_t(r,r') = 2\beta \int_0^t ds\, e^{-2\alpha s} \int dr_1 \int dr_2 e^{Gs}(r,r_1) e^{Gs}(r',r_2) J(r_1 - r_2) \tag{7.3}$$

where G is the generator of the following jump process on $\mathbb{R}^d$:

$$(Gf)(r) = \beta \int dr' J(r-r')[f(r') - f(r)]. \tag{7.4}$$

7.2. The Stable Case, $\beta < 1$. We have from (7.3)

$$\lim_{t\to\infty} C_t(r,r') = C_\infty(r,r') \tag{7.5}$$

By (6.21) and Theorem 6.2 we then have

(7.6)

$$\lim_{t\to\infty}\lim_{\gamma\to 0} E_\mu^{(\gamma,G)}\Big(X_t^{(\gamma,G)}(\phi)X_t^{(\gamma,G)}(\psi)\Big) = \int\int dr dr' \phi(r)\psi(r')\{\delta(r-r') + C_\infty(r,r')\}.$$

In [16] it is proved that

(7.7)

$$\lim_{\gamma\to 0} E_{\mu_{\beta,\gamma}}\Big(X_0^{(\gamma,G)}(\phi)X_0^{(\gamma,G)}(\psi)\Big) = \int\int dr dr' \phi(r)\psi(r')\{\delta(r-r') + C_\infty(r,r')\},$$

where $\mu_{\beta,\gamma}$ is the unique Gibbs measure in $\mathbb{Z}^d$, recall that $\beta < 1$.

In conclusion when $\beta < 1$ the fluctuations remain *small* and converge to their equilibrium value (7.7): we do not have in this case the desired growth of the fluctuations nor the nonlinear SPDE's.

7.3. The Unstable Case, $\beta > 1$. We have $\alpha = 1 - \beta < 0$, so that, contrary to the stable case, the covariance in (7.3) diverges exponentially as $t \to \infty$. This seems the best setup for finding nonlinear effects and indeed they will appear. But the instability is so strong that when they enter into play, the random effects are already gone. We will deal with this case in Section 8.

7.4. The Critical Case, $\beta = 1$. In this case $\alpha = 0$ so that the drift term in (7.1) is given by the generator G defined in (7.4) which, for what follows, should be thought of as a sort of Laplacian. The magnitude of the fluctuations $\xi(r,t)$ depends then in a highly non trivial way on the size of the system and when this is defined on the whole space, the number of dimensions of the space become critically relevant. We will consider first the system in a unit torus of $\mathbb{R}^d$, then in the whole $\mathbb{R}^d$, first with $d = 1$ and finally with $d = 2$.

In the unit torus $\mathbb{R}^d/\mathbb{Z}^d$ the total fluctuation is

$$\Xi_t = \int_{\mathbb{R}^d/\mathbb{Z}^d} dr\, \xi(r,t), \tag{7.8}$$

and its typical magnitude is

$$K_t = E\Big(\Xi_t^2\Big)^{1/2} = \Big(\int dr\, C_t(0,r)\Big)^{1/2}, \tag{7.9}$$

which by (7.3) grows like $t^{1/2}$. In the whole $\mathbb{R}^d$ the covariance $C_t(0,r)$ decays on the scale $r \approx \sqrt{t}$. We then replace (7.8) by averaging $\xi(r,t)$ over the ball $B(\sqrt{t})$ of volume $t^{d/2}$ which is centered at 0. We thus set

$$\Xi_t = t^{-d/2} \int_{B(\sqrt{t})} dr\, \xi(r,t). \tag{7.10}$$

Analogously to (7.9), its typical magnitude is

$$K_t = E\Big(\Xi_t^2\Big)^{1/2} \approx \Big(t^{-d/2}\int_{\mathbb{R}^d} dr\, C_t(0,r)\Big)^{1/2}. \tag{7.11}$$

Using (7.3) we then conclude that K_t grows proportionally to $t^{1/4}$ in $d=1$ while it remains bounded in $d=2$.

However while in $d=1$ (and in the unit torus as well) $\xi(r,t)$ is close to Ξ_t, this is no longer true in $d=2$ where in fact $\xi(r,t)$ is defined only as a distribution. This reflects on the fact that a *local fluctuation*

$$\Xi_t^\star = \int_B dr\, \xi(r,t) \tag{7.12}$$

B a unit ball, has typical magnitude of the order of $(\log t)^{1/2}$ (recall that $d=2$). When averaging over $B(\sqrt{t})$, all these logarithmic divergences compensate so that Ξ_t is typically bounded. But when the non linear terms become important there is no longer an exact compensation and, as we will see, such a logarithmic divergence will play a crucial role, with effects similar to those emerging in the Euclidean field theory.

7.5. Rescaled and Renormalized Fluctuations. Recalling the normalization in (6.21), the deviations from $m^{(G)}=0$ in the regime of validity of (7.1) are given by $M_t^{(\gamma)} = \gamma^{d/2}\Xi_t$. Since the non linearity in (6.17) around $m^{(G)}=0$ is cubic, we may expect this term to be relevant on a time interval t if $[M_t^{(\gamma)}]^3 t$ has the same order as $M_t^{(\gamma)}$ itself, i.e. what produced by (7.1) in the same time interval. Recalling the previous computations of the typical value, K_t, of Ξ_t, we find that the above criterion yields for the critical time scale in the unit torus of $\mathbb{R}^d$

$$\gamma^{d/2}t^{1/2} = [\gamma^{d/2}t^{1/2}]^3 t \tag{7.13}$$

namely

$$t = \gamma^{-d/2}, \qquad M_t^{(\gamma)} = \gamma^{d/4} \tag{7.14}$$

The same argument in $\mathbb{R}$ gives

$$\gamma^{1/2}t^{1/4} = [\gamma^{1/2}t^{1/4}]^3 t, \qquad t = \gamma^{-2/3},\ \ M_t^{(\gamma)} = \gamma^{1/3} \tag{7.15}$$

while, in $\mathbb{R}^2$, since K_t remains bounded,

$$\gamma = [\gamma]^3 t, \qquad t = \gamma^{-2},\ \ M_t^{(\gamma)} = \gamma. \tag{7.16}$$

In all the above cases $M_t^{(\gamma)}$ is still infinitesimal as $\gamma \to 0$, it is then natural to conjecture that the only relevant non linearity is cubic and that this critical regime is described by

$$\frac{\partial X}{\partial t} = -X + J\star X - \frac{1}{3}(J\star X)^3 + \gamma^{d/2}\dot{W} \tag{7.17}$$

Besides its relation with the spin system, equation (7.17) looks like a natural regularization of the Cahn–Allen equation with noise which is therefore interesting in its own right. The heuristics behind (7.14), (7.15) and (7.16) suggests to study (7.17) with the following change of variables:

$$Z_t^{(\gamma)}(r) = \begin{cases} \gamma^{-d/4} X(r, \gamma^{-d/2} t) & \text{in } \mathbb{R}^d/\mathbb{Z}^d \\ \gamma^{-1/3} X(\gamma^{-1/3} r, \gamma^{-2/3} t) & \text{in } \mathbb{R} \\ \gamma^{-1} X(\gamma^{-1} r, \gamma^{-2} t) & \text{in } \mathbb{R}^2 \end{cases} \tag{7.18}$$

According to the previous heuristics, $Z_t^{(\gamma)}$ should (or better may) converge as $\gamma \to 0$ to a process described by a nonlinear SPDE.

We conclude this subsection with the particle analog of $Z_t^{(\gamma)}$:

$$Y_t^{(\gamma)}(\phi) = \begin{cases} \gamma^{-d/4+d} \sum \sigma(x, \gamma^{-d/2} t) \phi(\gamma x) & \text{in } \mathbb{Z}^d/[\gamma^{-1}]\mathbb{Z}^d \\ \gamma^{-1/3+(1+1/3)} \sum \sigma(x, \gamma^{-2/3} t) \phi(\gamma^{1+1/3} x) & \text{in } \mathbb{Z} \\ \gamma^{-1+4} \sum \sigma(x, \gamma^{-2} t) \phi(\gamma^2 x) & \text{in } \mathbb{Z}^2. \end{cases} \tag{7.19}$$

7.6. The Limit Equations. The above conjecture is proved in the torus and in $d = 1$ dimensions for the spin system, we are not aware of the analogous result for $Z_t^{(\gamma)}$. The case in $\mathbb{R}$ is solved in [**7**],[**49**], the analysis however easily extends to the case of the torus. Starting from the latter, we have that $\{Y_t^{(\gamma)}\}_{t \geq 0}$ converges weakly to the solution of the ordinary stochastic differential equation

$$\frac{dX}{dt} = -\frac{1}{3} X^3 + \dot{W} \tag{7.20}$$

(i.e. long times in bounded regions suppress all the spatial structure). In $\mathbb{R}$ instead $\{Y_t^{(\gamma)}\}_{t \geq 0}$ converges weakly to the solution of the Cahn–Allen equation with noise

$$\frac{\partial X}{\partial t} = \Delta X - \frac{1}{3} X^3 + \dot{W}. \tag{7.21}$$

7.7. Critical Fluctuations in $d = 2$. It may look at first sight surprising that in $d = 2$, for any $a > 0$ $\{Y_t^{(\gamma)}\}_{t \geq a}$ converges weakly to the field identically equal to 0. The same phenomenon happens in equilibrium, [**16**]: calling $\mu_{\beta,\gamma}$, here $\beta = 1$, the (unique) Gibbs measure on $\{-1, 1\}^{\mathbb{Z}^2}$ (uniqueness at $\beta = 1$ is also proved in [**16**]), we have that for any test function ϕ

$$\lim_{\gamma \to 0} E_{\mu_{\beta,\gamma}} (Y_0^{(\gamma)}(\phi)^2) = 0, \tag{7.22}$$

where $Y_0^{(\gamma)}(\phi)$ is given by the last expression in (7.19). The same conclusions obviously apply to $Y_t^{(\gamma)}(\phi)$, for any $t > 0$, if the process starts from the invariant measure $\mu_{\beta,\gamma}$. This shows that the normalization and rescaling in (7.19) are not correct in $d = 2$ at $\beta = 1$. The question has been studied in [**48**] where it is shown that if the Glauber process starts from the equilibrium $\mu_{\beta,\gamma}$, then the correct equation that describes to leading orders the magnetic density field is

$$\frac{\partial X_t^{(\gamma)}}{\partial t} = [J \star X_t^{(\gamma)} - X_t^{(\gamma)}] - C\gamma^2 \log \gamma^{-1} J \star X_t^{(\gamma)} + \gamma \dot{W}, \qquad C = \frac{1}{\pi D} \tag{7.23}$$

$$D = \int J(r) r^2 \mathrm{d}r \tag{7.24}$$

Observe that as $\gamma \to 0$, the solution of (7.23) vanishes so that the 0-th order approximation to $X_t^{(\gamma)}$ is the stationary state $m_\beta \equiv 0$. By multiplying $X_t^{(\gamma)}$ by

$\gamma^{-d/2} = \gamma^{-1}$, we recover the fluctuation fields and in fact in the limit as $\gamma \to 0$ with time running in a compact interval, we obtain (7.1) (with $\beta = 1$). The new result in [**48**] is that the spin systems follows (6.22) also in the regime defined by the following renormalization procedure

$$u(r,t) = \gamma^{-1}\lambda^{-2}t(\lambda^{-1}r), \qquad \lambda = \gamma[\sqrt{\log\gamma^{-1}}]^{-1}. \tag{7.25}$$

If $X_t^{(\gamma)}$ solves (7.23), the function u in (7.25) is actually independent of γ and it solves

$$\frac{\partial u}{\partial t} = \Delta u - Cu + \dot{W}. \tag{7.26}$$

Thus the limit of the Glauber process under the renormalization as in (7.25) is described by the solution of (7.26).

The analogous limit procedure applied to (7.1) leads to (7.26) without the term $-Cu$ and since (7.1) describes the linear approximation, we may say that the extra term $-Cu$ is due to non linear effects. The time and space rescaling in the definition of $Y_t^{(\gamma)}$, see (7.19), is obtained by an extra rescaling with respect to that in (7.25). Again in [**48**] it is shown that there is still agreement between (7.23) and the Glauber process under this extra rescaling, thus the Glauber process is described by the long time behavior of (7.26). The *extra mass* term $-Cu$ determines a decay of correlations in (7.26) which therefore explains the triviality of the limit (7.22).

To summarize, we have seen that the non linearity in (7.17) reflects in an extra mass term $-Cu$ in (7.26) and that this is responsible for a decay of correlations not present in (7.1) which ultimately causes the triviality of the limit of $Y_t^{(\gamma)}$ and the failure of the heuristic argument for the derivation of a non linear SPDE. We can say a little more about this issue, but unfortunately only at a heuristic level. Indeed (7.23) can be formally derived from (7.17) by the substitution

$$-\frac{1}{3}(J \star X)^3 \longrightarrow C\gamma^2 \log\gamma^{-1}(J \star X). \tag{7.27}$$

On the other hand the covariance $C_t(0,0)$ when $\beta = 1$ and $d = 2$ behaves for $t = \gamma^{-2}$ just as $C\log\gamma^{-1}$ (the extra factor γ^2 comes from the normalization of the fluctuations). The result may thus be interpreted by saying that the leading contribution to $(J\star X)^3/3$ is given by $J\star X$ times the covariance of $(J\star X)$ computed using the linear approximation (at the times under consideration). As already observed, in $d = 2$ such a value is much larger (by a factor $\log\gamma^{-1}$) than the average magnitude K_t, $t = \gamma^{-2}$, given by (7.11) (which remains bounded). The non linear effects are here important and they occur when the noise is still effective, but they manifest via (7.27) only as a linear extra mass term in the final equation (7.26). To kill the extra mass term we go back to (6.17) and repeat the expansion that leads to (7.17) when the inverse temperature is $\beta = 1 + a\gamma^2\log\gamma^{-1}$, $a > 0$. We formally obtain to leading order in γ the new equation

$$\frac{\partial X}{\partial t} = \beta[J \star X - X] + [a\gamma^2\log\gamma^{-1}]X - \frac{1}{3}(J \star X)^3 + \gamma^{d/2}\dot{W}. \tag{7.28}$$

The conjecture is that by letting $a \to C$, C as in (7.23), the second term in (7.28) will cancel the extra mass term in (7.23). Unfortunately there is not yet a proof of such a statement, neither at the spin level nor starting from (7.28) itself.

The second term on the right hand side of (7.28) is the Wick regularization of the cubic term which provides a nice (and we think enlightening) interpretation of the Wick regularization in terms of critical phenomena in statistical mechanics. In fact the value $\beta = 1 + C\gamma^2 \log \gamma^{-1}$ has a particular meaning in the theory of Gibbs states. In [16] it is proven that the inverse critical temperature equal to 1 in the mean field limit, is for $\gamma > 0$ not smaller than $1 + C\gamma^2 \log \gamma^{-1}$. It is conjectured that this is the true value of the critical temperature, possibly modulo corrections of higher order in γ.

7.8. Kawasaki Dynamics. An analogous scheme of analysis (stable, critical, unstable) can be repeated for the Kawasaki case. Most of the rigorous results are however still missing. Here we look at the heuristics in the critical case ($\beta = 1$) and in the next section we will briefly consider the unstable case ($\beta > 1$).

As in the last part of the previous subsection we will consider temperatures which are γ–dependent, more precisely we will look at cases in which the temperature is close to the critical one: this is done because we want to obtain a more general form of the limiting equation. We will always work in $d = 1$. We set $\beta = \beta(\gamma) = 1 + a\gamma^{2/3}$ ($a \in \mathbb{R}$), $a \in \mathbb{R}$, and start the process with the product measures $\mu^{(\gamma)}$ such that for any $x \in \mathbb{Z}$

$$\mu^{(\gamma)}(\sigma(x)) = \gamma^{1/3} m^*(\gamma^{4/3} x), \tag{7.29}$$

where m^* is a continuous bounded function. The formal analysis below suggests that the law of the random function

$$\sigma_t^\gamma(r) \equiv \gamma^{-1/3} \sigma_{t\gamma^{-3-1/3}}([\gamma^{-4/3} r]), \tag{7.30}$$

converges (weakly and for the sake of definiteness in the same sense as in the Glauber case) to the law $\mathcal{P}$ on $C(\mathbb{R}^+; C^0(\mathbb{R}))$ of the solution of the following SPDE

$$\frac{\partial m}{\partial t} = \Delta \left[U'(m) - \frac{D}{2} \Delta m \right] + \sqrt{2} \nabla \dot{W}, \tag{7.31}$$

$$m(0, \cdot) = m^*(\cdot), \tag{7.32}$$

where D is defined in (7.24) and

$$U(m) = \frac{m^4}{4} - \frac{am^2}{2}. \tag{7.33}$$

It is already a nontrivial issue to give a meaning to (7.31) and hence to define $\mathcal{P}$. At first sight it may seem that the noise is too irregular to allow the existence of a solution which is continuous in space and time. This is however not true: the right–hand side of (7.31) contains the operator $-\Delta^2$, which has very strong regularizing properties. In particular it is easy to see that if U' is linear, m will be in $C(\mathbb{R}^+; C^0(\mathbb{R}))$. The nonlinear case is of course much less trivial, but in [50] it is shown that if $U(m) = m^2 + V(m)$, where V is a bounded C^3 function, (7.31) has a unique solution in $C(\mathbb{R}^+; C^0)$ for a suitable class of initial data. The extra problem that we encounter here is the fact that U' is not globally Lipschitz. Choosing properly the original lattice domain, we could have convergence to equation (7.31) on a torus and, in this case, one can deal with the lack of a global Lipschitz bound quite directly (see [42] for the case of stochastic Reaction–Diffusion equations and [22] for the case of a stochastic Cahn–Hilliard equation with additive noise, which, however, is not of the conservation law type). In the case of (7.31), as stated above, the method in [65] should apply.

The heuristics of the derivation of (7.31) goes as follows. In analogy with what we did above, to we start off with the following SPDE:

$$\partial_t m = \Delta m - \beta \nabla \left[(1-m^2) \nabla J * m \right] + \gamma^{1/2} \sqrt{2} \nabla \left(\sqrt{1-m^2} \dot{W}_\gamma \right). \tag{7.34}$$

Which is the analog of (4.1). where there is ϵ instead of γ: the *deterministic* part is given by (6.19) and the stochastic part is the same as in (4.1), apart for the factor $\sqrt{2}$. The system we are dealing with now, as explained after formula (6.20), is still a weakly asymmetric process and it is rather easy to see that, to highest order, the noise term is determined by the $O(1)$ part of the generator, which is a simple exclusion process, but speeded up of a factor 2 (whence the $\sqrt{2}$ in front of the noise part). The very same heuristics can be worked out directly for the particle model. As before, in order to use directly the exact scaling relations for the white noise without having to worry about the corrections given by the γ–regularization, we will replace $\dot{W}_\gamma$ with $\dot{W}$.

Now we scale space with $\gamma^{-1/3}$ and time with $\gamma^{-4/3}$: this scaling is non diffusive and reflects the fact that the limit equation is of 4^{th} order. The strange exponents are a consequence of the fact that the nonlinearity is given by a cubic power. More precisely we set

$$u(x,\tau) = \gamma^{-1/3} m(\gamma^{-1/3} x, \gamma^{-4/3} \tau), \tag{7.35}$$

with $x \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$, and we use the scaling properties of the white noise ($\dot{W}(\gamma^{-1/3}x, \gamma^{-4/3}\tau)$ is equal in law to $\gamma^{1/6}\gamma^{2/3}\dot{W}(x,\tau)$) we obtain (recall that $\beta = 1 + a\gamma^{2/3}$)

$$\begin{aligned} \partial_\tau u &= \gamma^{-2/3} \left[\Delta u - (1 + a\gamma^{2/3}) \nabla \left((1 - \gamma^{2/3} u^2) J_{\gamma^{1/3}} * \nabla u \right) \right] \\ &\qquad + \nabla \left(\sqrt{1 - \gamma^{2/3} u^2} \dot{W} \right), \end{aligned} \tag{7.36}$$

where $J_{\gamma^{1/3}}(\cdot) = \gamma^{-1/3} J(\gamma^{-1/3} \cdot)$. Expand $J_{\gamma^{1/3}} * \nabla u$ to obtain

$$\nabla u + \gamma^{2/3} \frac{D}{2} \nabla \Delta u + \dots, \tag{7.37}$$

where D is defined in (7.24). Once we insert (7.37) in (7.36), we realize that the terms of order $\gamma^{-2/3}$ cancel and, if we take only the terms of order 1, we find (7.31). Recalling that equation (6.19) was derived by scaling space with γ^{-1} and time with γ^{-2}, we have thus justified, at a formal level, our claim that the limit behavior of (7.30) is given by (7.31).

8. Macroscopic Effects of Small Fluctuations: the Origin of Spatial Patterns

We will now outline some of the results obtained in **[28]** as a prototype of the following situation: when a solution of a PDE, obtained as macroscopic limit of an IPS or a more general particle system, is *unstable* under small perturbations, the behavior of the IPS, on long macroscopic times, can deviate substantially from the solution of the PDE. Crucial, in the cases discussed here, are the small stochastic corrections which disappear on the space–time scaling of the hydrodynamic limit. Once again we are facing a problem involving the exchange of a limit: one can study the behavior of the macroscopic law, obtained by sending (in this case) γ to zero, for t large, but this may not coincide with the limit that we obtain as γ tends to zero by letting t depend on γ, with $t \to \infty$ as $\gamma \to 0$.

In the notation of Section 7, we are dealing with the case $\alpha < 0$. Therefore, for notational convenience we replace α by $-\alpha$ so that $\alpha = \beta - 1$ is strictly positive.

In **[28]** the authors consider the Glauber dynamics with a Kac potential as in Section 5. We recall that the hydrodynamic limit for this system is given in Theorem 6.1, equation (6.17), and we will look at the case $h = 0$. One can easily see that the generator of the linearized evolution around the stationary solution $m \equiv 0$, considered as an operator in $L^2(\mathbb{R}^d; dr)$, contains α in its spectrum which is contained in $(-\infty, \alpha]$ (recall that $\int J(r)dr = 1$) and therefore we are in an unstable case. If $\beta = \alpha + 1 > 1$, the nonlocal equation (6.16) has three stationary constant solutions $(-m_\beta, 0, +m_\beta)$, $m_\beta > 0$, of

$$m = \tanh(\beta m), \qquad m \in [-1, 1], \tag{8.1}$$

and a straightforward linear analysis shows that $\pm m_\beta$ are stable (m_β coincides with the limit as $\gamma \to 0$ of the equilibrium magnetization $m_{\beta,\gamma}$, **[11]**). We will call the states with magnetization $\pm m_\beta$ the *phases*. If we start with $m \equiv 0$, under small perturbations the system will leave this state and it will try to rearrange itself into regions of the two different phases $\pm m_\beta$. We will now study how the system makes this rearrangement. Let us stress however that Theorem 6.1 holds for this initial condition and therefore, for any *fixed* $\tau \geq 0$, $m = 0$ gives the large scale behavior of the system. On the other hand, Theorem 6.2 is telling us that fluctuations are present and that they are of order $\gamma^{1/2}$, if $d = 1$, and of order $\gamma^{d/2}$ in dimension d (see **[28]**). Due to the instability, the fluctuations grow at exponential rate, as one can easily understand from the linear analysis of (6.17). At time τ they are of order

$$\gamma^{d/2} \exp(\alpha\tau). \tag{8.2}$$

Theorem 6.2, like Theorem 6.1, is proven for τ finite. However, if we *assume* its validity at longer times, we obtain that the fluctuations will be $o(1)$ if $\tau = t|\log\gamma|$ and $t < d/2\alpha$. If the above heuristics are correct, the linearization of (6.16) should be a faithful description of the system up to $t_c|\log\gamma|$. We then set

$$t_c = \frac{d}{2\alpha}, \tag{8.3}$$

when the nonlinearity should take over.

Going back to the fluctuations results of Section 7 we can make a more accurate analysis, by starting again from (7.1), (7.2) and (7.3). In view of the growth of $C_t(r, r')$ it is convenient to change variables in (7.1), writing, for $\lambda > 0$,

$$X_t^{(\lambda)}(r) = \lambda^{-d/2} e^{-\lambda^2 \alpha t} \xi(\lambda r, \lambda^2 t). \tag{8.4}$$

It is not difficult to prove that for any $s > 0$ the process $\{X_t^{(\lambda)}\}_{t \in [s,\infty)}$ converges weakly in $D([s, \infty); \mathcal{S}')$ as $\lambda \to \infty$ to the restriction to $[s, \infty)$ of the process $\hat{\xi}(r, t)$ which is the solution of

$$\frac{d\hat{\xi}(r,t)}{dt} = \frac{D}{2} \Delta \hat{\xi}(r, t), \tag{8.5}$$

with $\hat{\xi}(r, 0)$ a Gaussian distribution with mean 0 and covariance

$$E\left(\hat{\xi}(r, 0)\hat{\xi}(r', 0)\right) = \delta(r - r')\left[1 + \frac{2}{\alpha}\right] \tag{8.6}$$

and D is defined in (7.24). Thus $\hat{\xi}(r,t)$ evolves deterministically and the only stochasticity comes from the initial datum. The result we just stated does not hold if $s=0$: observe in fact that the covariance of $\hat{\xi}(\cdot,0)$, given by (8.6), is larger than the limit as $\lambda\to\infty$ of the covariance of $X_0^{(\lambda)}$. This initial layer effect is the only remnance of the white noise present in (7.1) and absent in (8.5).

When we go beyond the linear analysis, see **[27]**,**[28]**, we eventually see the non linear effects, but they appear only when the deterministic regime described above has already been established. We thus have first small fluctuations and a linear SPDE, then a linear deterministic PDE and finally a non linear PDE still without noise. At this latter stage, described by the nonlinear equation, the stochasticity appears only as a random initial datum. The relation with the IPS is obtained by setting

$$\lambda=\sqrt{|\log\gamma|},\qquad \epsilon=\gamma/\lambda,\qquad \tau_\lambda=t\lambda^2, \tag{8.7}$$

and introducing the renormalized fluctuation field

$$Z_t^\gamma(\phi)=e^{-\alpha\tau_\lambda}\epsilon^{d/2}\sum_{x\in\mathbb{Z}^d}\phi(\epsilon x)\sigma_{\tau_\lambda}(x) \tag{8.8}$$

(to be compared with (6.21)). Then for any $0<s\le t_c=d/2\alpha$ the process $\{Z_t^\gamma\}_{t\in[s,t_c]}$ converges weakly in $D([s,t_c];\mathcal{S}')$ as $\gamma\to0$ to the restriction to $[s,t_c]$ of the process $\hat{\xi}(r,t)$ which solves (8.5). After t_c the non linear effects appear and the evolution is described by (6.17).

The above analysis gives a conjecture on the exit time from the unstable state $m=0$. It turns out that this conjecture can actually be proven and, more surprisingly, that the nonlinear stage of the escape can be treated in great detail too.

We recall that our initial condition for the IPS is the product measure μ with zero average magnetization. We write $P^{(\gamma)}=P_\mu^{(\gamma,G)}$ and $E^{(\gamma)}$ for its expectation.

THEOREM 8.1. *Set $R_\gamma=R\lambda/\gamma$. For any $n\in\mathbb{Z}^+$ and $R>0$ we have that*

$$\lim_{\gamma\to0}\sup_{x_1\ne\ldots\ne x_n,\ |x_i|\le R_\gamma}E^{(\gamma)}\left(\prod_{i=1}^n\sigma_{t\lambda^2}(x_i)\right)=0, \tag{8.9}$$

for all $t\in[0,t_c]$. Moreover

$$\lim_{\gamma\to0}\sup_{x_1\ne\ldots\ne x_n,\ |x_i|\le R_\gamma}\left|E^{(\gamma)}\left(\prod_{i=1}^n\sigma_{t_c\lambda^2+(\log\lambda)^2}(x_i)\right)-\mathcal{E}\left(\prod_{i=1}^n m_\beta\mathrm{sign}(\rho(x_i\gamma/\lambda))\right)\right|=0, \tag{8.10}$$

where $\{\rho(r)\}_{r\in\mathbb{R}^d}$ is a centered Gaussian field with covariance

$$\mathcal{E}\left(\rho(r)\rho(r')\right)=\exp\left\{-\frac{\alpha(r-r')^2}{2d\beta D}\right\}, \tag{8.11}$$

and D is defined in (7.24).

Theorem 8.1 is a simplified version of some of the results in **[28]**. The essence of its content can be summed up in the following four points:

1. The relevant time scale for the growth of the fluctuations is $|\log\gamma|$.
2. On this new time scale nothing really happens, i.e. $m = 0$, up to t_c.
3. *Slightly* after t_c the *phase separation* has taken place and the magnetization is locally concentrated around the values $\pm m_\beta$, i.e. on the phases.
4. The emergent spatial patterns have a typical length scale of $\gamma^{-1}\sqrt{|\log\gamma|} = \epsilon^{-1}$ lattice sites. On this spatial scale the zero-level set of the Gaussian field ρ gives, in distribution, the random (hyper–)surfaces which separate homogeneous regions. This level set is a.s. a smooth manifold (see [**57**]).

Remark: The evolution after the appearance of the interfaces is understood too. It is on the time scale λ^2 and it is given by motion *by mean curvature* of the interfaces between the phases (see [**68**],[**26**]).

We now go back to some simple heuristics in order to connect the results in Theorem 8.1 with PDEs with random initial condition. It would be more correct to consider SPDEs with random initial condition, since in the IPS there is both a dynamical noise and a random initial condition. We have already seen that the effect of the dynamical noise, on the time scale we are interested in, is only to *change the initial condition* and we will see that the observed geometrical patterns are independent of the change in the initial condition. We will thus for simplicity consider only the noise in the initial condition. This analogy with a PDE with random initial datum will turn out to be particularly helpful in understanding better Theorem 8.1 and in trying to outline the main difficulties which come up in the case with a conservation law.

We will exploit below the similarities between the integral equation (6.17) and the Reaction–Diffusion equation (RD). Equation (6.17) can be cast into the form

$$\frac{\partial m}{\partial \tau} = \mathcal{D}(m) + \mathcal{R}(m), \tag{8.12}$$

where

$$\mathcal{R}(m) = -m + \tanh(\beta m) \tag{8.13}$$

and

$$\mathcal{D}(m) = \tanh(\beta(J * m)) - \tanh(\beta m). \tag{8.14}$$

We will interpret $\mathcal{D}$ as a diffusion term and $\mathcal{R}$ as a reaction term. The form (8.12) hence suggests an analogy with the Reaction–Diffusion (or Cahn–Allen) equation [**2**],[**93**]]

$$\frac{\partial m}{\partial \tau} = \frac{\beta D}{2}\Delta m - U'(m), \tag{8.15}$$

with $-U'(m) = \mathcal{R}(m)$. Observe that $U(m)$ is a symmetric double–well potential with minima at $\pm m_\beta$ and $U(m) = -\alpha m^2/2 + O(m^4)$, near $m = 0$. We have drawn here an analogy between (8.12) and (8.15) at an intuitive level, but in fact the goes well beyond (see e.g, [**26**],[**27**],[**28**]). From now on we will stick to the RD equation: there would be no essential change in the heuristics below keeping the integral equation (6.17), but we expect that the reader may be more accustomed to (8.15).

We then consider equation (8.15) with random initial condition,

$$\begin{cases} \partial_\tau m = (\beta D/2)\Delta m - U'(m) \\ m(r,0) = \gamma^{d/2}(W * \varphi_\gamma)(r), \end{cases} \tag{8.16}$$

in which the random field $\{W(r)\}_{r\in\mathbb{R}^d}$ is a white noise. The choice $\varphi(r) = (1/\sqrt{2\pi}^d)$ $\exp(-r^2/2)$ simplifies the notation. Repeating in this context the analysis described above, we look at the solution $\ell(r,\tau)$ of the linearization of (8.16). We have

$$\ell(r,\tau) = \gamma^{d/2} e^{\alpha\tau} (\mathcal{G}_{\tau D\beta} * (W * \phi_\gamma))(r), \tag{8.17}$$

where $\mathcal{G}_\tau(r)$ is the heat kernel ($\mathcal{G}_0(r) = \delta(r)$ and $\partial_\tau \mathcal{G}_\tau(r) = (1/2)\Delta\mathcal{G}_\tau(r)$). Hence $\ell(r,t)$ is, for each $\tau \geq 0$, a centered spatial Gaussian field with covariance

$$E(\ell(r,\tau)\ell(r',\tau)) = \gamma^d \exp\{2\alpha\tau\} \mathcal{G}_{2(\tau D\beta+\gamma)}(r - r'). \tag{8.18}$$

As before, if we look at times of order $\lambda^2 (= |\log\gamma|)$, the natural space scale turns out to be λ. In fact, if we set $\ell_\gamma(x,t) = \ell(\lambda x, \lambda^2 t)$ the scaling properties of the heat kernel immediately yield

$$E(\ell_\gamma(x,t)\ell_\gamma(x',t)) = \gamma^{d-2\lambda t}\lambda^{-d}\mathcal{G}_{2[D\beta t+(\gamma/\lambda^2)]}(x - x'). \tag{8.19}$$

It is not too difficult to bound the difference between ℓ and the solution $m(r,t)$ of (8.16) for all times $t \leq t_c$ (see e.g. [**29**],[**56**]]) and what one obtains at t_c is that the solutions of (8.16) can then be roughly described as distributed according to

$$\lambda^{-d/2}\rho(x), \tag{8.20}$$

where ρ is the Gaussian random field introduced in Theorem 8.1.

What happens right after t_c? Loosely speaking the magnetization is of order $\lambda^{-d/2}$ and due to the instability, which gives an exponential growth, it will take a very short time ($O(|\log\lambda|)$) to make the magnetization profile of order 1, so that the nonlinearity will no longer be negligible. However this argument does not take into account that the simultaneous presence of regions with positive and negative magnetizations may cause, for example, a balance of forces preventing the final growth. To analyze this question, suppose that at time $\lambda^{-2}t_c$ we have a positive magnetization in r. Then with large probability, see (8.20), there will be a large ball (of radius $O(\lambda)$) in which the magnetization is also positive. If the magnetization was positive everywhere, a straightforward application of the monotonicity properties of (8.15) (see e.g. [**93**]) would show that it would approach the value $+m_\beta$ exponentially fast. The magnetization in our case is positive (at least) in a ball of radius $O(\lambda)$ centered in r, which diverges fast enough as γ goes to zero, to make the situation about the same as if the ball were infinite, because the evolution of (8.15) is almost local and one can control the effect of the magnetization outside the ball (see the Barrier Lemma in [**32**] and its applications in [**29**],[**56**],[**57**],[**28**]). The same conclusions hold of course starting from a negative magnetization. It is therefore clear that the one phase domains will be distributed at a time slightly longer than $\lambda^2 t_c$ (see Theorem 8.1) on the space scale λ^{-1} as $\mathrm{sign}(\rho(r))$.

The analysis we proposed is very simplified: in [**57**] and [**28**] one can find much finer results on the structure of the interface between the one–phase domains, but these results go beyond the aims of the present work. All these results, properly translated, should apply also to (8.16).

We conclude this section with three remarks.

Remark 1: We gave a formal analysis using the RD equation (8.15), which is not the right hydrodynamic equation for the long range IPS we are dealing with. RD equations can however be derived ([**24**],[**30**],[**70**],[**36**]) from dynamics of *mixed-type*, i.e. dynamics which are a superposition of Glauber and Kawasaki processes. The generator is chosen of the form

$$L_\gamma = \gamma^{-2} L_0 + L_G, \tag{8.21}$$

where L_0 is the generator of the simple exclusion process and L_G is a local Glauber dynamics:

$$L_G f(\sigma) = \sum_{x \in \mathbb{Z}^d} c(x;\sigma)\left[f(\sigma^x) - f(\sigma)\right], \tag{8.22}$$

where $c(x;\sigma) = c(0;\sigma(\cdot + x))$, with $c(0;\cdot)$ a local function from $\{-1,+1\}^{\mathbb{Z}^d}$ to $\mathbb{R}^+$. The freedom of choosing the Glauber rates c allows us to reproduce various potentials U appearing in (8.15), including the two well potentials considered here (see [**24**]) for an easy formula that links the rates c and the potential U).

The dynamics generated by (8.21) is essentially that of a local mean field. The Glauber process is short range, but the spins are stirred around by the exchange part of the generator, which is speeded up by a factor γ^{-2}. Hence every spin will effectively perform a random walk with jump rates γ^{-2}, so that in a time of order 1 (the time–scale for a Glauber event) each spin will have traveled a distance $O(\gamma^{-1})$. Therefore, loosely speaking, a spin is not really interacting with its immediate neighbors, but with some other spins chosen at random in a ball of radius γ^{-1}. The dynamics (8.21) has a nice interpretation in terms of birth–death spatial processes (see e.g. [**36**],[**71**] for connections with biological problems and [**36**] also for other results in the same spirit as the ones presented here).

Remark 2: The case with a conservation law, that is the pure Kawasaki case is very relevant for the applications, for example to certain coarsening processes in metallurgy (see e.g. [**12**],[**77**],[**86**]). Kawasaki dynamics are typically harder to analyze, but in this case most of the difficulties are present already at the PDE level (for a discussion with numerical simulations on the IPS see [**58**]). In analogy with the Glauber case, we look at the initial value problem (see Theorem 6.1)

$$\begin{cases} \partial_\tau m = \Delta m(r,\tau) - \beta \nabla\left[(1-m^2)(\nabla J * m)\right] \\ m(r,0) = \gamma^{d/2}(W * \varphi_\gamma)(r), \end{cases} \tag{8.23}$$

with $\beta > \beta_c = 1$, or, replacing the nonlocal equation in (8.23) with the Cahn–Hilliard equation [**13**]]

$$\begin{cases} \partial_\tau m = \Delta\left[U'(u) - \Delta u\right] \\ m(r,0) = \gamma^{d/2}(W * \varphi_\gamma)(r), \end{cases} \tag{8.24}$$

in which U is a symmetric double well potential, as before. We observe moreover that the random initial condition problems (8.16), (8.23), (8.24), once properly stated, are well posed even if the initial condition is not regularized, i.e. if $m(r,0) = \gamma^{d/2} W(r)$, and the solution at any $\tau > 0$ is smooth. We also note that the analogy between the integrodifferential equation in (8.23) and the CH equation (8.24) can be pushed quite far [**60**] and we do not expect any significant difference in the behavior of (8.23) and (8.24) on the relevant space–time scales.

Again it is easy to realize that $m = 0$ is an stationary unstable state for both the equation in (8.23) and the equation (8.24) ($m \equiv c$, with $c \in [-1, 1]$, is a stationary solution and it is unstable for $|c| \leq \bar{c}$, where $\bar{c}$ is easily computed). However, unlike the case without a conservation law, large positive values of the spectrum of the linearized evolution operator (the strongest instabilities) are not associated with long wavelength perturbations, but they are concentrated around a definite (finite) wavelength. In fact if we look at the linearization of (8.24) around $m = 0$ for the Fourier transform $\hat{m}(k, \tau)$ ($k \in \mathbb{R}^d$), we obtain ($U'(m) = \alpha m + O(m^3)$)

$$\frac{\partial \hat{m}(k,\tau)}{\partial \tau} = k^2[\alpha - k^2]\hat{m}(k,\tau), \tag{8.25}$$

and it is evident that the *most unstable* mode is associated with $k \in \mathbb{R}^d$ such that $|k| = \sqrt{\alpha/2}$. For what concerns the time–scale in which the phase segregation phenomenon should be observed, the situation is hence very similar to the one which we already discussed in the case without a conservation law, since the perturbation is $O(\gamma^{d/2})$ and the instability will produce a growth of the type $\exp(t\alpha^2/2)$: the exit time, meant as time at which the linear approximation breaks down, will then be given , on the time scale λ^2, by $t_c = d/\alpha^2$. The situation is however completely changed with respect to the spatial patterns that we observe (already in the linear analysis): in first approximation they will be a (random) superpositions of waves with wave numbers k with $|k| \approx \sqrt{\alpha/2}$ (the *Cahn field* [**12**]). In particular the spatial patterns will vary on the space scale 1 (since $\sqrt{\alpha/2}$ is a fixed number) and not on the very large scale λ (which diverges as $\gamma \to 0$), as we found in the Glauber case. The nonlinear analysis of the last stages of the escape is then much harder: as an extra difficulty, the conservation law makes the behavior of (8.24) and (8.23) essentially nonlocal and most of the tools available for RD and integral equations (8.15) and (6.16), notably domination techniques and barrier lemmas, do not apply to this case.

We remark that in some works in applied sciences (see e.g. [**12**], see also [**97**] and [**89**]) a procedure completely analogous to the one proven for systems without a conservation law is applied to the conserved case at a heuristic level: the random domain patterns after the appearances of the phases are taken to be distributed according to the sign of the random–wave field that one observes before the time of the escape, i.e. the Cahn field. This ansatz appears to give random domain patterns visually similar to those observed in experiments, computer simulations and numerical solutions of the Cahn–Hilliard equation. Detailed work in the direction we outlined has been done in $d = 1$ [**61**] and some recent work has been done also in higher dimensions [**82**].

Remark 3: The spatial patterns observed after the exit time evolve. As already remarked, in the case without a conservation law this evolution is by mean curvature [**69**],[**68**],[**9**],[**26**]. In the case with a conservation law, when the domain patterns are very large (they are not very large right after the exit time), the evolution is by the so called Mullins–Sekerka law (see [**85**],[**87**],[**1**],[**57**]). How the distribution of spatial patterns change under these laws is a very popular subject in physics and there are several conjectures, mostly based on the assumption that at very large time the patterns will be statistically self–similar (see [**45**] and references therein). A mathematical understanding of these problems is missing (and some of the physicists' conclusions are still controversial).

9. The Dynamics on Very Long Times: a Brief Look at Large Deviations

As we have seen in the previous sections, the hydrodynamic equations may fail completely to catch the behavior of an IPS beyond the time scale on which they are derived. While in the previous section we looked at situations in which the hydrodynamic solution is unstable, and thus particularly sensitive to the small perturbations arising from the discrete and stochastic nature of the system, we will now make some remarks about deviations from deterministic hydrodynamic behavior when the solution of the hydrodynamic equation is stable under small perturbations. We will restrict our analysis to cases in which the hydrodynamics is derived on the diffusive scaling. It is intuitively clear that the time that we have to wait to observe a *significant* (i.e. macroscopic) deviations from hydrodynamic behavior are much longer than the ones we have to wait in the presence of instabilities.

Several large deviation results for IPS, mostly associated with hydrodynamic limits, have been recently derived. We list here only a few references. The large deviation result for the simple exclusion process has been derived in [**70**]; many other results followed, dealing with more and more complex systems (see e.g. [**63**],[**76**],[**75**] ,[**90**]).

For the models introduced in Section 6, considered in finite macroscopic volumes (i.e. on boxes of diameter $\propto \gamma^{-1}$ lattice sites) , the large deviation statement associated with Theorem 6.1 can be found in [**20**], for the Glauber case, and [**3**] for the Kawasaki case. Large deviations for systems with long range interaction (Section 6) is of particular interest, since the hydrodynamic equations derived for these systems have for, $\beta > 1$, more than one stationary stable state. The standard approach of Freidlin and Wentzel [**46**], extended to this infinite dimensional setting, can then be applied to understand how the system can make a noise–induced transition from one of these stationary states to another (see for details [**20**],[**3**]). This analysis presents however several difficulties, mainly due to the great complexity of these IPS. An essential source of simplification comes from the fact that these two models are reversible with respect to a Gibbs measure and one can then obtain an expression for the *quasi-potential* (the infimum of the large deviation functional over the trajectories which lead to the escape from the basin of attraction of a stable stationary point of the macroscopic law) in terms of the equilibrium free energy and, moreover, something is known a priori about the *minimizing* trajectories. Even in very simple non reversible models, the situation is much less clear. We refer to [**63**] [**51**] for results in the Glauber+Kawasaki case generated by (8.21) and to [**52**] and [**80**] for another particular case.

The most relevant question, from the viewpoint taken in this chapter, is whether understanding the structure of the large deviations for an IPS can give some further insights toward understanding which SPDE better approximates an IPS. We believe that this is indeed the case. For example the results in [**70**] seem to suggest that the stochastic equation (4.1) is, also from the large deviations viewpoint, the *correct* stochastic improvement of the Burgers equation, when we are concerned with approximating the WASEP. We will not elaborate further on this here and we refer to [**40**], where a similar viewpoint is set forth, and to [**95**] where an informal, but very clear introduction to large deviations from hydrodynamic limits is given.

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