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Convergent Power Series for Anharmonic Chain with Periodic Forcing

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Abstract
We study the propagation of energy in one-dimensional anharmonic chains subject to a periodic, localized forcing. For the purely harmonic case, forcing frequencies outside the linear spectrum produce exponentially localized responses, preventing equi-distribution of energy per degree of freedom. We extend this result to anharmonic perturbations with bounded second derivatives and boundary dissipation, proving that for small perturbations and non-resonant forcing, the dynamics converges to a periodic stationary state with energy exponentially localized uniformly in the system size. The perturbed periodic state is described by a convergent power type expansion in the strength of the anharmonicity. This excludes chaoticity induced by anharmonicity, independently of the size of the system. Our perturbative scheme can also be applied in higher dimensions.

Keywords Anharmonic chain · Periodic force · Spectrum of the infinite harmonic chain · Perturbation series · Stability

Mathematics Subject Classification 80A19 · 80M22 · 82C22 · 82C70 · 70J35

1 Introduction

The approach to equilibrium which involves energy sharing among degrees of freedom of large systems with non-linear dynamics is one of the central problems in statistical mechanics. This motivated the numerical experiment in the seminal work of Fermi-Pasta-Ulam-Tsingou [18], which was perhaps the first application of digital computers to a non-military scientific goal. They considered a one-dimensional anharmonic chain of $N = 32$ oscillators, initially prepared in a periodic low-energy configuration. Surprisingly, the simulated evolution failed to show thermalization, i.e. an equidistribution of the energy among all other frequencies generated by the nonlinearity of the dynamics. The results of these simulations remain a source of investigation and a challenge to theoreticians [7, 16, 19, 33]. Later studies repeated

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the simulations for much larger systems and at higher energies, showing that thermalization can indeed occur. However, no complete analytical results in this direction are known.

In this article we approach the problem from a different perspective. Instead of starting the dynamics of the anharmonic chain in a given periodic configuration, we apply a periodic force of period θ at a single site of the chain. For the linear dynamics it is known that only frequencies within the spectral interval can propagate through the system. In fact, in [20] we proved that if the forcing frequency lies outside the frequency spectrum of the system, the perturbation induced by the forcing remains exponentially localized. This is not surprising, since linear dynamics is completely integrable and cannot thermalize.

Here we consider anharmonic perturbations of the linear dynamics, with potentials having bounded second derivatives. Dissipation is applied at the boundaries, ensuring the existence of a periodic stationary state for any system size N. We prove that, for sufficiently small perturbations and if the forcing frequency is outside the resonance spectrum of the unperturbed linear system, the dynamics, starting from any initial configuration, converges in time to a periodic stationary state with energy exponentially localized uniformly in N. In particular, this implies that no thermalization occurs under these circumstances. This result is obtained via a rigorous perturbative scheme around the purely harmonic case. Although the method can be extended to higher dimension and infinite systems, we refrain from doing so here for simplicity.

An open question remains as to what happens when the forcing frequency lies within the resonance interval of the unperturbed linear system, or when the anharmonicity is unbounded (as in the FPUT case); see some numerical results in the companion article [21].

As far as we know, there are no other analytical results of this kind for non-linear dynamics that are uniform in the system size. There has also been considerable work on transport properties of anharmonic chains, both for systems coupled to thermal reservoirs and for systems driven by external periodic forces but at positive temperature, see [1, 2, 5, 11–14, 17, 23, 24, 31, 35, 36] and references therein.

Computer simulations on similar systems, where a quartic anharmonicity is added to a pinned harmonic potential, reveal a rich variety of steady states emerging as the driving parameters of the external force are varied [5, 6, 8, 22, 25, 27, 28, 34].

2 Microscopic dynamics

The configuration of the system is given by

$$(\mathbf{q}, \mathbf{p}) = (q_{-N}, \ldots, q_N, p_{-N}, \ldots, p_N) \in \Omega_N := \mathbb{R}^{2N+1} \times \mathbb{R}^{2N+1}. \tag{2.1}$$

We denote $\mathbb{Z}_N := \{-N, \ldots, N\}$. The microscopic dynamics of the chain are given by the forced Hamiltonian system with the Hamiltonian given by

$$\mathcal{H}_N(\mathbf{q}, \mathbf{p}) = \sum_{x \in \mathbb{Z}_N} \left[\frac{p_x^2}{2} + \frac{1}{2}(q_x - q_{x-1})^2 + \frac{\omega_0^2 q_x^2}{2} + \nu\Big(V(q_x) + U(q_x - q_{x-1})\Big) \right], \tag{2.2}$$

friction on both endpoints of the lattice interval and a driving periodic force at $x = 0$. This yields

$$\ddot{q}_x(t;\nu) = \Delta q_x(t;\nu) - \omega_0^2 q_x(t;\nu) - \gamma\big(\delta_{-N}(x) + \delta_N(x)\big)\dot{q}_x(t;\nu) \\ - \nu\Big(V'(q_x(t;\nu)) - \nabla U'\big(q_x(t;\nu) - q_{x-1}(t;\nu)\big)\Big) + \mathcal{F}(t/\theta)\delta_{x,0}, \quad x \in \mathbb{Z}_N, \tag{2.3}$$

with strictly positive parameters $\gamma,\ \omega_0,\ \theta$. The Neumann laplacian Δ and the discrete gradient ∇ are defined as

$$\Delta f_x = f_{x+1} + f_{x-1} - 2f_x, \quad \nabla f_x = f_{x+1} - f_x, \quad x \in \mathbb{Z}_N, \tag{2.4}$$

with the boundary condition

$$f_{N+1} = f_N, \quad f_{-N-1} = f_{-N}. \tag{2.5}$$

The assumption that $\gamma > 0$ is our standing hypothesis and, unless otherwise stated, is in force throughout the paper.

We let the force be of the form

$$\mathcal{F}(t/\theta) = \sum_{m\in\mathbb{Z}} \widehat{\mathrm{F}}_m e^{im\omega t}, \quad \text{where } \widehat{\mathrm{F}}_0 = 0 \quad \text{and} \\ \widehat{\mathrm{F}}_m^\star = \widehat{\mathrm{F}}_{-m} \quad \text{and} \quad 0 < \sum_{m\in\mathbb{Z}} \big(m|\widehat{\mathrm{F}}_m|\big)^2 < +\infty. \tag{2.6}$$

Here

$$\omega = \frac{2\pi}{\theta}. \tag{2.7}$$

In the special instance $N = 0$, $\mathbb{Z}_0 := \{0\}$ the above model can be formulated as follows

$$\ddot{q}_0(t;\nu) = -\omega_0^2 q_0(t;\nu) - 2\gamma\dot{q}_0(t;\nu) - \nu V'(q_0(t;\nu)) + \mathcal{F}(t/\theta) \tag{2.8}$$

and it constitutes a case of its own interest, see ref. [9, 15] . We will discuss it in Section 4.

We consider the case where the non-quadratic part of the pinning and interacting potentials $V(q)$, and $U(r)$ respectively, are of $\mathcal{C}^2$ class of regularity [1] with bounded second derivatives, i.e.

$$\|V''\|_\infty + \|U''\|_\infty < +\infty. \tag{2.9}$$

Here for a given function $G : \mathbb{R} \to \mathbb{R}$ we denote by

$$\|G\|_\infty := \sup_q |G(q)|. \tag{2.10}$$

Examples of such potentials are furnished by

$$V(q) = \frac{q^{2n}}{1 + \alpha q^{2n}} \quad \text{for some } \alpha > 0, \tag{2.11}$$

or

$$V(q) = (\sin q)^{2n}, \qquad V(q) = (1 + \alpha q^2)^{\delta/2}, \tag{2.12}$$

where $n \geqslant 1$ and $\delta \in (0, 2)$. Analogous examples of interaction potentials, e.g. $U(r) = \cos r$, are also covered.

[1] Our argument actually requires only $\mathcal{C}^1$ smoothness and the bounded Lipschitz assumption on the derivative.

We prove that when all integer multiplicities of $\omega = \frac{2\pi}{\theta}$ are outside the interval containing the spectrum of the harmonic chain $\mathcal{I} := [\omega_0, \omega_u]$, with $\omega_u := \sqrt{\omega_0^2 + 4}$, a unique state of the system is achieved, as $t \to +\infty$, for an arbitrary initial condition. In addition, the state is θ-periodic and given by a perturbative expansion in the parameter ν, that converges for $|\nu| < \nu_0$, uniformly in the size of the system N. More precisely

$$\nu_0 = \frac{\delta_*}{\mathfrak{V}}. \tag{2.13}$$

where $\delta_* := \inf\left[|(m\omega)^2 - w^2| : m \in \mathbb{Z},\ w \in \mathcal{I}\right]$, and $\mathfrak{V} = \|V''\|_\infty + 3\|U''\|_\infty$.

Finding a periodic stable solution of (2.3) by a convergent perturbative series is, we believe, unlikely for nonlinear anharmonicity that grows faster than quadratic, considered in [3, 16, 30], and, as far as we know, not proven for any other non-equilibrium system. Our results do not extend to the case where some integer multiplicity of ω lies inside spectrum of the harmonic chain or $|\nu| > \nu_0$.

3 Main Results

3.1 Definition of norms

To make the above statements precise we define the Euclidean norm

$$\|f\|_N := \left(\sum_{x\in\mathbb{Z}_N} f_x^2\right)^{1/2} < +\infty$$

on the space $\mathbb{R}^{2N+1}$ of all real valued sequences $(f_x)_{x\in\mathbb{Z}_N}$. Consider the space $L^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$ of all Borel measurable functions $F : [0,\theta] \to \mathbb{R}^{2N+1}$ equipped with the norm

$$|||F|||_N := \left(\frac{1}{\theta}\int_0^\theta \|F(t)\|_N^2 \mathrm{d}t\right)^{1/2} \tag{3.1}$$

for any Borel measurable $F : [0,\theta] \to \ell_2(\mathbb{Z}_N)$. We shall also consider spaces $H^k\big([0,\theta];\mathbb{R}^{2N+1}\big)$, for $k = 1, 2, \ldots$, obtained by the completion of the space of C^∞-smooth functions F in the norm

$$|||F|||_{N,k} := \left(|||F|||_N^2 + \frac{1}{\theta}\sum_{\ell=1}^k \int_0^\theta \|F^{(\ell)}(t)\|_N^2 \mathrm{d}t\right)^{1/2}. \tag{3.2}$$

Here $F^{(\ell)}(t)$ denotes the ℓ-th derivative. In the following, when no ambiguity would arise, we drop the subscript N.

3.2 A periodic solution for an anharmonic chain

In the case when $\gamma > 0$ we state the following existence result, whose proof is fairly routine and relies on the application of the Schauder fixed point theorem (see Appendix C).

Theorem 3.1 *Assume $\gamma > 0$. For each $\nu \in \mathbb{R}$ there exists a θ-periodic solution $\mathbf{q}_\mathrm{p}(t;\nu) = \Big(q_{x,\mathrm{p}}(t;\nu)\Big)_{x\in\mathbb{Z}_N}$ of* (2.3).

3.3 Construction of a periodic solution for an anharmonic chain by perturbative series

Given a sequence $f := (f_x)_{x\in\mathbb{Z}_N}$ define

$$W_x(f) := V'(f_x) - \nabla U'(f_x - f_{x-1}). \tag{3.3}$$

We consider a sequence $\left(Q^{(L)}_{x,\mathrm{p}}(t;\nu)\right)$, $x \in \mathbb{Z}_N$, $L = 0, 1, \ldots$ of functions, θ-periodic in t, that satisfy

$$\begin{aligned} \ddot{Q}^{(L)}_{x,\mathrm{p}}(t;\nu) =& \big(\Delta - \omega_0^2\big) Q^{(L)}_{x,\mathrm{p}}(t;\nu) - \gamma(\delta_{-N}(x) + \delta_N(x))\dot{Q}^{(L)}_{x,\mathrm{p}}(t;\nu) \\ &- \nu W_x\big(Q^{(L-1)}_{\mathrm{p}}(t;\nu)\big) + \mathcal{F}(t/\theta)\delta_0(x), \quad x \in \mathbb{Z}_N. \end{aligned} \tag{3.4}$$

We let

$$W_x\big(Q^{(-1)}_{\mathrm{p}}(t;\nu)\big) \equiv 0 \quad \text{and} \quad Q^{(0)}_{x,\mathrm{p}}(t;\nu) := q^{(0)}_{x,\mathrm{p}}(t;\nu), \tag{3.5}$$

where $q^{(0)}_{x,\mathrm{p}}(t;\nu) = q^{(0)}_{x,\mathrm{p}}(t)$ is the unique θ-periodic solution of (2.3) for $\nu = 0$. For $L \geqslant 1$ we define

$$q^{(L)}_{x,\mathrm{p}}(t;\nu) := \nu^{-L}\left(Q^{(L)}_{x,\mathrm{p}}(t;\nu) - Q^{(L-1)}_{x,\mathrm{p}}(t;\nu)\right) \tag{3.6}$$

i.e.

$$Q^{(L)}_{x,\mathrm{p}}(t;\nu) = \sum_{\ell=0}^{L} q^{(\ell)}_{x,\mathrm{p}}(t;\nu)\nu^{\ell}. \tag{3.7}$$

A straightforward calculation (see (8.17) below) shows that $q^{(L)}_x(t;\nu)$, $x \in \mathbb{Z}_N$ is a θ-periodic solution of

$$\begin{aligned} \ddot{q}^{(L)}_x(t;\nu) &= \Delta q^{(L)}_x(t;\nu) - \omega_0^2 q^{(L)}_x(t;\nu) \\ &\quad - \gamma\dot{q}^{(L)}_x(t;\nu)\delta_{-N}(x) - \gamma\dot{q}^{(L)}_x(t;\nu)\delta_N(x) - v_{x,L-1}(t;\nu), \quad x \in \mathbb{Z}_N, \end{aligned} \tag{3.8}$$

where

$$v_{x,L-1}(t;\nu) = \frac{1}{\nu^{L-1}}\Big[W_x\big(Q^{(L-1)}_{\mathrm{p}}(t;\nu)\big) - W_x\big(Q^{(L-2)}_{\mathrm{p}}(t;\nu)\big)\Big]. \tag{3.9}$$

Note that $v_{x,0}(t;\nu) = W_x\big(q^{(0)}_{\mathrm{p}}(t;\nu)\big)$.

Since (3.8) can be treated as a harmonic chain with a prescribed periodic forcing, a standard argument, similar to the one used in Section 5.1, proves that the scheme described above has a unique θ-periodic solution.

An essential assumption used in our next result is that no integer multiplicity of the forcing frequency belongs to the spectrum of the harmonic chain on the lattice $\mathbb{Z}$, with no boundary condition, i.e.

$$m\omega \notin \mathcal{I} := [\omega_0, \sqrt{\omega_0^2 + 4}] \quad \text{for all } m \in \mathbb{Z}. \tag{3.10}$$

We recall that

$$\begin{aligned} \delta_* &= \inf\Big[|(m\omega)^2 - w^2| : \ m \in \mathbb{Z},\ w \in \mathcal{I}\Big], \\ \mathfrak{V} &= \|V''\|_\infty + 3\|U''\|_\infty. \end{aligned} \tag{3.11}$$

By taking $m = 0$ we observe that $\delta_* \leqslant \omega_0^2$. Let

$$\nu_0 := \frac{\delta_*}{\mathfrak{V}}. \tag{3.12}$$

Our main result can be formulated as follows.

Theorem 3.2 *Suppose that $\delta_* > 0$. Then, the following estimate holds*

$$\|\!|q_{\rm p}^{(\ell+1)}(\cdot;\nu)\|\!| \leqslant \frac{1}{\nu_0}\|\!|q_{\rm p}^{(\ell)}(\cdot;\nu)\|\!|, \quad \nu\in\mathbb{R},\ \ell=0,1,2,\ldots. \tag{3.13}$$

In consequence, for $|\nu|<\nu_0$ the sums $Q_{x,{\rm p}}^{(L)}(t;\nu)$, given by (3.7) converge in the $\|\cdot\|_{N,1}$-norm (see (3.2)), as $L\to+\infty$. The convergence is uniform in N and

$$\|\!|\sum_{\ell=L}^{+\infty} q_{\rm p}^{(\ell)}(\cdot;\nu)\nu^\ell\|\!| \leqslant \frac{|\nu/\nu_0|^L\|\!|q_{\rm p}^{(0)}(\cdot;\nu)\|\!|}{1-|\nu/\nu_0|}, \quad L=1,2,\ldots,\ |\nu|<\nu_0. \tag{3.14}$$

Consequently, for $|\nu|<\nu_0$ the series,

$$q_{x,{\rm p}}(t;\nu)=\sum_{\ell=0}^{+\infty} q_{x,{\rm p}}^{(\ell)}(t;\nu)\nu^\ell. \tag{3.15}$$

defines a θ-periodic solution of (2.3).

Our next result concerns the uniqueness of the θ-periodic solution of (2.3).

Theorem 3.3 *For $|\nu|<\nu_0$ the system* (2.3) *has a unique θ-periodic solution.*

Hence, for $|\nu|<\nu_0$, (3.15) defines the unique θ-periodic solution of (2.3).
The proofs of Theorems 3.2 and 3.3 are presented in Section 5.

3.4 Global stability of the unique periodic solution in the anharmonic case

Theorems 3.2 and 3.3 hold even in the absence of dissipation at the boundaries ($\gamma=0$) in the case of an infinite chain ($N=+\infty$). Of course in the case $\gamma=0$ the unique periodic solution has no stability properties (as can be seen even for $\nu=0$), it is just a very special solution of (2.3). Stability holds only when $\gamma>0$.

For $t\geqslant s$ denote by $\mathbf{q}(t,s,\mathbf{q},\mathbf{p};\nu)=\big(q_x(t,s,\mathbf{q},\mathbf{p};\nu)\big)$ the solution of (2.3) satisfying the initial condition

$$\mathbf{q}(s,s,\mathbf{q},\mathbf{p};\nu)=\mathbf{q}, \qquad \dot{\mathbf{q}}(s,s,\mathbf{q},\mathbf{p};\nu)=\mathbf{p}. \tag{3.16}$$

Then $\big(\mathbf{q}(t,s,\mathbf{q},\mathbf{p};\nu),\dot{\mathbf{q}}(t,s,\mathbf{q},\mathbf{p};\nu)\big)$ tends to the periodic solution constructed in Theorem 3.2, as $s\to-\infty$.

Theorem 3.4 *Suppose that $(\mathbf{q}_{\rm p}(t;\nu))$ is the unique periodic solution of* (2.3) *constructed in Theorem 3.2 and $\gamma>0$. Then,*

$$\lim_{s\to-\infty}\Big(\|\mathbf{q}(t,s,\mathbf{q},\mathbf{p};\nu)-\mathbf{q}_{\rm p}(t;\nu)\|+\|\dot{\mathbf{q}}(t,s,\mathbf{q},\mathbf{p};\nu)-\dot{\mathbf{q}}_{\rm p}(t;\nu)\|\Big)=0 \tag{3.17}$$

for any $t\in\mathbb{R}$, $(\mathbf{q},\mathbf{p})\in\Omega_N$ and $|\nu|<\nu_0$, with ν_0 given by (3.12).

We present the proof of this result in Section 8.2.

3.5 Work done by periodic forcing

The work performed by the force on the system over the period θ is given by the formula

$$\mathcal{W}_N(\nu) = \frac{1}{\theta}\int_0^\theta \mathcal{F}(t/\theta)p_{0,\mathrm{p}}(t;\nu)\mathrm{d}t. \tag{3.18}$$

Here $p_{x,\mathrm{p}}(t;\nu) = \dot{q}_{x,\mathrm{p}}(t;\nu)$.

We have the following identity

$$\begin{aligned}&\mathcal{H}_N\big(\mathbf{q}_\mathrm{p}(t;\nu),\dot{\mathbf{q}}_\mathrm{p}(t;\nu);\nu\big) - \mathcal{H}_N\big(\mathbf{q}_\mathrm{p}(0;\nu),\dot{\mathbf{q}}_\mathrm{p}(0;\nu);\nu\big)\\ &+\gamma\Big(\int_0^t p^2_{-N,\mathrm{p}}(s;\nu)\mathrm{d}s + \int_0^t p^2_{N,\mathrm{p}}(s;\nu)\mathrm{d}s\Big) = \int_0^t p_{0,\mathrm{p}}(s;\nu)\mathcal{F}\Big(\frac{s}{\theta}\Big)\mathrm{d}s.\end{aligned} \tag{3.19}$$

Here $\mathcal{H}_N(\cdot,\cdot)$ is the Hamiltonian of the system, see (2.2). A direct consequence of (3.19) is the following.

Theorem 3.5 *Suppose that $\gamma > 0$ and $\big(q_{x,\mathrm{p}}(\cdot;\nu)\big)$ is a θ-periodic solution of* (2.3)*. Furthermore assume that the potential U is such that the equation*[2]

$$r + \nu U'(r) = 0 \tag{3.20}$$

has only a finite number of solutions. Then,

$$\mathcal{W}_N(\nu) = \frac{\gamma}{\theta}\Big(\int_0^\theta p^2_{-N,\mathrm{p}}(s)\mathrm{d}s + \int_0^\theta p^2_{N,\mathrm{p}}(s)\mathrm{d}s\Big) > 0. \tag{3.21}$$

In addition,

$$\lim_{N\to+\infty}\mathcal{W}_N(\nu) = 0. \tag{3.22}$$

The proof of this result is presented in Section 7.

3.6 Spatial decay of the periodic solution as $N \to \infty$

Theorem 3.6 *There exist positive constants C_1, A, ρ depending only on ω, γ and independent of N such that for $|\nu| < \nu_0 \wedge C_1$*

$$\begin{aligned}&|q_{x,p}(t;\nu)| \leqslant A\exp\{-\rho|x|\}, \quad t\in[0,\theta] \quad \textit{and}\\ &\int_0^\theta p^2_{x,p}(t;\nu)\mathrm{d}t \leqslant A\exp\{-\rho|x|\}, \quad x\in\mathbb{Z}_N.\end{aligned} \tag{3.23}$$

The proof of this result is presented in Section B.2.

3.7 The case of odd harmonic modes: loss of uniqueness

It turns out that in the special case when both potentials $V(\cdot)$, $U(\cdot)$ are even and

$$\widehat{\mathrm{F}}_{2m} = 0 \quad \text{for all} \quad m\in\mathbb{Z} \tag{3.24}$$

[2] Condition 3.20 means that the interaction potential $\frac{1}{2}r^2 + \nu U(r)$, between the neighboring atoms does not become ”flat” in some intervals, which would prevent the energy transfer in the chain.

the convergence of the series (3.15) can be extended to all $|\nu| < \overline{\nu}_0 = \frac{\overline{\delta}_*}{\mathfrak{V}}$, where we can define

$$\overline{\delta}_* := \inf\Big[\big|\big((2m-1)\omega\big)^2 - w^2\big| : m \in \mathbb{Z},\ w \in \mathcal{I}\Big] \geqslant \delta_*. \tag{3.25}$$

Theorem 3.7 *Under hypothesis* (3.24) *estimates* (3.13) *are in force for* $|\nu| < \overline{\nu}_0$*, Consequently in this case the series* (3.15) *defines a* θ*-periodic solution of* (2.3)*.*

The proof of this result is contained in Section 6.

Remark 3.8 We stress here the fact that Theorem 3.7 does not claim either the uniqueness, or stability of the solution given by the series (3.15), in the case $\nu_0 \leqslant |\nu| < \overline{\nu}_0$. In this interval the periodic solution defined by (3.15) still exists but it may be neither stable nor unique. As we shall see in Section 6.1, the system (2.3) may admit more than one θ-periodic solution. Obviously, in that case, the stability in the sense claimed in Theorem 3.4 cannot hold.

4 The case $N = 0$: a single anharmonic oscillator with damping

The case when $N = 0$, i.e. a single anharmonic oscillators with damping and a one mode periodic forcing (see (2.8)), is interesting in its own right and is an example of what happens for small N, [15]. It is instructive to look first at the harmonic case ($\nu = 0$), where the equation can be solved explicitly. Assuming for simplicity that $\mathcal{F}(t) = F\cos(\omega t)$, equation (2.8) is given by

$$\ddot{q}(t) = -\omega_0^2 q(t) - 2\gamma\dot{q}(t) + F\cos(\omega t) \quad , \quad q(0) = q_0, \quad \dot{q}(0) = p_0. \tag{4.1}$$

Its solution when $\omega_0 \neq \omega$ and $\gamma = 0$ is given by:

$$q(t) = q_0\cos(\omega_0 t) + \frac{p_0}{\omega_0}\sin(\omega_0 t) + F\,\frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0^2 - \omega^2}. \tag{4.2}$$

When $\omega = \omega_0$ (the resonance case) we interpret the solution as the limit $\omega \to \omega_0$.

If $\gamma > 0$, then denoting $\lambda_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ we can write the solution of (4.1)

$$\begin{aligned} q(t) &= \alpha_0 e^{-\lambda_+ t} + \alpha_1 e^{-\lambda_- t} \\ &\quad + \frac{F}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}\left[(\omega_0^2 - \omega^2)\cos(\omega t) + 2\gamma\omega\sin(\omega t)\right] \end{aligned} \tag{4.3}$$

and

$$\alpha_0 = \frac{1}{2\sqrt{\gamma^2 - \omega_0^2}}\Big[\frac{2\gamma F\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} - p_0\Big] - \frac{\lambda_-}{2\sqrt{\gamma^2 - \omega_0^2}}\Big[q_0 - \frac{F(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}\Big],$$

$$\alpha_1 = \frac{\lambda_+}{2\sqrt{\gamma^2 - \omega_0^2}}\Big[q_0 - \frac{F(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}\Big] - \frac{1}{2\sqrt{\gamma^2 - \omega_0^2}}\Big[\frac{2\gamma F\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} - p_0\Big].$$

For $\gamma = \omega_0$ ($\lambda_+ = \lambda_- = \gamma$) we take the limit $\gamma \to \omega_0$ to get a finite value.

Remark 4.1 Notice that when $\gamma > 0$, the system tends to a θ-periodic state for any initial condition ($\mathrm{Re}(\lambda_\pm) > 0$). However, when $\gamma = 0$ and $\omega \neq \omega_0$ only the specific initial condition

$$q_0 = \frac{F}{\omega_0^2 - \omega^2}, \quad p_0 = 0$$

drives the system to such a periodic state. For other initial data there is no θ-periodic solution. For $\omega = \omega_0$ no θ-periodic solution exists.

Under the anharmonic perturbation Theorems 3.2 – 3.4 hold with the following modifications:

$$\begin{aligned} \delta_* &:= \inf_{m=0,1,\ldots} \phi(m), \quad \text{where} \\ \phi(m) &:= \left\{ \left(\omega_0^2 - (m\omega)^2\right)^2 + 4\gamma^2 (m\omega)^2 \right\}^{1/2}. \end{aligned} \tag{4.4}$$

A simple calculation shows that for $\omega_0 > \sqrt{2}\gamma$

$$\delta_* = \begin{cases} \min\{\phi(m_*), \phi(m_*+1)\}, \text{ where} \\ m_* := \left\lfloor \frac{(\omega_0^2 - 2\gamma^2)^{1/2}}{\omega} \right\rfloor \end{cases} \tag{4.5}$$

and for $\omega_0 \leqslant \sqrt{2}\gamma$ we have

$$\delta_* = \omega_0^2. \tag{4.6}$$

For the situation examined in Section 3.7 (V and U even and only odd forcing modes), in the case $N = 0$ the value of $\overline{\delta}_*$ is given by formula (4.5), if $\omega_0 \geqslant \sqrt{\omega^2 + 2\gamma^2}$ and

$$\overline{\delta}_* := \left\{ \left(\omega^2 - \omega_0^2\right)^2 + 4(\gamma\omega)^2 \right\}^{1/2}, \tag{4.7}$$

if $\omega_0 < \sqrt{\omega^2 + 2\gamma^2}$. Then we define $\overline{\nu}_0 := \frac{\overline{\delta}_*}{\mathfrak{Y}}$.

Finally, observe that the values ν_0 (or $\overline{\nu}_0$) for $N = 0$ depend explicitly on the friction coefficient γ. We expect that an optimal ν_0 would generally depends on γ and N and, therefore, our estimation (3.12) is just a lower bound of all of them, that is indepent of N and γ.

5 Proof of the convergence of the perturbative series.

We prove in this section the convergence of the perturbative scheme, Theorem 3.2 and the corresponding uniqueness result, Theorem 3.3.

5.1 Time harmonics of a periodic solution

Consider a θ-periodic solution to (2.3), i.e. such a solution $\left(q_{x,\mathrm{p}}(t;\nu)\right)$ that satisfies

$$q_{x,\mathrm{p}}(t+\theta;\nu) = q_{x,\mathrm{p}}(t;\nu), \quad t \in \mathbb{R}, \tag{5.1}$$

Then,

$$q_{x,\mathrm{p}}(t;\nu) =: \sum_{m\in\mathbb{Z}} \widetilde{q}_x(m;\nu) e^{im\omega t}. \tag{5.2}$$

where

$$\widetilde{q}_x(m;\nu) = \frac{1}{\theta} \int_0^\theta e^{-im\omega t} q_{x,\mathrm{p}}(t;\nu) \mathrm{d}t, \quad m \in \mathbb{Z},\ x \in \mathbb{Z}_N \tag{5.3}$$

are the Fourier coefficients of the θ-periodic solution. They satisfy the system of equations

$$\begin{aligned} 0 = &\big[(\omega m)^2 - \omega_0^2 + \Delta - i\gamma\omega m(\delta_{x,-N} + \delta_{x,N})\big]\widetilde{q}_x(m;\nu) + \widehat{\mathrm{F}}_m\delta_{x,0} \\ &+ \nu\widetilde{v}_x(m;\nu), \quad x \in \mathbb{Z}_N, \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} &\widetilde{v}_x(m;\nu) = \frac{1}{\theta}\int_0^\theta e^{-im\omega t} v_x(t;\nu)\mathrm{d}t \quad \text{and} \\ &v_x(t;\nu) := -V'\big(q_{x,\mathrm{p}}(t;\nu)\big) + \nabla U'\big(q_{x,\mathrm{p}}(t;\nu) - q_{x-1,\mathrm{p}}(t;\nu)\big), \quad (m,x) \in \mathbb{Z}\times\mathbb{Z}_N. \end{aligned} \tag{5.5}$$

Denote by $\ell_{2,N}$ the space of all square integrable complex valued sequences $\widetilde{\mathbf{f}} := \big(\widetilde{f}_x(m)\big)_{(m,x)\in\mathbb{Z}\times\mathbb{Z}_N}$ satisfying $\widetilde{f}_x^\star(m) = \widetilde{f}_x(-m)$ equipped with the Hilbert space norm

$$\|\widetilde{\mathbf{f}}\|_{\ell^2,N} := \left(\sum_{(m,x)\in\mathbb{Z}\times\mathbb{Z}_N} |\widetilde{f}_x(m)|^2\right)^{1/2}. \tag{5.6}$$

Immediately, from (5.4) and Green's function estimate (A.21) we conclude the following.

Proposition 5.1 *Suppose that $F \in L^2[0,\theta]$ and $\big(\widetilde{q}_x(m;\nu)\big)_{(m,x)\in\mathbb{Z}\times\mathbb{Z}_N} \in \ell_{2,N}$ is a solution of the system (5.4). Then, $\mathbf{q}(t) := \big(q_{x,\mathrm{p}}(t;\nu)\big)_{(t,x)\in\mathbb{Z}\times\mathbb{Z}_N}$ given by (5.2) belongs to $H^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$ and is a periodic solution of (2.3).*

5.2 The case of a harmonic chain

Consider first the case of a harmonic chain, i.e. when $\nu = 0$. Then equation (2.3) takes the form

$$\ddot{q}_x(t) = \big(\Delta - \omega_0^2\big)q_x(t) - \gamma\big(\delta_{-N}(x) + \delta_N(x)\big)\dot{q}_x(t) + \mathcal{F}(t/\theta)\delta_{x,0}, \quad x \in \mathbb{Z}_N. \tag{5.7}$$

Proposition 5.2 *For $\nu = 0$ there exists a unique θ-periodic solution.*

Proof The Fourier coefficients of a θ-periodic solution satisfy

$$\Big(\omega_0^2 - \Delta - (m\omega)^2 + i\gamma m\omega(\delta_{-N} + \delta_N)\Big)\widetilde{q}_x(m) = \widehat{\mathrm{F}}_m\delta_{x,0}, \quad x \in \mathbb{Z}_N, \quad m \in \mathbb{Z}. \tag{5.8}$$

Then, the unique solution of (5.8) is given by the formula

$$\widetilde{q}_x(m) = \widehat{\mathrm{F}}_m H^{(N)}_{-(m\omega)^2,\gamma m\omega}(x,0) \tag{5.9}$$

and

$$q_{\mathrm{p},x}(t) = \sum_{m\in\mathbb{Z}} \widetilde{q}_x(m)e^{im\omega t}, \quad p_{\mathrm{p},x}(t) = \sum_{m\in\mathbb{Z}} im\omega\widetilde{q}_x(m)e^{im\omega t}. \tag{5.10}$$

Here $H^{(N)}_{-(m\omega)^2,\sigma}(x,y)$ is the Green's function of $\lambda + \omega_0^2 - i\sigma(\delta_{-N} + \delta_N) - \Delta$, i.e.

$$H^{(N)}_{\lambda,\sigma}(x,y) = \big(\lambda + \omega_0^2 - \Delta + i\sigma(\delta_{-N} + \delta_N)\big)^{-1}\delta_y(x), \tag{5.11}$$

where we assume that $\sigma, \lambda \in \mathbb{R}$. The computation of the Green's function is carried out in Section A.3. □

Remark 5.3 By (A.15) and Proposition A.1, the solution given by (5.9) is well defined even if $\gamma = 0$. However, without the dissipation at the boundaries this will be just a very special solution without any stability properties, see Remark 4.1. When $\gamma > 0$ this is a globally stable periodic solution, c.f. (4.1) and (4.2).

5.3 Proof of Theorem 3.2

Consider the Fourier coefficients of the approximating scheme (3.8)

$$\widetilde{q}_x^{(L)}(m;\nu) = \frac{1}{\theta}\int_0^{\theta} e^{-im\omega t} q_{x,\mathrm{p}}^{(L)}(t;\nu)dt, \quad m \in \mathbb{Z}_N. \tag{5.12}$$

Coefficients $\widetilde{q}_x^{(0)}(m)$ satisfy (5.4) for $\nu = 0$, and for $L = 1, 2, \ldots$

$$0 = \Big[\Delta + (m\omega)^2 - \omega_0^2 - i\gamma m\omega(\delta_{-N}(x) + \delta_N(x))\Big]\widetilde{q}_x^{(L)}(m) - \widetilde{v}_{x,L-1}(m;\nu), \quad x \in \mathbb{Z}_N, \tag{5.13}$$

with (see (3.9))

$$\widetilde{v}_{x,L-1}(m;\nu) = \frac{1}{\theta}\int_0^{\theta} e^{-im\omega t} v_{x,L-1}(t;\nu)\mathrm{d}t. \tag{5.14}$$

We have, see (5.11),

$$\widetilde{q}_x^{(0)}(m;\nu) = \widehat{\mathrm{F}}_m H_{-(m\omega)^2,\gamma m\omega}(x,0) \tag{5.15}$$

and

$$\widetilde{q}_x^{(L)}(m;\nu) = \sum_{y\in\mathbb{Z}_N} H_{-(m\omega)^2,\gamma m\omega}(x,y)\widetilde{v}_{y,L-1}(m), \quad x \in \mathbb{Z}_N,\ \ell = 1, 2, \ldots. \tag{5.16}$$

Multiplying both sides of (5.13) by $\Big(\widetilde{q}_x^{(L)}(m)\Big)^\star$ and summing over x we get

$$\begin{aligned}\sum_{x\in\mathbb{Z}_N}\Big(\omega_0^2 - \Delta - (m\omega)^2\Big)\widetilde{q}_x^{(L)}(m)\Big(\widetilde{q}_x^{(L)}(m)\Big)^\star &= -i\gamma(m\omega)\widetilde{q}_{-N}^{(L)}(m)\Big(\widetilde{q}_{-N}^{(L)}(m)\Big)^\star \\ &- i\gamma(m\omega)\widetilde{q}_N^{(0)}(m)\Big(\widetilde{q}_N^{(0)}(m)\Big)^\star + \sum_{x\in\mathbb{Z}_N}\widetilde{v}_{x,L-1}(m;\nu)\Big(\widetilde{q}_x^{(L)}(m)\Big)^\star,\end{aligned} \tag{5.17}$$

for each $m \in \mathbb{Z}$ and $L = 1, 2, \ldots$. Observe that

$$0 < -\sum_{x\in\mathbb{Z}_N}\big(\Delta\widetilde{q}_x^{(L)}(m)\big)\widetilde{q}_x^{(L)}(m)^\star = \sum_{x\in\mathbb{Z}_N}\Big|\nabla\widetilde{q}_x^{(L)}(m)\Big|^2 \leqslant 4\sum_{x\in\mathbb{Z}_N}\big(\widetilde{q}_x^{(L)}(m)\big)^2. \tag{5.18}$$

Consequently for the real part of (5.17) we have

$$\begin{aligned}\sum_{x\in\mathbb{Z}_N}\Big(\omega_0^2 - (m\omega)^2\Big)|\widetilde{q}_x^{(L)}(m)|^2 &\leqslant \mathrm{Re}\sum_{x\in\mathbb{Z}_N}\widetilde{v}_{x,L-1}(m;\nu)\Big(\widetilde{q}_x^{(L)}(m)\Big)^\star \\ &\leqslant \sum_{x\in\mathbb{Z}_N}\big|\widetilde{v}_{x,L-1}(m;\nu)\big|\,\Big|\widetilde{q}_x^{(L)}(m)\Big|.\end{aligned} \tag{5.19}$$

Since $\widetilde{q}_x^{(L)}(-m) = \left(\widetilde{q}_x^{(L)}(m)\right)^\star$ we have $|\widetilde{q}_x^{(L)}(-m)|^2 = |\widetilde{q}_x^{(L)}(m)|^2$. Suppose that $0 \leqslant |m|\omega < \omega_0$. Then

$$\sum_{0\leqslant(m\omega)^2<\omega_0^2}\sum_{x\in\mathbb{Z}_N}\left(\omega_0^2-(m\omega)^2\right)|\widetilde{q}_x^{(L)}(m)|^2 \leqslant \sum_{0\leqslant(m\omega)^2<\omega_0^2}\sum_{x\in\mathbb{Z}_N}\left|\widetilde{v}_{x,L-1}(m;\nu)\right|\left|\widetilde{q}_x^{(L)}(m)\right|. \tag{5.20}$$

In the case when $|m|\omega > \sqrt{\omega_0^2+4}$ we argue as follows. Using inequality (5.18) we have

$$\begin{aligned}
&\sum_{(m\omega)^2>\omega_0^2+4}\sum_{x\in\mathbb{Z}_N}\left((m\omega)^2-\omega_0^2-4\right)|\widetilde{q}_x^{(L)}(m)|^2\\
&\leqslant \sum_{(m\omega)^2>\omega_0^2+4}\sum_{x\in\mathbb{Z}_N}\left[\left((m\omega)^2-\omega_0^2\right)|\widetilde{q}_x^{(L)}(m)|^2-\left|\nabla\widetilde{q}_x^{(L)}(m)\right|^2\right] \qquad (5.21)\\
&= \sum_{(m\omega)^2>\omega_0^2+4}\sum_{x\in\mathbb{Z}_N}\underbrace{\left((m\omega)^2+\Delta-\omega_0^2-i\gamma m(\delta_{-N}(x)+\delta_{x,N})\right)\widetilde{q}_x^{(L)}(m)}_{=-\widetilde{v}_{x,L-1}(m;\nu)}\left(\widetilde{q}_x^{(L)}(m)\right)^\star\\
&= -\sum_{(m\omega)^2>\omega_0^2+4}\sum_{x\in\mathbb{Z}_N}\widetilde{v}_{x,L-1}(m;\nu)\left(\widetilde{q}_x^{(L)}(m)\right)^\star.
\end{aligned}$$

Hence we can now sum over all m and by (5.20), (5.21), the definition of δ_* (3.11) and the Cauchy-Schwarz inequality we obtain

$$\delta_*\|\widetilde{q}^{(L)}(\cdot;\nu)\|_{\ell^2,N}^2 \leqslant \left(\|\widetilde{v}_{L-1}(\cdot;\nu)\|_{\ell^2,N}\right)^{1/2}\left(\|\widetilde{q}^{(L)}(\cdot;\nu)\|_{\ell^2,N}^2\right)^{1/2}. \tag{5.22}$$

Then, by the Plancherel identity,

$$\begin{aligned}
|\!|\!|q_{\rm p}^{(L)}(\cdot;\nu)|\!|\!|^2 &= \|\widetilde{q}^{(L)}(\cdot;\nu)\|_{\ell^2,N}^2 \leqslant \frac{1}{\delta_*}\|\widetilde{v}_{L-1}(\cdot;\nu)\|_{\ell^2,N}^2\\
&= \frac{1}{\delta_*\theta}\sum_{x\in\mathbb{Z}_N}\int_0^\theta [v_{x,L-1}(t;\nu)]^2 \mathrm{d}t.
\end{aligned}$$

Using (3.9) and (2.9) we conclude that

$$\begin{aligned}
|v_{x,L-1}(t;\nu)| &\leqslant \frac{1}{|\nu|^{L-1}}\Big[\|V''\|_\infty\left|Q_{x,\rm p}^{(L-1)}(t;\nu)-Q_{x,\rm p}^{(L-2)}(t;\nu)\right|\\
&+\|U''\|_\infty\Big(\left|\nabla^\star\left(Q_{x,\rm p}^{(L-1)}(t;\nu)-Q_{x,\rm p}^{(L-2)}(t;\nu)\right)\right|+\left|\nabla\left(Q_{x,\rm p}^{(L-1)}(t;\nu)-Q_{x,\rm p}^{(L-2)}(t;\nu)\right)\right|\Big)\\
&= \|V''\|_\infty|q_{x,\rm p}^{(L-1)}(t;\nu)|+\|U''\|_\infty\left(|\nabla^\star q_{x,\rm p}^{(L-1)}(t;\nu)|+|\nabla q_{x,\rm p}^{(L-1)}(t;\nu)|\right) \qquad (5.23)\\
&\leqslant (\|V''\|_\infty+\|U''\|_\infty)|q_{x,\rm p}^{(L-1)}(t;\nu)|+\|U''\|_\infty\left(|q_{x-1,\rm p}^{(L-1)}(t;\nu)|+|q_{x+1,\rm p}^{(L-1)}(t;\nu)|\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&[v_{x,L-1}(t;\nu)]^2 \leqslant (\|V''\|_\infty + \|U''\|_\infty)^2 [q_{x,\mathrm{p}}^{(L-1)}(t;\nu)]^2 \\
&+ (\|V''\|_\infty + \|U''\|_\infty)\|U''\|_\infty \Big(2[q_{x,\mathrm{p}}^{(L-1)}(t;\nu)]^2 + [q_{x-1,\mathrm{p}}^{(L-1)}(t;\nu)]^2 + [q_{x+1,\mathrm{p}}^{(L-1)}(t;\nu)]^2\Big) \\
&+ 2\|U''\|_\infty \Big([q_{x-1,\mathrm{p}}^{(L-1)}(t;\nu)]^2 + [q_{x+1,\mathrm{p}}^{(L-1)}(t;\nu)]^2\Big).
\end{aligned} \tag{5.24}
$$

As a result, see (3.11),

$$
|||q_{\mathrm{p}}^{(L)}(\cdot;\nu)|||_N^2 \leqslant \frac{\mathfrak{V}^2}{\delta_*^2 \theta} \sum_{x\in\mathbb{Z}_N} \int_0^\theta [q_{x,\mathrm{p}}^{(L-1)}(t;\nu)]^2 \mathrm{d}t
$$

and estimate (3.13) follows for $N = 1, 2, \ldots$. The remaining conclusions of the theorem follow from the estimate. ◻

5.4 Regularity of the periodic solution

From equality (5.13) we conclude that there exists a constant $C > 0$ such that

$$
(m\omega)^2 \|\widetilde{q}^{(L)}(m;\nu)\|_N \leqslant C\big(\|\widetilde{q}^{(L)}(m;\nu)\|_N + \|\widetilde{v}_{L-1}(m;\nu)\|_N\big), \quad m \in \mathbb{Z}. \tag{5.25}
$$

From (5.24) we obtain in turn that

$$
|||v_L(\cdot;\nu)||| \leqslant \mathfrak{V} |||q_{\mathrm{p}}^{(L)}(\cdot;\nu)|||, \quad L = 1, 2, \ldots. \tag{5.26}
$$

Combining (5.25), (5.26) and (3.13) we conclude that there exist constants $C_1, C_2 > 0$ such that

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}} [1 + (m\omega)^2]^2 \|\widetilde{q}^{(L)}(m;\nu)\|_N^2 \leqslant C_1\big(|||q_{\mathrm{p}}^{(L)}(\cdot;\nu)|||^2 + |||v_L(\cdot;\nu)|||^2\big) \\
&\leqslant C_2 |||q_{\mathrm{p}}^{(L)}(\cdot;\nu)|||^2 \leqslant \frac{C_2}{\nu_0^{2L}}, \quad L = 1, 2, \ldots.
\end{aligned} \tag{5.27}
$$

Proposition 5.4 *Under the assumptions of Theorem 3.2 there exists a constant $C > 0$ such that*

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} \|q_{\mathrm{p}}^{(L)}(t;\nu)\|_N \leqslant \frac{C}{\nu_0^L} \quad \text{and} \\
&\sup_{t\in\mathbb{R}} \|\dot{q}_{\mathrm{p}}^{(L)}(t;\nu)\|_N \leqslant \frac{C}{\nu_0^L}, \quad L = 1, 2, \ldots.
\end{aligned} \tag{5.28}
$$

In consequence, there exists $C > 0$

$$
\sup_{t\in\mathbb{R}} \Big(\|Q_{\mathrm{p}}^{(L)}(t;\nu)\|_N + \|\dot{Q}_{\mathrm{p}}^{(L)}(t;\nu)\|_N\Big) \leqslant C, \quad L = 0, 1, 2, \ldots. \tag{5.29}
$$

Furthermore $Q_{\mathrm{p}}^{(L)}(\cdot;\nu) \in C^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$ and the series given by (3.15) converges uniformly in t to a θ-periodic function $\big(q_{x,\mathrm{p}}(t;\nu)\big)$ whose Fourier coefficients $\big(\widetilde{q}_x(m;\nu)\big)$ satisfy the system (5.4).

Proof From bound (5.27) and the Cauchy-Schwarz inequality we have

$$\sup_{t\in\mathbb{R}} \|q_{\mathrm{p}}^{(L)}(t;\nu)\|_N \leqslant \sum_{m\in\mathbb{Z}} \|\widetilde{q}^{(L)}(m;\nu)\|_N$$
$$\leqslant \left(\sum_{m\in\mathbb{Z}}[1+(m\omega)^2]\|\widetilde{q}^{(L)}(m;\nu)\|_N^2\right)^{1/2}\left(\sum_{m\in\mathbb{Z}}[1+(m\omega)^2]^{-1}\right)^{1/2} \leqslant \frac{C}{\nu_0^L}$$

for some constant $C > 0$. The proof of the second estimate in (5.28) is similar. These estimates obviously imply the uniform convergence of the series in (3.15) and bound (5.29).

From (5.27) and its definition (3.7) we conclude also that $Q_{\mathrm{p}}^{(L)}(\cdot;\nu)$ belongs to $H^2\Big([0,\theta];\mathbb{R}^{2N+1}\Big)$ and satisfies (3.4). In fact $Q_{\mathrm{p}}^{(L)}(\cdot;\nu) \in C^2\Big([0,\theta];\mathbb{R}^{2N+1}\Big)$. □

From the proof of Theorem 3.2 we conclude also the following.

Corollary 5.5 *Under the assumptions of Theorem 3.2 for any $|\nu| < \nu_0$ there exists a constant $C > 0$ independent of N and $\gamma > 0$ such that the Hamiltonian $\mathcal{H}_N(\cdot,\cdot)$ defined in (2.2) satisfies*

$$\frac{1}{\theta}\int_0^\theta \mathcal{H}_N\big(\mathbf{q}_{\mathrm{p}}(t;\nu),\mathbf{p}_{\mathrm{p}}(t;\nu)\big)\mathrm{d}t \leqslant C. \tag{5.30}$$

5.5 Uniqueness: Proof of Theorem 3.3

Proof Consider the mapping $\mathcal{T}$ of $L^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$ into itself, assigning to each $f(t) = \big(f_x(t)\big)$ belonging to $L^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$ an element

$$(\mathcal{T}f)_x(t) = \sum_{m\in\mathbb{Z}} \widetilde{\mathcal{T}f}_x(m)e^{im\omega t}, \quad x \in \mathbb{Z}_N$$

given by

$$\begin{aligned}\widetilde{\mathcal{T}f}_x(m) &= H^{(N)}_{-(m\omega)^2,\gamma m\omega}(x,0)\widehat{\mathrm{F}}_m \\ &\quad + \nu\sum_{y\in\mathbb{Z}_N} H^{(N)}_{-(m\omega)^2,\gamma m\omega}(x,y)\widetilde{v}_y(m;f), \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N.\end{aligned} \tag{5.31}$$

Here

$$f_x(t) = \sum_{m\in\mathbb{Z}} \widetilde{f}_x(m)e^{im\omega t}, \quad x \in \mathbb{Z}_N \tag{5.32}$$

and

$$\widetilde{v}_y(m;f) = -\frac{1}{\theta}\int_0^\theta e^{-im\omega t}\Big[V'(f_x(t)) - \nabla U\big(f_x(t) - f_{x-1}(t)\big)\Big]\mathrm{d}t.$$

We have $\Big(\widetilde{\mathcal{T}f}_x(m)\Big)^\star = \widetilde{\mathcal{T}f}_x(-m)$.

Suppose that $f^{(1)}, f^{(2)} \in L^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$. Then, $\delta\widetilde{\mathcal{T}f}_x(m) := \widetilde{\mathcal{T}f}_x^{(2)}(m) - \widetilde{\mathcal{T}f}_x^{(1)}(m)$ satsify

$$\begin{aligned}0 &= \Big(\Delta + (m\omega)^2 - \omega_0^2 - i\gamma m\omega(\delta_{-N}+\delta_N)\Big)\Big(\delta\widetilde{\mathcal{T}f}_x(m)\Big) + \\ &\quad + \nu\Big(\widetilde{v}_x(m;f^{(2)}) - \widetilde{v}_x(m;f^{(1)})\Big), \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N.\end{aligned} \tag{5.33}$$

Multiplying both sides by $\left(\delta\widetilde{Tf}_x^{(j)}(m)\right)^\star$ and summing over x we get

$$\begin{aligned}&\sum_{x\in\mathbb{Z}_N}\Big[\big(\omega_0^2-(m\omega)^2\big)|\delta\widetilde{Tf}_x(m)|^2+|\nabla\delta\widetilde{Tf}_x(m)|^2\Big]\\&+i\gamma m\omega\Big(|\delta\widetilde{Tf}_{-N}(m)|^2+|\delta\widetilde{Tf}_N(m)|^2\Big)\\&=\nu\sum_{x\in\mathbb{Z}_N}\Big(\widetilde{v}_x(m;f^{(2)})-\widetilde{v}_x(m;f^{(1)})\Big)\Big(\delta\widetilde{Tf}_x(m)\Big)^\star.\end{aligned}$$

Using the assumption (3.10) and applying the Cauchy-Schwarz inequality on the right hand side we conclude that

$$\begin{aligned}&\delta_*\sum_{m\in\mathbb{Z}}\sum_{x\in\mathbb{Z}_N}\left|\delta\widetilde{Tf}_x(m)\right|^2\\&\leqslant|\nu|\left(\sum_{m\in\mathbb{Z}}\sum_{x\in\mathbb{Z}_N}\left|\widetilde{v}_x(m;f^{(2)})-\widetilde{v}_x(m;f^{(1)})\right|^2\right)^{1/2}\left(\sum_{m\in\mathbb{Z}}\sum_{x\in\mathbb{Z}_N}\left|\delta\widetilde{Tf}_x(m)\right|^2\right)^{1/2}.\end{aligned}$$

In consequence

$$|||Tf^{(2)}-Tf^{(1)}|||_N\leqslant\frac{|\nu|}{\nu_0}|||f^{(2)}-f^{(1)}|||_N.$$

By the Banach contraction principle there exists a unique solution of the equation $Tf=f$ in $L^2\big([0,\theta];\mathbb{R}^{2N+1}\big)$. Then $f(t)$ given by (5.32) is the unique θ-periodic solution of (2.3). Hence the conclusion of the theorem follows. □

6 Proof of Theorem 3.7

Using the argument from the proof of Theorem 3.2 one can infer the conclusions of Theorem 3.7, provided it can be shown that

$$\widetilde{q}_x^{(L)}(2m;\nu)=0,\quad x\in\mathbb{Z}_N,\ L,m=0,1,2,\ldots. \tag{6.1}$$

This would follow, if we can prove that

$$\widetilde{v}_{x,L}(2m;\nu)=0,\quad x\in\mathbb{Z}_N,\ L,m=0,1,2,\ldots, \tag{6.2}$$

see (5.14). To see (6.2) (and (6.1)) assume for simplicity that $U\equiv 0$. The argument in the general case is similar. Since $V'(\cdot)$ is odd and real valued, its Fourier transform $\widehat{V'}(\xi)$ is also an odd function. To further simplify the argument we suppose that its support is compact. The general case can be treated by an approximation argument. Thanks to (5.9) and the fact that $\widehat{F}_{2m}=0$ for all m, we conclude that (6.1) holds for $L=0$. Suppose now that (6.2) is true for some non-negative integer $L-1$. Since

$$q_x^{(L-1)}(t)=\sum_{m=1}^{+\infty}\widetilde{q}_x^{(L-1)}(2m-1)e^{i(2m-1)\omega t}+\sum_{m=1}^{+\infty}\Big(\widetilde{q}_x^{(L-1)}(2m-1)\Big)^\star e^{-i(2m-1)\omega t}$$

we have

$$\widetilde{v}_{x,L}(2m) = \frac{1}{2\pi\theta}\int_{\mathbb{R}} \widehat{V}'(\xi)\mathrm{d}\xi \int_0^\theta e^{-2i\omega mt}\exp\left\{i\xi q_x^{(L-1)}(t)\right\}\mathrm{d}t$$
$$= \lim_{M\to+\infty}\frac{\theta}{2\pi i}\int_{\mathbb{R}} \widehat{V}'(\xi)\mathrm{d}\xi \int_0^\theta e^{-2i\omega mt}\exp\left\{i\sum_{m'=1}^{M}\xi\widetilde{q}_x^{(L-1)}\big(2m'-1\big)e^{i(2m'-1)\omega t}\right\}$$
$$\times \exp\left\{i\sum_{m''=1}^{M}\xi(\widetilde{q}_x^{(L-1)}(2m''-1))^\star e^{-i(2m''-1)\omega t}\right\}\mathrm{d}t. \tag{6.3}$$

Denote by $C(1) = [z \in \mathbb{C} : |z| = 1]$, a contour oriented counter-clockwise. Using contour integration we can rewrite the utmost right hand side of (6.3) as

$$\lim_{M,N\to+\infty}\int_{\mathbb{R}} \widehat{V}'(\xi)\widehat{v}^{(L)}_{M,N}(2m;\xi)\mathrm{d}\xi, \quad \text{where}$$
$$\widehat{v}^{(L)}_{M,N}(2m;\xi) := \frac{1}{2\pi i}\int_{C(1)}\prod_{m'=1}^{M}\sum_{n_{m'}=0}^{N}\frac{\big(i\xi\widetilde{q}_x^{(L-1)}(2m'-1)z^{2m'-1}\big)^{n_{m'}}}{n_{m'}!}$$
$$\times\prod_{m''=1}^{M}\sum_{n'_{m''}=0}^{N}\frac{(i\xi(\widetilde{q}_x^{(L-1)}(2m''-1))^\star)^{n'_{m''}}}{(z^{2m''-1})^{n'_{m''}}(n'_{m''})!}\frac{\mathrm{d}z}{z^{2m+1}}$$

Performing the contour integration we get

$$\widehat{v}^{(L)}_{M,N}(2m;\xi) = \sum_{n_1=0}^{N}\dots\sum_{n_M=0}^{N}\sum_{n'_1=0}^{N}\dots\sum_{n'_M=0}^{N}(i\xi)^{n_1+\ldots+n_M+n'_1+\ldots+n'_M}$$
$$\times\,\delta\Big(n_1+3n_2+\ldots+(2M-1)n_M-2m-\big(n'_1+3n'_2+\ldots+(2M-1)n'_M\big)\Big)$$
$$\times\prod_{m'=1}^{M}\frac{\big(\widetilde{q}_x^{(L-1)}(2m'-1)\big)^{n_{m'}}}{n_{m'}!}\prod_{m''=1}^{M}\frac{\big((\widetilde{q}_x^{(L-1)}(2m''-1))^\star\big)^{n'_{m''}}}{(n'_{m''})!}.$$

Here $\delta(k) = 0$, if $k \neq 0$ and 1 if $k = 0$. The expression above is non-zero only if

$$n'_1 + n'_2 + \ldots + n'_M + n_1 + n_2 + \ldots + n_M \quad \text{is even.}$$

Since $\xi \mapsto \widehat{V}'(\xi)$ is odd and $\xi \mapsto \widehat{v}^{(L)}_{M,N}(2m;\xi)$ is even we conclude therefore that

$$\int_{\mathbb{R}} \widehat{V}'(\xi)\widehat{v}^{(L)}_{M,N}(2m;\xi)\mathrm{d}\xi = 0$$

and (6.2) follows by an induction argument. □

6.1 Remark about non-uniqueness

Suppose that the interaction potential $U \equiv 0$ and V is even. If ν is as in the statement of Theorem 3.3, then the derivative of the pinning potential $q \mapsto W(q;\nu) := \frac{(\omega_0 q)^2}{2} + \nu V(q)$ is strictly increasing. Therefore, the potential attains its unique minimum at $q = 0$. Our result states that the periodic solution of (2.3) constructed in Theorem 3.2 is unique.

However, as it has been observed in Remark 3.7, the perturbative scheme may work for a larger range of the anharmonicity parameter ν than considered in Theorem 3.3. In the latter case there could exist some periodic solutions of (2.3) other than defined by (3.15). Consider for example the forcing given by

$$\mathcal{F}\left(\frac{t}{\theta}\right) = 2\mathrm{F}\cos(\omega t). \tag{6.4}$$

Assume that $V(\cdot)$ is of C^2 class of regularity. According to Remark 3.7, by adjusting ω to be sufficiently large, we can make $\overline{\nu}_0$ - the range of the convergence of the perturbative scheme - as large as we wish. Then for some $|\nu| < \overline{\nu}_0$ the potential $W(q;\nu)$ could admit some other local minimum, say at $\overline{q} \neq 0$. Then $W'(\overline{q};\nu) = \omega_0^2\overline{q} + \nu V'(\overline{q}) = 0$. Assume furthermore that $W''(\overline{q};\nu) = \omega_0^2 + \nu V''(\overline{q}) > 0$. Then, obviously

$$\overline{q}_x(t;0) \equiv \overline{q}, \quad x \in \mathbb{Z}_N \tag{6.5}$$

solves equation (2.3) with $\mathcal{F}(\cdot) \equiv 0$ (corresponding to $\mathrm{F} = 0$). It can be seen, see Section D of the Appendix, that there will be $\theta = 2\pi/\omega$-periodic solutions $\left(\overline{q}_x(t;F)\right)_{x\in\mathbb{Z}_N}$ of (2.3) that are close to $\left(\overline{q}_x(t;0)\right)_{x\in\mathbb{Z}_N}$ for small values of F.

7 Proof of Theorem 3.5

7.1 Proof of positivity of the work functional

The fact that $\mathcal{W}_N(\nu) \geqslant 0$ is a consequence of the identity (3.19). The only remaining part is the proof of strict positivity of the work functional. Suppose that $\mathcal{W}_N(\nu) = 0$. Then $p_{-N,\mathrm{p}}(t) = p_{N,\mathrm{p}}(t) \equiv 0$, $t \geqslant 0$. This implies that $q_{-N,\mathrm{p}}(t) \equiv q_{-N,\mathrm{p}}$, $q_{N,\mathrm{p}}(t) \equiv q_{N,\mathrm{p}}$, for all $t \geqslant 0$, and some constants $q_{-N,\mathrm{p}}$ and $q_{N,\mathrm{p}}$. Then $q_{-N+1,\mathrm{p}}(t)$ must solve the equation

$$0 = q_{-N+1,\mathrm{p}}(t) - q_{-N,\mathrm{p}} - \omega_0^2 q_{-N,\mathrm{p}} - \nu\left[V'(q_{-N,\mathrm{p}}) - U'(q_{-N+1,\mathrm{p}}(t) - q_{-N,\mathrm{p}})\right]$$

that implies that $q_{-N+1,\mathrm{p}}(t)$ is constant in t and $\dot{p}_{-N+1,\mathrm{p}}(t) = 0$. Consequently

$$\begin{aligned} 0 = \dot{p}_{-N+1,\mathrm{p}}(t) &= q_{-N+2,\mathrm{p}}(t) + q_{-N,\mathrm{p}} - (1+\omega_0^2) q_{-N+1,\mathrm{p}} \\ &- \nu\left[V'(q_{-N+1,\mathrm{p}}) + U'(q_{-N+2,\mathrm{p}}(t) - q_{-N+1,\mathrm{p}}) - U'(q_{-N+1,\mathrm{p}} - q_{-N,\mathrm{p}})\right]. \end{aligned}$$

that implies $q_{-N+2,\mathrm{p}}(t)$ is constant in t and $\dot{p}_{-N+2,\mathrm{p}}(t) = 0$. Iterating we have

$$q_{-N+j,\mathrm{p}}(t) \equiv q_{-N+j,\mathrm{p}}, \qquad \dot{p}_{-N+j,\mathrm{p}}(t) = 0, \quad j = 0,\ldots,N.$$

Repeating the same argument starting from the other side we obtain

$$q_{N-j,\mathrm{p}}(t) \equiv q_{N-j,\mathrm{p}}, \qquad \dot{p}_{N-j,\mathrm{p}}(t) = 0, \quad j = 0,\ldots,N.$$

This implies for the 0-site

$$\begin{aligned} 0 = \dot{p}_{0,\mathrm{p}}(t) =& (q_{1,\mathrm{p}} - q_{-1,\mathrm{p}} - 2q_{0,\mathrm{p}}) - \omega_0^2 q_{0,\mathrm{p}} \\ & - \nu\left[V'(q_{0,\mathrm{p}}) + U'(q_{1,\mathrm{p}} - q_{0,\mathrm{p}}(t)) - U'(q_{0,\mathrm{p}} - q_{-1,\mathrm{p}})\right] + \mathcal{F}(t/\theta) \end{aligned}$$

and we conclude that $\mathcal{F}(t/\theta) \equiv \text{const}$, which contradicts (2.6). □

7.2 Proof of (3.22)

To conclude (3.22) it suffices to show that

$$\lim_{N\to+\infty}\frac{1}{\theta}\int_0^\theta \left[p_{\pm N,\mathrm{p}}(t;\nu)\right]^2 \mathrm{d}t = 0. \tag{7.1}$$

From Proposition 5.4 we have

$$p_{\pm N,\mathrm{p}}(t;\nu) = \sum_{L=0}^{+\infty} \nu^L \dot{q}^{(L)}_{\pm N,\mathrm{p}}(t;\nu),$$

where the series converges uniformly in t, as $N\to+\infty$, for $|\nu|<\nu_0$. Using this and estimate (5.27), to conclude (7.1) it suffices to prove that

$$\lim_{N\to+\infty}\frac{1}{\theta}\int_0^\theta \left[q^{(L)}_{\pm N,\mathrm{p}}(t;\nu)\right]^2 \mathrm{d}t = \lim_{N\to+\infty}\sum_{m\in\mathbb{Z}} |\widetilde{q}^{(L)}_{\pm N}(m)|^2 = 0 \tag{7.2}$$

for each $L=0,1,\ldots$. The latter is a consequence of formulas (5.15) and (5.16) and estimate (B.10).

8 Global stability of the periodic solution

In the present section we assume that N is finite and $\mathbf{q}\big(t;\nu,\mathbf{q},\mathbf{p}\big) = \Big(q_x\big(t;\nu,\mathbf{q},\mathbf{p}\big)\Big)$, is the solution of (2.3) with the initial condition at s

$$q_x\big(s;\nu,\mathbf{q},\mathbf{p}\big) = q_x \quad \text{and} \quad \dot{q}_x\big(s;\nu,\mathbf{q},\mathbf{p}\big) = p_x, \quad x\in\mathbb{Z}_N \tag{8.1}$$

for any $\nu\in\mathbb{R}$. Let $\mathbf{p}\big(t;\nu,\mathbf{q},\mathbf{p}\big) = \Big(p_x\big(t;\nu,\mathbf{q},\mathbf{p}\big)\Big)$, where $p_x\big(t;\nu,\mathbf{q},\mathbf{p}\big) = \dot{q}_x\big(t;\nu,\mathbf{q},\mathbf{p}\big)$. We omit writing the initial data $(\mathbf{q},\mathbf{p})$, if they are obvious from the context.

We shall also denote by

$$\mathbf{q}_{\mathrm{p}}(t;\nu) = \Big(q_{x,\mathrm{p}}(t;\nu)\Big)_{x\in\mathbb{Z}_N}, \qquad \mathbf{p}_{\mathrm{p}}(t;\nu) = \Big(p_{x,\mathrm{p}}(t;\nu)\Big)_{x\in\mathbb{Z}_N}$$

a θ-periodic solution of (2.3).

8.1 Global stability of the periodic solution in the case of a harmonic chain on $\mathbb{Z}_N$

In the case $\nu=0$ equation (5.7), with the initial data at s given by $(\mathbf{q},\mathbf{p})$, can be explicitly solved. Let us introduce some auxiliary notation. Define a 2×2 bloc matrix, with each block a $(2N+1)\times(2N+1)$ matrix,

$$A = \begin{pmatrix} 0_{2N+1} & \mathrm{Id}_{2N+1} \\ \Delta - \omega_0^2 \mathrm{Id}_{2N+1} & -\gamma E_{2N+1} \end{pmatrix}, \tag{8.2}$$

where Id_n is the $n\times n$ identity matrix and

$$E_{2N+1} = [\delta_{-N-1}(x)\delta_{-N-1}(y) + \delta_{N+1}(x)\delta_{N+1}(y)]_{x,y=-N-1,\ldots,N+1}.$$

We denote by $\sigma(A)$ the set of all eigenvalues of A and by $\mathrm{Re}\,\sigma(A)$ the set of all their real parts. Both here and in what follows by the norm of the matrix $M=[m_{i,j}]$ we understand its operator norm $\|M\| := \sup_{\|x\|=1}\|Mx\|$, where $\|\cdot\|$ is the euclidean norm of a vector.

Suppose that $(\mathbf{q},\mathbf{p})\in\Omega_N$ and $s\leqslant t$. Denote by $\mathbf{q}(t,s)=\big(q_x(t,s)\big)$ the solution of (5.7) satisfying

$$q_x(s,s)=q_x,\quad \dot q_x(s,s)=p_x,\quad x\in\mathbb{Z}_N. \tag{8.3}$$

Using this notation we can write

$$\begin{aligned}&\begin{pmatrix}\dot{\mathbf{q}}(t;s)\\ \dot{\mathbf{p}}(t;s)\end{pmatrix}=A\begin{pmatrix}\mathbf{q}(t;s)\\ \mathbf{p}(t;s)\end{pmatrix}+\mathbb{F}\Big(\frac{t}{\theta}\Big),\\ &\mathbf{q}(s;s)=\mathbf{q},\quad \mathbf{p}(s;s)=\mathbf{p}.\end{aligned} \tag{8.4}$$

The solution can be written as

$$\begin{pmatrix}\mathbf{q}(t;s)\\ \mathbf{p}(t;s)\end{pmatrix}=e^{A(t-s)}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix}+\int_s^t e^{A(t-s')}\mathbb{F}\Big(\frac{s'}{\theta}\Big)\mathrm{d}s', \tag{8.5}$$

where $\mathbb{F}(t)$ is a vector valued function, with $2(2N+1)$ components that are all 0, except the one corresponding to the momentum co-ordinate at $x=0$, where it equals $\mathcal{F}(t)$.

In particular, $\mathbf{q}_{\mathrm{p}}(t)$ the θ-periodic solution of (2.3) is given by

$$\begin{pmatrix}\mathbf{q}_{\mathrm{p}}(t)\\ \mathbf{p}_{\mathrm{p}}(t)\end{pmatrix}=\int_{-\infty}^t e^{A(t-s)}\mathbb{F}\Big(\frac{s}{\theta}\Big)\mathrm{d}s. \tag{8.6}$$

Define

$$\lambda_N:=-\lim_{t\to+\infty}\frac{\log\|e^{At}\|_N}{t}. \tag{8.7}$$

It can be shown, see [32, Proposition 1.2] that there exists $C_0>0$ such that

$$\lambda_N\geqslant\frac{C_0}{N^3},\quad N=1,2,\dots. \tag{8.8}$$

As an immediate consequence of (8.7) we conclude the following.

Proposition 8.1 *There exist $C_A>0$ such that*

$$\|e^{At}\|\leqslant C_A e^{-\lambda_N t},\quad t\geqslant 0. \tag{8.9}$$

We claim that, with the presence of dissipation on both ends of a chain the solution $\big(\mathbf{q}(t,s,\mathbf{q},\mathbf{p}),\dot{\mathbf{q}}(t,s,\mathbf{q},\mathbf{p})\big)$ tends to the periodic solution as $s\to-\infty$.

Theorem 8.2 *Suppose that $(\mathbf{q}_{\mathrm{p}}(t))$ is the unique periodic solution given by (5.10). Then, for $\gamma>0$*

$$\lim_{s\to-\infty}\Big(\|\mathbf{q}(t,s,\mathbf{q},\mathbf{p})-\mathbf{q}_{\mathrm{p}}(t)\|_N+\|\dot{\mathbf{q}}(t,s,\mathbf{q},\mathbf{p})-\dot{\mathbf{q}}_{\mathrm{p}}(t)\|_N\Big)=0 \tag{8.10}$$

for any $t\in\mathbb{R}$ and $(\mathbf{q},\mathbf{p})\in\Omega_N$.

Proof Using (8.5) and (8.6) we get

$$\begin{pmatrix}\mathbf{q}(t;s)-\mathbf{q}_{\mathrm{p}}(t)\\ \mathbf{p}(t;s)-\mathbf{p}_{\mathrm{p}}(t)\end{pmatrix}=e^{A(t-s)}\Big[\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix}-\begin{pmatrix}\mathbf{q}_{\mathrm{p}}(s)\\ \mathbf{p}_{\mathrm{p}}(s)\end{pmatrix}\Big]. \tag{8.11}$$

The result is then an immediate conclusion from Proposition 8.1. □

8.2 Global stability via the approximation scheme

Theorem 8.3 *Suppose that* $\mathbf{q}_{\mathrm{p}}(t;\nu)$ *is the periodic solution defined in Theorem 3.2 and* $\gamma > 0$. *Then, for each* $|\nu| < \nu_0$ *(see (3.11)) and* $\xi > 0$ *large enough we have*

$$\lim_{s\to-\infty}\int_0^{+\infty} e^{-\xi t}\|\mathbf{q}\big(t;s,\nu\big)-\mathbf{q}_{\mathrm{p}}(t;\nu)\|^2 \mathrm{d}t = 0. \tag{8.12}$$

The proof of the theorem is given in Section 8.2.3 below but first we apply it to prove Theorem 3.4.

Proof of Theorem 3.4 in the case $|\nu| < \nu_0$

As a corollary of Theorem 8.3 we conclude that $\Big(\mathbf{q}\big(\cdot;s,\nu\big),\mathbf{p}\big(\cdot;s,\nu\big)\Big)$ converges to the periodic solution $\big(\mathbf{q}_{\mathrm{p}}(\cdot;\nu),\mathbf{p}_{\mathrm{p}}(\cdot;\nu)\big)$, as $s\to-\infty$, in $L^2_{\mathrm{loc}}[0,+\infty)$. Since the functions are bounded, together with their derivatives, the family is compact in the topology of uniform convergence on compact subsets of $[0,+\infty)$. The L^2_{loc} convergence allows us to identify the limit, as the periodic solution, which in particular implies the conclusion of Theorem 3.4. □

8.2.1 Approximation scheme

For fixed $(\mathbf{q},\mathbf{p})\in\Omega_N$ and $s<0$ define

$$\mathbf{q}^{(L)}(t;s,\mathbf{q},\mathbf{p},\nu)=\Big(q_x^{(L)}(t;s,\mathbf{q},\mathbf{p},\nu)\Big)_{x\in\mathbb{Z}_N},\quad t\geqslant 0,\quad L=0,1,2,\ldots$$

as the solution of

$$\begin{aligned}
&\ddot{q}_x^{(0)}(t;s,\nu)=\big(\Delta-\omega_0^2\big)q_x^{(0)}(t;s,\nu)-\gamma\big(\delta_{-N}(x)+\delta_N(x)\big)\dot{q}_x^{(0)}(t;s,\nu)+\mathcal{F}(t/\theta)\delta_0(x),\\
&q_x^{(0)}(0;s,\nu)=q_x(0;s,\mathbf{q},\mathbf{p},\nu)-\sum_{L=1}^{+\infty}q_{x,\mathrm{p}}^{(L)}(0;\nu)\nu^L,\\
&\dot{q}_x^{(0)}(0;\nu)=\dot{q}_x(0;s,\mathbf{q},\mathbf{p},\nu)-\sum_{L=1}^{+\infty}\dot{q}_{x,\mathrm{p}}^{(L)}(0;\nu)\nu^L,\quad x\in\mathbb{Z}_N
\end{aligned}\tag{8.13}$$

and for $L=1,2,\ldots$

$$\begin{aligned}
&\ddot{q}_x^{(L)}(t;s,\nu)=\big(\Delta-\omega_0^2\big)q_x^{(L)}(t;s,\nu)-\gamma\big(\delta_{-N,x}+\delta_{N,x}\big)\dot{q}_x^{(L)}(t;s,\nu)-v_{x,L-1}(t;s,\nu)\\
&q_x^{(L)}(0;s,\nu)=q_{x,\mathrm{p}}^{(L)}(0;\nu),\qquad \dot{q}_x^{(L)}(0;s,\nu)=\dot{q}_{x,\mathrm{p}}^{(L)}(0;\nu),\quad x\in\mathbb{Z}_N.
\end{aligned}\tag{8.14}$$

Here

$$v_{x,L-1}(t;s,\nu)=\frac{1}{\nu^{L-1}}\Big[W_x\big(\mathbf{Q}^{(L-1)}(t;s,\nu)\big)-W_x\big(\mathbf{Q}^{(L-2)}(t;s,\nu)\big)\Big] \tag{8.15}$$

where W_x is defined by (3.3) and $\mathbf{Q}^{(L)}(t;s,\nu)=\Big(Q_x^{(L)}(t;s,\nu)\Big)_{x\in\mathbb{Z}_N}$ is given by an analogue of (3.7):

$$Q_x^{(L)}(t;s,\nu)=\sum_{\ell=0}^{L}q_x^{(\ell)}(t;s,\nu)\nu^{\ell},\qquad t\geqslant 0,\qquad L\geqslant 0. \tag{8.16}$$

By convention $W_x\big(\mathbf{Q}^{(-1)}(t;s,\nu)\big)\equiv 0$. Both here and below we suppress writing the initial data $\mathbf{q}$, $\mathbf{p}$ in the notation, when they are clear from the context.

Notice that $\sum_{L=0}^{+\infty} q_x^{(L)}(0;s,\nu)\nu^L = q_x(0;s,\mathbf{q},\mathbf{p},\nu)$. Then for $L\geqslant 0$

$$\begin{aligned}\ddot{Q}_x^{(L)}(t;s,\nu) &= \big(\Delta-\omega_0^2\big)Q_x^{(L)}(t;s,\nu)-\gamma(\delta_{-N}(x)+\delta_N(x))\dot{Q}_x^{(L)}(t;s,\nu)\\ &\quad -\nu W\big(Q_x^{(L-1)}(t;s,\nu)\big)+\mathcal{F}(t/\theta)\delta_{x,0},\\ Q_x^{(L)}(0;s,\nu) &= q_x(0;s,\nu)-\sum_{\ell=L+1}^{+\infty} q_{x,\mathrm{p}}^{(\ell)}(0;\nu)\nu^{\ell}\\ \dot{Q}_x^{(L)}(0;\nu) &= q_x(0;s,\nu)-\sum_{\ell=L+1}^{+\infty}\dot{q}_{x,\mathrm{p}}^{(\ell)}(0;\nu)\nu^{\ell},\quad x\in\mathbb{Z}_N.\end{aligned}\tag{8.17}$$

Let

$$\mathfrak{Q}_L(t,s):=\|\mathbf{Q}^{(L)}(t,s;\nu)-\mathbf{Q}^{(L-1)}(t,s;\nu)\|+\|\dot{\mathbf{Q}}^{(L)}(t,s;\nu)-\dot{\mathbf{Q}}^{(L-1)}(t,s;\nu)\|$$

for $L=1,2,\ldots$.

Lemma 8.4 *For fixed $t_*>0>s$ and $|\nu|<\nu_0$ we have*

$$\sum_{L=1}^{+\infty}\sup_{t\in[0,t_*]}\mathfrak{Q}_L(t,s)<+\infty.\tag{8.18}$$

Proof The solutions of (8.17) satisfy

$$\begin{aligned}&\begin{pmatrix}\mathbf{Q}^{(L)}(t;s,\nu)-\mathbf{Q}^{(L-1)}(t;s,\nu)\\ \dot{\mathbf{Q}}^{(L)}(t;s,\nu)-\dot{\mathbf{Q}}^{(L-1)}(t;s,\nu)\end{pmatrix}=\nu^L e^{At}\begin{pmatrix}\mathbf{q}_{\mathrm{p}}^{(L)}(0;\nu)\\ \dot{\mathbf{q}}_{\mathrm{p}}^{(L)}(0;\nu)\end{pmatrix}\\ &-\nu\int_0^t e^{A(t-s')}\Big[\mathbb{V}\big(\mathbf{Q}^{(L-1)}(s';s,\nu)\big)-\mathbb{V}\big(\mathbf{Q}^{(L-2)}(s';s,\nu)\big)\Big]\mathrm{d}s'\end{aligned}\tag{8.19}$$

for any $L=1,2,\ldots$, $t\geqslant 0$. Here $\mathbb{V}:\Omega_N\to\Omega_N\times\Omega_N$ and $\mathcal{W}:\Omega_N\to\Omega_N$ are functions defined as

$$\mathbb{V}(f)=\begin{pmatrix}0\\ \mathcal{W}(f),\end{pmatrix},\qquad \mathcal{W}_x(f)=W_x(f),\quad x\in\mathbb{Z}_N\tag{8.20}$$

(cf (3.3)) and, by convention, $\mathbb{V}\big(\mathbf{Q}^{(-1)}(s';s,\nu)\big)\equiv 0$. Hence

$$\begin{aligned}\mathfrak{Q}_L(t,s)&\leqslant|\nu|^L\big(\|\mathbf{q}_{\mathrm{p}}^{(L)}(\cdot;\nu)\|_\infty+\|\dot{\mathbf{q}}_{\mathrm{p}}^{(L)}(\cdot;\nu)\|_\infty\big)+|\nu|\mathfrak{V}\int_0^t\mathfrak{Q}_{L-1}(s',s)\mathrm{d}s'\\ &\leqslant\ldots\leqslant|\nu|^L\big(\|\mathbf{q}_{\mathrm{p}}^{(L)}(\cdot;\nu)\|_\infty+\|\dot{\mathbf{q}}_{\mathrm{p}}^{(L)}(\cdot;\nu)\|_\infty\big)+|\nu|^{L-1}\big(\|\mathbf{q}_{\mathrm{p}}^{(L-1)}(\cdot;\nu)\|_\infty+\|\dot{\mathbf{q}}_{\mathrm{p}}^{(L-1)}(\cdot;\nu)\|_\infty\big)\frac{|\nu|\mathfrak{V}t}{1!}\\ &+\ldots+\big(\|\mathbf{q}_{\mathrm{p}}^{(0)}(\cdot;\nu)\|_\infty+\|\dot{\mathbf{q}}_{\mathrm{p}}^{(0)}(\cdot;\nu)\|_\infty\big)\frac{(|\nu|\mathfrak{V}t)^L}{L!}.\end{aligned}\tag{8.21}$$

Summing over L we conclude that

$$\sum_{L=1}^{+\infty}\sup_{t\in[0,t_*]}\mathfrak{Q}_L(t,s)\leqslant\Big(\sum_{L=0}^{+\infty}|\nu|^L\big(\|\mathbf{q}_{\mathrm{p}}^{(L)}(\cdot;\nu)\|_\infty+\|\dot{\mathbf{q}}_{\mathrm{p}}^{(L)}(\cdot;\nu)\|_\infty\big)\Big)\exp\{|\nu|\mathfrak{V}t_*\}.$$

The conclusion of the lemma then follows from Proposition 5.4. ◻

As a consequence of Lemma 8.4 we have for $t > 0$

$$\begin{aligned} \sum_{\ell=0}^{+\infty} q_x^{(\ell)}(t; s, \nu)\nu^\ell &= \lim_{L\to+\infty} Q_x^{(L)}(t; s, \nu) = q_x(t; s, \nu), \\ \sum_{\ell=0}^{+\infty} \dot q_x^{(\ell)}(t; s, \nu)\nu^\ell &= \lim_{L\to+\infty} \dot Q_x^{(L)}(t; s, \nu) = \dot q_x(t; s, \nu). \end{aligned} \tag{8.22}$$

8.2.2 Approximation scheme for the Laplace transform

Consider the Laplace transforms

$$\widehat{q}_x^{(L)}(\lambda; s, \nu) = \int_0^{+\infty} e^{-\lambda t} q_x^{(L)}(t; s, \nu)\mathrm{d}t.$$

They satisfy the system

$$\begin{aligned} 0 &= \left[\Delta - \omega_0^2 - \lambda^2 - \gamma\lambda(\delta_{-N}(x) + \delta_N(x))\right]\widehat{q}_x^{(0)}(\lambda; s, \nu) \\ &+ \left[\lambda + \gamma(\delta_{-N}(x) + \delta_N(x))\right]q_x^{(0)}(0; s, \nu) + \dot q^{(0)}(0; s, \nu) + \ \delta_{x,0}\sum_{m\in\mathbb{Z}} \frac{\widehat{\mathrm{F}}_m}{\lambda - im\omega}, \quad x \in \mathbb{Z}_N \end{aligned} \tag{8.23}$$

and for $L = 1, 2, \ldots$

$$\begin{aligned} 0 &= \left[\Delta - \omega_0^2 - \lambda^2 - \gamma\lambda(\delta_{-N}(x) + \delta_N(x))\right]\widehat{q}_x^{(L)}(\lambda; s, \nu) \\ &+ \left[\lambda + \gamma(\delta_{-N}(x) + \delta_N(x))\right]q_{x,\mathrm{p}}^{(L)}(0; \nu) + \dot q_{x,\mathrm{p}}^{(L)}(0; \nu) - \widehat{v}_{x,L-1}(\lambda; s, \nu), \quad x \in \mathbb{Z}_N, \end{aligned} \tag{8.24}$$

with

$$\widehat{v}_{x,L-1}(\lambda; s, \nu) = \int_0^{+\infty} e^{-\lambda t} v_{x,L-1}(t; s, \nu)\mathrm{d}t, \tag{8.25}$$

where $v_{x,L-1}(t; s, \nu)$ is given by (8.15).

Let $\widehat{q}_{x,\mathrm{p}}^{(L)}(\lambda; \nu)$ be the Laplace transform of $q_{x,\mathrm{p}}^{(L)}(t; \nu)$ and

$$\begin{aligned} \delta q_x^{(L)}(t; s, \nu) &:= q_x^{(L)}(t; s, \nu) - q_{x,\mathrm{p}}^{(L)}(t; \nu), \\ \delta \widehat{q}_x^{(L)}(\lambda; s, \nu) &:= \widehat{q}_x^{(L)}(\lambda; s, \nu) - \widehat{q}_{x,\mathrm{p}}^{(L)}(\lambda; \nu), \\ \delta Q_x^{(L)}(t; s, \nu) &:= Q_x^{(L)}(t; \nu) - Q_{x,\mathrm{p}}^{(L)}(t; \nu). \end{aligned} \tag{8.26}$$

For $L = 1, 2, \ldots$ we get

$$\begin{aligned} &0 = \left[\Delta - \omega_0^2 - \lambda^2 - \gamma\lambda(\delta_{-N}(x) + \delta_N(x))\right]\delta\widehat{q}_x^{(L)}(\lambda; s, \nu) - \delta\widehat{v}_{x,L-1}(\lambda; s, \nu), \quad x \in \mathbb{Z}_N, \quad \text{with} \\ &\delta\widehat{v}_{x,L-1}(\lambda; s, \nu) := \widehat{v}_{x,L-1}(\lambda; s, \nu) - \widehat{v}_{x,L-1}^{(\mathrm{p})}(\lambda; \nu). \end{aligned} \tag{8.27}$$

Let $\lambda = \xi + i\eta$, where $\xi > 0$ and $\eta \in \mathbb{R}$. Multiplying both sides by $\left(\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right)^\star$ and summing over x we get

$$\begin{aligned}&\left(\omega_0^2 + \xi^2 - \eta^2 + 2i\xi\eta\right) \sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 + \sum_{x\in\mathbb{Z}_N} \left|\nabla_x \delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 \\ &+ \gamma\,(\xi + i\eta)\left[\left|\delta\widehat{q}_{-N}^{(L)}(\lambda; s, \nu)\right|^2 + \left|\delta\widehat{q}_N^{(L)}(\lambda; s, \nu)\right|^2\right] \\ &= -\sum_{x\in\mathbb{Z}_N} \delta\widehat{v}_{x,L-1}(\lambda; s, \nu)\left(\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right)^\star.\end{aligned} \tag{8.28}$$

Comparing the real and imaginary parts we conclude that

$$\begin{aligned}&\left(\omega_0^2 + \xi^2 - \eta^2\right) \sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 + \sum_{x\in\mathbb{Z}_N} \left|\nabla_x \delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 \\ &+ \gamma\xi\left[\left|\delta\widehat{q}_{-N}^{(L)}(\lambda; s, \nu)\right|^2 + \left|\delta\widehat{q}_N^{(L)}(\lambda; s, \nu)\right|^2\right] = -\sum_{x\in\mathbb{Z}_N} \mathrm{Re}\left[\delta\widehat{v}_{x,L-1}(\lambda; s, \nu)\left(\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right)^\star\right]\end{aligned} \tag{8.29}$$

and

$$\begin{aligned}&2\xi\eta \sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 + \gamma\eta\left[\left|\delta\widehat{q}_{-N}^{(L)}(\lambda; s, \nu)\right|^2 + \left|\delta\widehat{q}_N^{(L)}(\lambda; s, \nu)\right|^2\right] \\ &= -\sum_{x\in\mathbb{Z}_N} \mathrm{Im}\left[\delta\widehat{v}_{x,L-1}(\lambda; s, \nu)\left(\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right)^\star\right].\end{aligned} \tag{8.30}$$

Recall that $\delta_* < \omega_0^2$. Choose $a \in (\delta_*, \omega_0^2)$. For $\eta^2 \leqslant \omega_0^2 - a$ and all $\xi > 0$ we conclude from (8.29) that

$$a \sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 \leqslant \left\{\sum_{x\in\mathbb{Z}_N} \left[\delta\widehat{v}_{x,L-1}(\lambda; s, \nu)\right]^2\right\}^{1/2} \left\{\sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2\right\}^{1/2}. \tag{8.31}$$

For $\eta \geqslant \sqrt{\omega_0^2 - a}$ we get from (8.30) that

$$2\xi\sqrt{\omega_0^2 - a} \sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 \leqslant \left\{\sum_{x\in\mathbb{Z}_N} \left[\delta\widehat{v}_{x,L-1}(\lambda; s, \nu)\right]^2\right\}^{1/2} \left\{\sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2\right\}^{1/2}. \tag{8.32}$$

If $\eta \leqslant -\sqrt{\omega_0^2 - a}$, then we multiply both sides of (8.30) by -1 and (8.32) is still in force. Consequently choosing ξ such that $2\xi\sqrt{\omega_0^2 - a} \geqslant a$ we have that (8.31) holds for any η.

From the above consideration we conclude the following.

Lemma 8.5 *For $\lambda = \xi + i\eta$, $\eta \in \mathbb{R}$, $\delta_* < a < \omega_0^2$ and $\xi > \frac{a}{2\sqrt{\omega_0^2 - a}}$ we have*

$$a^2 \sum_{x\in\mathbb{Z}_N} \left|\delta\widehat{q}_x^{(L)}(\lambda; s, \nu)\right|^2 \leqslant \sum_{x\in\mathbb{Z}_N} \left[\delta\widehat{v}_{x,L-1}(\lambda; s, \nu)\right]^2. \tag{8.33}$$

Corollary 8.6 *For a and ξ as in the statement of Lemma 8.5 we have*

$$\begin{aligned} &a^2 \sum_{x\in\mathbb{Z}_N} \int_0^{+\infty} e^{-2\xi t}\big[\delta q_x^{(L)}(t;s,\nu)\big]^2 \mathrm{d}t \\ &\leqslant \mathfrak{V}^2 \sum_{x\in\mathbb{Z}_N} \int_0^{+\infty} e^{-2\xi t}[\delta q_x^{(L-1)}(t;s,\nu)]^2 \mathrm{d}t, \quad L=1,2,\ldots. \end{aligned} \tag{8.34}$$

Proof Integrating over η in (8.33) and using the Parseval identity we conclude that

$$a^2 \sum_{x\in\mathbb{Z}_N} \int_0^{+\infty} e^{-2\xi t}\big[\delta q_x^{(L)}(t;s,\nu)\big]^2 \mathrm{d}t \leqslant \sum_{x\in\mathbb{Z}_N} \int_0^{+\infty} e^{-2\xi t}\Big[\delta v_{x,L-1}(t;s,\nu)\Big]^2 \mathrm{d}t. \tag{8.35}$$

Using (8.15) and (2.9) we conclude in the same way as in (5.23) that

$$\begin{aligned} &|\delta v_{x,L-1}(t;s,\nu)| \leqslant (\|V''\|_\infty + \|U''\|_\infty)|\delta q_x^{(L-1)}(t;s,\nu)| \\ &+ \|U''\|_\infty\Big(|\delta q_{x-1}^{(L-1)}(t;s,\nu)| + |\delta q_{x+1}^{(L-1)}(t;s,\nu)|\Big). \end{aligned}$$

Hence,

$$\begin{aligned} &[\delta v_{x,L-1}(t;s,\nu)]^2 \leqslant (\|V''\|_\infty + \|U''\|_\infty)^2[\delta q_x^{(L-1)}(t;s,\nu)]^2 \\ &+ (\|V''\|_\infty + \|U''\|_\infty)\|U''\|_\infty\Big(2[\delta q_x^{(L-1)}(t;s,\nu)]^2 + [\delta q_{x-1}^{(L-1)}(t;s,\nu)]^2 + [\delta q_{x+1}^{(L-1)}(t;s,\nu)]^2\Big) \\ &+ 2\|U''\|_\infty^2\Big([\delta q_{x-1}^{(L-1)}(t;s,\nu)]^2 + [\delta q_{x+1}^{(L-1)}(t;s,\nu)]^2\Big). \end{aligned} \tag{8.36}$$

Combining with (8.35) we conclude estimate (8.34) with $\mathfrak{V}$ defined in (3.11). □

8.2.3 Proof of Theorem 8.3

We start with the following.

Lemma 8.7 *For fixed $(\mathbf{q},\mathbf{p}) \in \Omega_N$ and ν we have*

$$\sup_{t\geqslant s}\Big(\|\mathbf{q}(t,s;\mathbf{q},\mathbf{p},\nu)\| + \|\mathbf{p}(t,s;\mathbf{q},\mathbf{p},\nu)\|\Big) := \mathfrak{Q}(\mathbf{q},\mathbf{p}) < +\infty. \tag{8.37}$$

Proof The solution of (2.3) satisfies

$$\begin{aligned} &\begin{pmatrix} \mathbf{q}(t,s;\nu) \\ \mathbf{p}(t,s;\nu) \end{pmatrix} = e^{A(t-s)}\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} - \nu\int_s^t e^{A(t-s')}\mathbb{V}(\mathbf{q}(s',s;\nu))\mathrm{d}s' \\ &+ \int_s^t e^{A(t-s')}\mathbb{F}\Big(\frac{s'}{\theta}\Big)\mathrm{d}s'. \end{aligned} \tag{8.38}$$

Here $\mathbb{V} : \Omega_N \to \Omega_N \times \Omega_N$ is a function defined in (8.20). Using (8.38) we conclude that

$$\begin{aligned} &\Big(\|\mathbf{q}(t,s;\nu)\|^2 + \|\mathbf{p}(t,s;\nu)\|^2\Big)^{1/2} \leqslant \|e^{A(t-s)}\|\Big(\|\mathbf{q}\|^2 + \|\mathbf{p}\|^2\Big)^{1/2} \\ &+ |\nu|\int_s^t \|e^{A(t-s')}\|\|\mathbb{V}(\mathbf{q}(s',s;\nu))\|_{\ell_2(\mathbb{Z}_N)}\mathrm{d}s' \\ &+ \int_s^t \|e^{A(t-s')}\|\|\mathbb{F}\Big(\frac{s'}{\theta}\Big)\|_{\ell_2(\mathbb{Z}_N)}\mathrm{d}s'. \end{aligned} \tag{8.39}$$

Thanks to (8.8) we can write

$$\left(\|\mathbf{q}(t,s;\nu)\|^2+\|\mathbf{p}(t,s;\nu)\|^2\right)^{1/2}\leqslant Ce^{-\lambda_N(t-s)}\left(\|\mathbf{q}\|^2+\|\mathbf{p}\|^2\right)^{1/2} + C\int_s^t e^{-\lambda_N(t-s')}\mathrm{d}s'. \tag{8.40}$$

Hence (8.37) follows. □

Consider the equation

$$\ddot{\mathfrak{q}}_x(t)=\left(\Delta-\omega_0^2\right)\mathfrak{q}_x(t)-\gamma\left(\delta_{-N}(x)+\delta_N(x)\right)\dot{\mathfrak{q}}_x(t),\quad x=-N,\dots,N. \tag{8.41}$$

Denote by B_R the ball of radius $R>0$, centered at 0 in $\mathbb{R}^{2(2N+1)}$ and by $\mathfrak{q}(t)=\left(\mathfrak{q}_x(t)\right)$, $\dot{\mathfrak{q}}(t)=\left(\dot{\mathfrak{q}}_x(t)\right)$.

Lemma 8.8 *For fixed $R>0$, ν and $\xi\geqslant 0$ we have*

$$\lim_{s\to-\infty}\sup_{(\mathfrak{q}(0),\dot{\mathfrak{q}}(0))\in B_R}\sum_{x\in\mathbb{Z}_N}\int_0^{+\infty}e^{-2\xi t}\mathfrak{q}_x^2(t)\mathrm{d}t=0. \tag{8.42}$$

Proof Using (8.9) we conclude that for $t\geqslant 0\geqslant s$ we have

$$\left(\sum_{x\in\mathbb{Z}_N}\mathfrak{q}_x^2(t)\right)^{1/2}\leqslant Ce^{-\lambda_N t}\mathfrak{Q}(\mathbf{q},\mathbf{p}). \tag{8.43}$$

From this

$$\sum_{x\in\mathbb{Z}_N}\int_0^{+\infty}e^{-2\xi t}\mathfrak{q}_x^2(t)\mathrm{d}t\leqslant C\mathfrak{Q}^2(\mathbf{q},\mathbf{p})\int_0^{+\infty}e^{-2\lambda_N t}\mathrm{d}t=\frac{C}{2\lambda_N}\mathfrak{Q}^2(\mathbf{q},\mathbf{p}) \tag{8.44}$$

and (8.45) follows. □

We restore the dependence on the initial data $(\mathbf{q},\mathbf{p})\in\mathbb{R}^{2(2N+1)}$ in the notation of $\delta q_x^{(0)}(t;s,\mathbf{q},\mathbf{p},\nu)$. From Lemmas 8.7 and 8.8 we immediately conclude the following.

Corollary 8.9 *We have*

$$\lim_{s\to-\infty}\sum_{x\in\mathbb{Z}_N}\int_0^{+\infty}e^{-2\xi t}\left[\delta q_x^{(0)}(t;s,\mathbf{q},\mathbf{p},\nu)\right]^2\mathrm{d}t=0. \tag{8.45}$$

Proof Note that $\delta q^{(0)}(0;s,\mathbf{q},\mathbf{p},\nu)=\left(\delta q_x^{(0)}(t;s,\mathbf{q},\mathbf{p},\nu)\right)_{x\in\mathbb{Z}_N}$ solves (8.41). According to Lemma 8.7 there exists $R>0$ such that

$$\left(\delta q^{(0)}(0;s,\mathbf{q},\mathbf{p},\nu),\delta\dot{q}^{(0)}(0;s,\mathbf{q},\mathbf{p},\nu)\right)\in B_R\quad\text{for all } s<0.$$

The conclusion of the corollary then follows directly from Lemma 8.8. □

Notice that $\frac{\mathfrak{V}}{a} < \frac{\mathfrak{V}}{\delta_*} := \nu_0$. Suppose that $|\nu| < \nu_0$. Using Corollary 8.6 we conclude that for ξ large enough we have

$$\begin{aligned}&\left\{\sum_{x\in\mathbb{Z}_N}\int_0^{+\infty} e^{-2\xi t}\left[\delta q_x^{(L)}(t;s,\nu)\right]^2\right\}^{1/2} \mathrm{d}t\\&\leqslant \left(\frac{\mathfrak{V}}{a}\right)^L\left\{\int_0^{+\infty} e^{-2\xi t}[\delta q_x^{(0)}(t;s,\nu)]^2\mathrm{d}t\right\}^{1/2}, \quad L=1,2,\ldots.\end{aligned} \tag{8.46}$$

Hence,

$$\begin{aligned}&\left\{\sum_{x\in\mathbb{Z}_N}\int_0^{+\infty} e^{-2\xi t}\left[q_x(t;s,\nu)-q_{x,\mathrm{p}}(t,\nu)\right]^2\mathrm{d}t\right\}^{1/2}\\&=\left\{\sum_{x\in\mathbb{Z}_N}\int_0^{+\infty} e^{-2\xi t}\Big[\sum_{\ell\geqslant 0}\nu^\ell\delta q_x^{(\ell)}(t;s,\nu)\Big]^2\mathrm{d}t\right\}^{1/2}\\&\leqslant\sum_{\ell\geqslant 0}|\nu|^\ell\left\{\int_0^{+\infty} e^{-2\xi t}\sum_{x\in\mathbb{Z}_N}\left[\delta q_x^{(\ell)}(t;s,\nu)\right]^2\mathrm{d}t\right\}^{1/2}\\&\leqslant\frac{1}{1-(|\nu|\mathfrak{V}/a)}\left\{\int_0^{+\infty} e^{-2\xi t}[\delta q_x^{(0)}(t;s,\nu)]^2\mathrm{d}t\right\}^{1/2}.\end{aligned} \tag{8.47}$$

The conclusion of Theorem 8.3 then follows from Corollary 8.9. □

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Data Availability We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

Declarations

Conflicts of Interest The authors declare no conflicts of interest regarding this manuscript.

Appendix A Formulas for the Green's function

A.1 Green's function of the operator $\lambda + \omega_0^2 - \Delta$ on $\mathbb{Z}$

Suppose that $-\lambda \in \mathbb{C} \setminus \mathcal{I}$. The Green's function of the operator $\lambda - \omega_0^2 + \Delta$, where Δ is the free Laplacian on $\mathbb{Z}$ is given by

$$\mathrm{G}_\lambda(x) = \big(\lambda + \omega_0^2 - \Delta\big)^{-1} \delta_0(x) \quad = \int_0^1 \frac{\cos(2\pi u x)\mathrm{d}u}{\lambda + 4\sin^2(\pi u) + \omega_0^2}. \tag{A.1}$$

Note that

$$\mathrm{G}_{\lambda^\star}(x) = \mathrm{G}_\lambda^\star(x), \quad x \in \mathbb{Z},\ -\lambda \in \mathbb{C} \setminus \mathcal{I}. \tag{A.2}$$

Let $\zeta = \chi(\lambda) := \frac{1}{2}(2 + \omega_0^2 + \lambda)$ and let Φ_+ be the inverse of the Joukowski function

$$\zeta = J(z) = \frac{1}{2}\Big(z + \frac{1}{z}\Big), \quad z \in \mathbb{C}$$

considered for $|z| > 1$. Then $z = \Phi_+(\zeta)$, $\zeta \notin [-1, 1]$. Condition $-\lambda \notin \mathcal{I}$ is equivalent with $\zeta \notin [-1, 1]$. Furthermore $|\Phi_+\big(\chi(\lambda)\big)| > 1$. We have then

$$\mathrm{G}_\lambda(x) = \frac{\Phi_+^{1-|x|}\big(\chi(\lambda)\big)}{\Phi_+^2\big(\chi(\lambda)\big) - 1} = \frac{\Phi_+^{-|x|}\big(\chi(\lambda)\big)}{2[\Phi_+\big(\chi(\lambda)\big) - \chi(\lambda)]}, \quad x \in \mathbb{Z} \tag{A.3}$$

We can use the mapping $\zeta \mapsto \Phi_+(\zeta)$, $\zeta \notin [-1, 1]$ to define the mapping $\zeta \mapsto w = (\zeta^2 - 1)^{1/2}$, $\zeta \notin [-1, 1]$ by letting

$$(\zeta^2 - 1)^{1/2} := \Phi_+(\zeta) - \zeta, \quad \zeta \notin [-1, 1]. \tag{A.4}$$

In particular,

$$\{\zeta^2 - 1\}^{1/2} = \begin{cases} \sqrt{\zeta^2 - 1}, & \text{when } \zeta \in [1, +\infty), \\ i\sqrt{1 - \zeta^2}, & \text{when } \zeta \in i(0, +\infty), \\ -\sqrt{\zeta^2 - 1}, & \text{when } \zeta \in (-\infty, -1], \\ -i\sqrt{1 - \zeta^2}, & \text{when } \zeta \in i(-\infty, 0) \end{cases}$$

Here $w \mapsto \sqrt{w}$ is the branch of the inverse of $z \mapsto z^2$ such that $\mathrm{Re}\sqrt{w} > 0$, when $w \in \mathbb{C} \setminus (-\infty, 0]$. Furthermore

$$\Phi_+\big(\chi(\lambda)\big) = \zeta + \{\zeta^2 - 1\}^{1/2} = \Phi_+(\zeta).$$

Formula (A.3) can be rewritten in the form

$$\mathrm{G}_\lambda(x) = \frac{\big\{1 + \frac{1}{2}(\lambda + \omega_0^2) + \frac{1}{2}\{(\lambda + \omega_0^2)(\lambda + \omega_u^2)\}^{1/2}\big\}^{-|x|}}{\{(\lambda + \omega_0^2)(\lambda + \omega_u^2)\}^{1/2}}, \quad x \in \mathbb{Z} \tag{A.5}$$

recall that $\omega_u = \sqrt{\omega_0^2 + 4}$.

Suppose that $\omega \in \mathbb{R}$. In case $|m|\omega < \omega_0$ we have

$$
\begin{aligned}
G_{-(m\omega)^2}(x) &= \frac{\xi_+^{-|x|}(m\omega)}{D(m\omega)}, \quad x \in \mathbb{Z}, \quad \text{with} \\
\xi_+(\Omega) &:= 1 + \frac{1}{2}\big(\omega_0^2 - \Omega^2 + D(\Omega)\big) \quad \text{and} \\
D(\Omega) &:= \sqrt{(\omega_0^2 - \Omega^2)(\omega_u^2 - \Omega^2)}, \quad \Omega \in [\omega_0, \omega_u]
\end{aligned}
\tag{A.6}
$$

When $|m|\omega > \omega_u$ we have

$$
\begin{aligned}
G_{-(m\omega)^2}(x) &= -\frac{\xi_-^{-|x|}(m\omega)}{D(m\omega)}, \quad x \in \mathbb{Z}, \quad \text{with} \\
\xi_-(\Omega) &:= 1 + \frac{1}{2}\big(\omega_0^2 - \Omega^2 - D(\Omega)\big), \quad x \in \mathbb{Z}.
\end{aligned}
\tag{A.7}
$$

We have

$$
\begin{aligned}
&\xi_+(\Omega) > 1, \quad \text{when} \quad 0 \leqslant \Omega < \omega_0 \quad \text{and} \\
&\xi_-(\Omega) < -1, \quad \text{when} \quad \Omega > \sqrt{4 + \omega_0^2}.
\end{aligned}
$$

A.2 Green's function on $\mathbb{Z}_N$

Suppose that $G_\lambda^{(N)}(x, y) = \big(\lambda + \omega_0^2 - \Delta\big)^{-1}\delta_y(x)$, $x, y \in \mathbb{Z}_N$ is the Green's function for the Neumann laplacian on $\mathbb{Z}_N$. We have

$$
G_\lambda^{(N)}(x, y) = \sum_{j=0}^{2N} \frac{\psi_j(x)\psi_j(y)}{\lambda + \omega_j^2}, \quad x, y \in \mathbb{Z}_N.
\tag{A.8}
$$

Here $\mu_j = \omega_j^2$, $j = 0, \ldots, 2N$ are the eigenvalues of $\omega_0^2 - \Delta$, with

$$
\omega_j = \omega\left(\frac{j}{2N+1}\right)
\tag{A.9}
$$

and the dispersion relation $\omega(k) = \sqrt{\omega_0^2 + 4\sin^2(\pi k)}$. The eigenvectors are given by

$$
\begin{aligned}
\psi_0(x) &= \left(\frac{1}{2N+1}\right)^{1/2}, \\
\psi_j(x) &= \left(\frac{2}{2N+1}\right)^{1/2} \cos\left(\frac{\pi j}{2} \cdot \frac{2(x+N)+1}{2N+1}\right) \quad x = -N, \ldots, N, \; j = 1, \ldots, 2N.
\end{aligned}
\tag{A.10}
$$

Using the method of images we obtain the following formula for the Green's function

$$
G_\lambda^{(N)}(x, y) = \sum_{\ell \in \mathbb{Z}} \Big[G_\lambda\big(x - y + 2\ell(2N+1)\big) + G_\lambda\big(x + y + (2\ell+1)(2N+1)\big)\Big], \quad x, y \in \mathbb{Z}_N,
\tag{A.11}
$$

with G_λ given by formula (A.5) From (A.3) and (A.11) we conclude the following.

Proposition A.1 *Suppose that there exists* $\delta > 0$ *such that*

$$|\lambda + w| \geqslant \delta > 0, \quad \forall w \in [\omega_0^2, \omega_0^2 + 4]. \tag{A.12}$$

Then, there exist constants $C, c > 0$*, depending only on* δ *and such that*

$$\left|G_\lambda^{(N)}(x, y)\right| \leqslant \frac{C}{1 + |\lambda|} \exp\{-c|x - y|\}, \quad x, y \in \mathbb{Z}_N. \tag{A.13}$$

A.3 Green's function for $\lambda + \omega_0^2 + i\sigma(\delta_{-N} + \delta_N) - \Delta$

As before Δ is the Neumann laplacian on $\mathbb{Z}_N$. From (5.11) we conclude that $H_{\lambda,\sigma}^{(N)}(x, y)$ satisfies

$$\left(\lambda + \omega_0^2 - \Delta_x\right) H_{\lambda,\sigma}^{(N)}(x, y) = \delta_y(x) - i\sigma\left(\delta_{-N}(x) + \delta_N(x)\right) H_{\lambda,\sigma}^{(N)}(x, y). \tag{A.14}$$

Hence

$$H_{\lambda,\sigma}^{(N)}(x, y) = G_\lambda^{(N)}(x, y) - i\sigma H_{\lambda,\sigma}^{(N)}(-N, y) G_\lambda^{(N)}(x, -N) - i\sigma H_{\lambda,\sigma}^{(N)}(N, y) G_\lambda^{(N)}(x, N). \tag{A.15}$$

For $x = -N$ and $x = N$ we get in particular the following system of equations on $H_\lambda^{(N)}(\pm N, y)$

$$\begin{aligned} &H_{\lambda,\sigma}^{(N)}(-N, y)\Big(1 + i\sigma G_\lambda^{(N)}(N, N)\Big) + i\sigma H_{\lambda,\sigma}^{(N)}(N, y) G_\lambda^{(N)}(-N, N) = G_\lambda^{(N)}(-N, y) \\ &i\sigma H_{\lambda,\sigma}^{(N)}(-N, y) G_\lambda^{(N)}(N, -N) + H_{\lambda,\sigma}^{(N)}(N, y)\Big(1 + i\sigma G_\lambda^{(N)}(N, N)\Big) = G_\lambda^{(N)}(N, y). \end{aligned} \tag{A.16}$$

Let

$$\begin{aligned} \mathfrak{Q}_N &:= \Big(1 + i\sigma G_\lambda^{(N)}(N, N)\Big)^2 + \sigma^2\Big(G_\lambda^{(N)}(-N, N)\Big)^2, \\ \mathfrak{Q}(-N, y) &:= \Big(1 + i\sigma G_\lambda^{(N)}(N, N)\Big) G_\lambda^{(N)}(-N, y) - i\sigma G_\lambda^{(N)}(-N, N) G_\lambda^{(N)}(N, y), \\ \mathfrak{Q}(N, y) &:= \Big(1 + i\sigma G_\lambda^{(N)}(N, N)\Big) G_\lambda^{(N)}(N, y) - i\sigma G_\lambda^{(N)}(-N, N) G_\lambda^{(N)}(-N, y), \end{aligned}$$

Note that if $\lambda \in \mathbb{R}$ and $\sigma \neq 0$ we have

$$\operatorname{Im} \mathfrak{Q}_N = 2\sigma G_\lambda^{(N)}(N, N) = 2\sigma \sum_{j=0}^{2N} \frac{\psi_j^2(0)}{\lambda + \omega_j^2} \neq 0. \tag{A.17}$$

We also have $\mathfrak{Q}_N \neq 0$ in the case when $\lambda \in \mathbb{C}$, and $|\operatorname{Im} \lambda|$ is sufficiently small.

Then,

$$H_{\lambda,\sigma}^{(N)}(-N, y) = \frac{\mathfrak{Q}(-N, y)}{\mathfrak{Q}_N}, \quad H_{\lambda,\sigma}^{(N)}(N, y) = \frac{\mathfrak{Q}(N, y)}{\mathfrak{Q}_N} \tag{A.18}$$

Substituting into (A.15) we get

$$H_{\lambda,\sigma}^{(N)}(x, y) = G_\lambda^{(N)}(x, y) - i\sigma G_\lambda^{(N)}(x, -N) \frac{\mathfrak{Q}(-N, y)}{\mathfrak{Q}_N} - i\sigma G_\lambda^{(N)}(x, N) \frac{\mathfrak{Q}(N, y)}{\mathfrak{Q}_N}. \tag{A.19}$$

Lemma A.2 *Suppose that* $\lambda \in \mathbb{R}$ *and*

$$|\lambda + w| \geqslant \delta > 0, \quad w \in [\omega_0^2, \omega_u^2]. \tag{A.20}$$

Then, there exist constants $C_*, c_* > 0$, *depending only on* δ *and such that*

$$\left|H_{\lambda,\sigma}^{(N)}(x, y)\right| \leqslant \frac{C_*}{1 + |\lambda|} \exp\{-c_*|x - y|\}, \quad x, y \in \mathbb{Z}_N. \tag{A.21}$$

Appendix B Exponential decay of the periodic solution in *x*

B.1 Some auxiliaries

Suppose that $\widetilde{q}_x^{(L)}(m; \nu)$, $L = 0, 1, 2, \ldots$ are determined by the scheme described in (5.13)–(5.14), see Section 5.3. Let $c > 0$. Define, by recurrence

$$e_c(x) := e_c^{\star,1}(x) := e^{-c|x|}$$

and

$$e_c^{\star,L+1}(x) := \sum_{y\in\mathbb{Z}} e^{-c|x-y|} e_c^{\star,L}(y), \quad x \in \mathbb{Z}.$$

We also adopt the convention that $e_0(x) := \delta_0(x), \quad x \in \mathbb{Z}$. Let

$$\widehat{e}_c(k) := \sum_{x\in\mathbb{Z}} e^{-c|x|} \exp\{-2\pi i k x\} = \frac{1 - e^{-2c}}{|1 - e^{-c} e^{2\pi i k}|^2}, \quad k \in \mathbb{T}. \tag{B.1}$$

Then,

$$\widehat{e}_c^{\star,L}(k) := \sum_{x\in\mathbb{Z}} e_c^{\star,L}(x) \exp\{2\pi i k x\} = \left(\frac{1 - e^{-2c}}{|1 - e^{-c} e^{2\pi i k}|^2}\right)^L. \tag{B.2}$$

Since

$$e_c^{\star,L}(x) = \int_0^1 \left(\frac{1 - e^{-2c}}{|1 - e^{-c} e^{2\pi i k}|^2}\right)^L \exp\{2\pi i k x\} \, \mathrm{d}k,$$

we can easily conclude that

$$0 \leqslant e_c^{\star,L}(x) \leqslant \left(\frac{1 + e^{-c}}{1 - e^{-c}}\right)^L, \quad x \in \mathbb{Z}. \tag{B.3}$$

For any q such that

$$|q| h(c) < 1, \quad \text{where} \quad h(c) := \frac{1 + e^{-c}}{1 - e^{-c}}$$

we can define therefore

$$(\delta_0 - q e_c)^{-1,\star}(x) = \sum_{L=0}^{+\infty} q^L e_c^{\star,L}(x). \tag{B.4}$$

The series is uniform convergent for all $x \in \mathbb{Z}$. We also have the following.

Lemma B.1 *For any* $c > 0$ *and* $|q| < 1/h(c)$ *there exist contstants* $A(c, q), \rho(c, q) > 0$, *depending only on the indicated parameters, such that*

$$0 \leqslant (\delta_0 - q e_c)^{-1,\star}(x) \leqslant A(c, q) \exp\{-\rho(c, q)|x|\}, \quad x \in \mathbb{Z}. \tag{B.5}$$

Proof From (B.4) and (B.2) we conclude that

$$\widehat{(\delta_0 - qe_c)}^{-1,\star}(k) = \frac{1}{1 - q\left(\frac{1-e^{-2c}}{|1-e^{-c}e^{2\pi ik}|^2}\right)}. \tag{B.6}$$

It is an analytic function in $k \in \mathbb{T}$. Therefore, see e.g. [26, p. 27], there exist constants A and $\rho > 0$ such that

$$0 \leqslant (\delta_0 - qe_c)^{-1,\star}(x) \leqslant A \exp\{-\rho|x|\}, \quad x \in \mathbb{Z}. \tag{B.7}$$

□

Lemma B.2 *Suppose that $c_* > 0$ is as in the statement of Lemma A.2. Then, there exist $C, c_1 > 0$ such that*

$$|\widetilde{q}_x^{(L)}(m;\nu)| \leqslant \frac{Cc_1^L e_{c_*}^{\star,L}(x)}{1+(m\omega)^2}, \quad x \in \mathbb{Z}_N \tag{B.8}$$

for all $N = 1, 2, \ldots, L = 0, 1, \ldots$ and $m \in \mathbb{Z}$.

Proof The estimate for $L = 0$ follows from (5.15) and estimate (A.21) with the constant

$$C := C_* \sup_{m\in\mathbb{Z}} |\widehat{F}_m|,$$

where $C_* > 0$ is as in the statement of Lemma A.2 and $e_{c_*}^{\star,0}(x) = e^{-c_*|x|}$. Suppose that we have shown that

$$|\widetilde{q}_x^{(L)}(m;\nu)| \leqslant \frac{Cc_1^L e_{c_*}^{\star,L}(x)}{1+(m\omega)^2}, \quad x \in \mathbb{Z}_N \tag{B.9}$$

for some L, with the constant c_1 to be determined later on. Then,

$$q_x^{(L)}(t;\nu) = \sum_{m\in\mathbb{Z}} \widetilde{q}_x^{(L)}(m;\nu)e^{im\omega t}$$

we have

$$|q_x^{(L)}(t;\nu)| \leqslant \sum_{m\in\mathbb{Z}} |\widetilde{q}_x^{(L)}(m;\nu)| \leqslant Cc_1^L e_{c_*}^{\star,L}(x) \sum_{m\in\mathbb{Z}} \frac{1}{1+(m\omega)^2}. \tag{B.10}$$

Therefore

$$\begin{aligned}
|\widetilde{v}_{x,L}(m)| &\leqslant \frac{1}{|\nu|^L\theta}\left\{\int_0^\theta \left|V'\big(Q_{x,\mathrm{p}}^{(L)}(t;\nu)\big) - V'\big(Q_x^{(L-1)}(t;\nu)\big)\right| \mathrm{d}t\right.\\
&\left.+ \int_0^\theta \left|\nabla U'\big(Q_x^{(L)}(t;\nu)\big) - \nabla U'\big(Q_x^{(L-1)}(t;\nu)\big)\right| \mathrm{d}t\right\} \leqslant \frac{\|V\|_{\mathrm{Lip}} + 2\|U\|_{\mathrm{Lip}}}{\theta} \int_0^\theta |q_x^{(L)}(t;\nu)| \mathrm{d}t\\
&\leqslant Cc_1^L(\|V\|_{\mathrm{Lip}} + 2\|U\|_{\mathrm{Lip}}) e_{c_*}^{\star,L}(x) \sum_{m\in\mathbb{Z}} \frac{1}{1+(m\omega)^2}
\end{aligned} \tag{B.11}$$

and, thanks to (5.16) and (A.21),

$$\begin{aligned}|\widetilde{q}_x^{(L+1)}(m;\nu)| &\leqslant \sum_{y\in\mathbb{Z}_N}|H_{-(m\omega)^2,\gamma m\omega}(x,y)||\widetilde{v}_{y,L}(m)| \\ &\leqslant CC_* c_1^L(\|V\|_{\text{Lip}}+2\|U\|_{\text{Lip}})\Big(\sum_{m'\in\mathbb{Z}}\frac{1}{1+(m'\omega)^2}\Big)\frac{e_{c_*}^{\star,L+1}(x)}{1+(m\omega)^2} \\ &= \frac{Cc_1^{L+1}e_{c_*}^{\star,L+1}(x)}{1+(m\omega)^2},\end{aligned} \tag{B.12}$$

provided we choose

$$c_1 := C_*\sum_{m'\in\mathbb{Z}}\frac{\|V\|_{\text{Lip}}+2\|U\|_{\text{Lip}}}{1+(m'\omega)^2}.$$

Thus (B.8) follows. □

B.2 Proof of Theorem 3.6

We have

$$q_{x,p}(t;\nu) = \sum_{m\in\mathbb{Z}}\widetilde{q}_{x,p}(m;\nu)\exp\{2\pi i m\omega t\},$$

where

$$\widetilde{q}_{x,p}(m;\nu) = \sum_{L=0}^{+\infty}\widetilde{q}_{x,p}^{(L)}(m;\nu)\nu^L.$$

Using estimate (B.8) we conclude that

$$\begin{aligned}|\widetilde{q}_{x,p}(m;\nu)| &\leqslant \frac{C}{1+(m\omega)^2}\sum_{L=0}^{+\infty}(|\nu|c_1)^L e_{c_*}^{\star,L}(x) \\ &= \frac{C}{1+(m\omega)^2}\Big(\delta_0-(|\nu|c_1)e_{c_*}\Big)^{-1,\star}(x).\end{aligned}$$

This estimate together with (B.5) imply the first bound of (3.23). The second bound follows from the fact that

$$\frac{1}{\theta}\int_0^\theta p_{x,p}^2(t;\nu)\mathrm{d}t = \sum_{m\in\mathbb{Z}}(m\omega)^2|\widetilde{q}_{x,p}(m;\nu)|^2.$$

□

Appendix C The existence of a periodic solution for a finite anharmonic chain. Proof of Theorem 3.1

Using the notation of Section 8 for any θ-periodic Ω_N-valued function $\mathbf{q}(t)$ we can write

$$\int_{-\infty}^{t} e^{A(t-s)}\mathbb{V}\big(\mathbf{q}(s)\big)\mathrm{d}s = \sum_{\ell=0}^{+\infty}\int_0^\theta e^{A(s+\ell\theta)}\mathbb{V}\big(\mathbf{q}(t-s-\ell\theta)\big)\mathrm{d}s$$
$$= \sum_{\ell=0}^{+\infty}\int_0^\theta e^{A(s+\ell\theta)}\mathbb{V}\big(\mathbf{q}(t-s)\big)\mathrm{d}s = \int_0^\theta (I-e^{A\theta})^{-1}e^{As}\mathbb{V}\big(\mathbf{q}(t-s)\big)\mathrm{d}s.$$

Suppose that $t \in [0,\theta]$. Then we can write the utmost right hand side of the above equality as being equal to

$$\int_0^\theta K(t-s)\mathbb{V}(\mathbf{q}(s))\mathrm{d}s,$$

where K is the θ-periodic extension of

$$K(s) = \begin{cases} (I-e^{A\theta})^{-1}e^{As}, & s \in [0,\theta/2], \\ (I-e^{A\theta})^{-1}e^{A(\theta+s)}, & s \in [-\theta/2, 0]. \end{cases}$$

We conclude the following.

Proposition C.1 *If $\mathbf{q}_{\mathrm{p}}(t;\nu)$ a θ-periodic solution of (2.3), then it satisfies the equation*

$$\begin{pmatrix}\mathbf{q}_{\mathrm{p}}(t;\nu)\\ \mathbf{p}_{\mathrm{p}}(t;\nu)\end{pmatrix} = -\nu\int_0^\theta K(t-s)\mathbb{V}(\mathbf{q}_{\mathrm{p}}(s;\nu))\mathrm{d}s + \int_{-\infty}^{t} e^{A(t-s)}\mathbb{F}\Big(\frac{s}{\theta}\Big)\mathrm{d}s. \tag{C.1}$$

Conversely, if $\mathbf{q}(t;\nu)$ is a θ-periodic solution of (C.1), then it is a θ-periodic solution of (2.3).

Proof of Theorem 3.1

On the space $\mathcal{C}_{\mathrm{p}} := \mathcal{C}_{\mathrm{p}}\Big([0,\theta];\Omega_N\Big)$ of θ-periodic, Ω_N-valued functions, equipped with the topology of the uniform convergence on compact intervals, we introduce the operator

$$\mathcal{T}[F](t) := -\nu\int_0^\theta K(t-s)\mathbb{V}(\Pi_1 F(s))\mathrm{d}s + \int_{-\infty}^{t} e^{A(t-s)}\mathbb{F}\Big(\frac{s}{\theta}\Big)\mathrm{d}s.$$

Here $\Pi_1 : \mathbb{R}^{2(2N+1)} \to \mathbb{R}^{2N+1}$ is given by $\Pi_1\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} = \mathbf{q}$. We have $\mathcal{T} : \mathcal{C}_{\mathrm{p}} \to \mathcal{C}_{\mathrm{p}}$. Moreover, $\mathcal{T}\big(\mathcal{C}_{\mathrm{p}}\big)$ is bounded in $\mathcal{C}_{\mathrm{p}}$ and, since $t \mapsto K(t)$ is Lipschitz, the set is compact. By the Schauder fixed point theorem we conclude the existence of a fixed point for $\mathcal{T}$, which by virtue of Proposition C.1 is a periodic solution to (2.3). □

Appendix D An application of the implicit function theorem in the proof of non-uniqueness of periodic solution

Recall that $\big(\overline{q}_x(t;F)\big)_{x\in\mathbb{Z}_N}$ is a $\theta = 2\pi/\omega$-periodic solutions of (2.3) corresponding to the single mode forcing (6.4). Its time harmonics $\widetilde{\overline{q}}_x(m;F)$ solve

$$0 = \big[(\omega m)^2 - \omega_0^2 + \Delta - i\gamma\omega m(\delta_{-N}+\delta_N)\big]\widetilde{\overline{q}}_x(m;\mathrm{F}) + \mathrm{F}\big(\delta_1(m)+\delta_{-1}(m)\big)\delta_0(x) \\ + \nu\widetilde{v}_x(m;\mathrm{F}), \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N, \tag{D.1}$$

with

$$\widetilde{v}_x(m;\mathrm{F}) = \frac{1}{\theta}\int_0^\theta e^{-im\omega t}v_x(t;\mathrm{F})\mathrm{d}t \quad \text{and} \\ v_x(t;\mathrm{F}) := -V'\big(q_{x,\mathrm{p}}(t;\mathrm{F})\big), \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N. \tag{D.2}$$

We can rewrite the equation in the space $\ell_{2,N}$ (see (5.6)) as follows

$$\Phi\Big(\mathrm{F},\widetilde{\overline{q}}(\cdot;\mathrm{F})\Big) = 0, \tag{D.3}$$

where $\Phi:\mathbb{R}\times\ell_{2,N}\to\ell_{2,N}$ is a mapping given by

$$\Phi\big(\mathrm{F},\widetilde{f}\big)_x(m) := \widetilde{f}_x(m) - H^{(N)}_{-(m\omega)^2,\gamma m\omega}(x,0)\mathrm{F}\big(\delta_1(m)+\delta_{-1}(m)\big) \\ -\ \nu\sum_{y\in\mathbb{Z}_N}H^{(N)}_{-(m\omega)^2,\gamma m\omega}(x,y)\widetilde{v}(\widetilde{f})_y(m), \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N, \\ \widetilde{v}(\widetilde{f})_x(m) = \frac{1}{\theta}\int_0^\theta e^{-im\omega t}V'(f_x(t))\mathrm{d}t \quad \text{and} \\ f_x(t) := \sum_{m\in\mathbb{Z}}\widetilde{f}_x(m)e^{im\omega t}, \quad t\geqslant 0. \tag{D.4}$$

For $\mathrm{F}=0$ equation (D.3) is satisfied by $\widetilde{\overline{q}} = \big(\widetilde{\overline{q}}_x(m;0)\big)$ given by

$$\widetilde{\overline{q}}_x(m;0) = \overline{q}\delta_{m,0}, \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N.$$

The Fréchet derivative of the right hand side of (D.3) with respect to the q variable, at $(0,\widetilde{\overline{q}})$ is a linear operator $\mathrm{L}:\ell_{2,N}\to\ell_{2,N}$ given by

$$\mathrm{L}\widetilde{h}_x(m) := \widetilde{h}_x(m) - \mathrm{K}\widetilde{h}_x(m), \quad \text{where} \\ \mathrm{K}\widetilde{h}_x(m) := \nu V''(\overline{q})\sum_{y\in\mathbb{Z}_N}H^{(N)}_{-(m\omega)^2,\gamma m\omega}(x,y)\widetilde{h}_y(m), \quad (m,x)\in\mathbb{Z}\times\mathbb{Z}_N.$$

It is easy to see that the operator $\mathrm{K}:\ell_{2,N}\to\ell_{2,N}$ is compact. We shall show that its null space is trivial. This would imply, see e.g. [29, Theorem 21.2.6, 238], that L is onto and its inverse is bounded. Using the implicit function theorem, see e.g. [10, Theorem 4.15.1], we conclude then that there exists $\overline{F}_0>0$ such that for any $\mathrm{F}\in(-\overline{F}_0,\overline{F}_0)$ equation (D.4) has a unique solution that gives rise to a θ periodic solution $\overline{q}_{\mathrm{p},x}(t;\mathrm{F})$ of (2.3) that satisfies

$$\lim_{\mathrm{F}\to 0}\overline{q}_{\mathrm{p},x}(t;\mathrm{F}) = \overline{q}, \quad x\in\mathbb{Z}_N.$$

To see that the kernel of L is trivial note that if $\mathrm{L}\widetilde{h}=0$, then

$$0 = \big[(\omega m)^2 - \omega_0^2 - \nu V''(\overline{q}) + \Delta - i\gamma\omega m(\delta_{-N}+\delta_N)\big]\widetilde{h}_x(m), \quad x\in\mathbb{Z}_N. \tag{D.5}$$

Multiplying both sides pf (D.5) by $\widetilde{h}^{\star}_{x}(m)$ and summing over x we conclude that

$$\sum_{m\in\mathbb{Z}}(m\omega)^2|\widetilde{h}_{\pm N}(m)|^2 = 0.$$

Hence $\widetilde{h}_{\pm N}(m) = 0$ for $m \neq 0$. Arguing as in Section 7, from (D.5) we conclude that $\widetilde{h}_x(m) \equiv 0$ for all $x \in \mathbb{Z}_N$ and $m \neq 0$. For $m = 0$ we can multiply both sides of equation (D.5) by $\widetilde{h}^{\star}_{x}(0)$ and sum over x. Since $\omega_0^2 + \nu V''(\overline{q}) > 0$ this allows us to conclude that also in the case $\widetilde{h}_x(0) \equiv 0$ for all $x \in \mathbb{Z}_N$.

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