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PAPER: Classical statistical mechanics, equilibrium and non-equilibrium

Exact solution of the facilitated totally asymmetric simple exclusion process

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Abstract. We obtain the exact solution of the facilitated totally asymmetric simple exclusion process (F-TASEP) in 1D. The model is closely related to the conserved lattice gas (CLG) model and to some cellular automaton traffic models. In the F-TASEP a particle at site j in \mathbb{Z} jumps, at integer times, to site $j + 1$, provided site $j - 1$ is occupied and site $j + 1$ is empty. When started with a Bernoulli product measure at density ρ , the system approaches a stationary state. This non-equilibrium steady state (NESS) has phase transitions at $\rho = 1/2$ and $\rho = 2/3$. The different density regimes $0 < \rho < 1/2$, $1/2 < \rho < 2/3$, and $2/3 < \rho < 1$ exhibit many surprising properties; for example, the pair correlation $g(j) = \langle \eta(i)\eta(i+j) \rangle$ satisfies, for all $n \in \mathbb{Z}$, $\sum_{j=kn+1}^{k(n+1)} g(j) = k\rho^2$, with $k = 2$ when $0 \leq \rho \leq 1/2$, $k = 6$ when $1/2 \leq \rho \leq 2/3$, and $k = 3$ when $2/3 \leq \rho \leq 1$. The quantity $\lim_{L \rightarrow \infty} V_L/L$, where V_L is the variance in the number of particles in an interval of length L , jumps discontinuously from $\rho(1 - \rho)$ to 0 when $\rho \rightarrow 1/2$ and when $\rho \rightarrow 2/3$.

Keywords: absorbing states, exact results, exclusion processes

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1. Introduction

Phase transitions with absorbing states are a well-established field of study [1]. In particular, the facilitated time evolution of particles on a lattice has been the subject of extensive study in both the physics and mathematics literature. In two or more dimensions, much is known from numerical simulation [2–4], but there are few analytic results (but see [5]); in one dimension both numerical and analytic studies are available [6–8]. Here we present exact results for the *facilitated totally asymmetric simple exclusion process* (F-TASEP); for more details and proofs, see [9]. The F-TASEP is a variation of the conserved lattice gas model of particles moving on the one dimensional lattice \mathbb{Z} . If we think of the empty sites as cars and the occupied sites as empty spaces then the model corresponds to one of the traffic models considered in [10, 11], as we discuss later in this section.

A configuration of the model is an arrangement of particles on \mathbb{Z} , with each site either empty or occupied by a single particle, that is, the configuration space is $\{0, 1\}^{\mathbb{Z}}$; we write $\eta = (\eta(i))_{i \in \mathbb{Z}}$ for a given configuration. We study the discrete time dynamics, in which every particle whose left-hand neighboring site is occupied attempts, at each integer time, to jump to the neighboring site on its right; the jump takes place if the target is unoccupied. The model is thus a deterministic cellular automaton. We let η_t denote the configuration at time t ; the evolution is deterministic and we can determine the ultimate fate of any initial configuration with a well defined density ρ [9]. We will describe the translation invariant (TI) states of the system, at any density $\rho \in [0, 1]$, which are stationary under the dynamics (the TIS states). In particular we will determine, completely or (in one phase) partially, the final TIS state when the system is started in a Bernoulli measure: an initial state $\mu^{(\rho)}$ in which each site is independently occupied with probability ρ .

We will make use of a closely related model, the *totally asymmetric stack model* (TASM), another particle system on \mathbb{Z} evolving in discrete time. In the TASM there are no restrictions on the number of particles at any site, so that the configuration space

is $\{0, 1, 2, \dots\}^{\mathbb{Z}}$. We denote stack configurations by boldface letters. The dynamics is as follows: at each integer time, every stack with at least two particles ($\mathbf{n}(i) \geq 2$) sends one particle to the neighboring site to its right. There is a natural but somewhat loose correspondence of this model with the F-TASEP, with a stack configuration \mathbf{n} corresponding to a particle configuration in which strings of $\mathbf{n}(i)$ particles are separated by single holes (a similar correspondence is often introduced for zero range processes, see e.g. [12]). At the configuration level this correspondence is not one to one, but one may show that it gives rise to a bijective correspondence between the TI states, and also between the TIS states, of the two models [13]. In particular, if $\hat{\mu}$ is a TI state for the TASM, with density $\hat{\rho}$, then the corresponding state μ of the F-TASEP has density $\rho = \hat{\rho}/(1 + \hat{\rho})$. If $\mu = \mu^{(\rho)}$ then in the corresponding TASM measure $\hat{\mu} = \hat{\mu}^{(\rho)}$ the $\mathbf{n}(i)$ are i.i.d. with geometric distribution: $\hat{\mu}^{(\rho)}\{\mathbf{n}(i) = k\} = (1 - \rho)\rho^k$.

We note some simple properties of the dynamics in the TASM. Letting $\mathbf{n}_t(k)$ denote the height at time t of the stack of particles on site k , we have

- If $\mathbf{n}_t(k) \geq 2$ then $\mathbf{n}_{t+1}(k) = \mathbf{n}_t(k)$ unless $\mathbf{n}_t(k-1) \leq 1$, in which case $\mathbf{n}_{t+1}(k) = \mathbf{n}_t(k) - 1$;
- If $\mathbf{n}_t(k) \leq 1$ then $\mathbf{n}_{t+1}(k) = \mathbf{n}_t(k)$ unless $\mathbf{n}_t(k-1) \geq 2$, in which case $\mathbf{n}_{t+1}(k) = \mathbf{n}_t(k) + 1$.

Thus the possible changes in the value of $\mathbf{n}(k)$ in one step of the dynamics, say from t to $t + 1$, may be summarized as

$$0 \rightarrow 1 \leftrightarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow \dots \quad (1.1)$$

The indicated increases occur if and only if $\mathbf{n}_t(k-1) \geq 2$, and the decreases occur if and only if $\mathbf{n}_t(k-1) \leq 1$.

Suppose now that $\hat{\mu}$ is a TIS state for the TASM, with density $\hat{\rho}$. The discussion above then implies that at most three types of stack configurations can occur with non-zero probability in $\hat{\mu}$: (a) all stacks have height 0 or 1; (b) all stacks have height 1 or 2; (c) all stacks have height 2 or more. This follows from the observation that any stack of size 0 or 1 will increase if it is preceded by a stack of size 2 or more and any stack of size 3 or more will decrease if it is preceded by a stack of size 0 or 1, while according to (1.1) no new stacks of size 0 or of size 3 or more can be formed.

Now, given $\hat{\mu}$ as above, let μ denote the corresponding TIS state of the F-TASEP and let ρ be the particle density in μ . The possibilities discussed above then become: (a) no two adjacent sites are occupied by particles; (b) no two consecutive sites are empty and no three consecutive sites are occupied; (c) no two sites which are adjacent or at a distance of 2 from each other are both empty. It is now obvious that case (a) occurs when $\rho \leq 1/2$ ($\hat{\rho} \leq 1$), (b) when $1/2 \leq \rho \leq 2/3$ ($1 \leq \hat{\rho} \leq 2$), and (c) when $\rho \geq 2/3$ ($\hat{\rho} \geq 2$). At density $\rho = 1/2$ there is a unique TIS state of the F-TASEP, the superposition of the two (stationary but not TI) states concentrated on the two periodic configurations with alternating 1's and 0's, while in the TASM there is a unique TIS state, with density $\hat{\rho} = 1$ and $\mathbf{n}(j) = 1$ for all j . Similarly at $\rho = 2/3$ the F-TASEP has a unique TIS state, the superposition of three states which are periodic with repeating unit 110, and the TASM at $\hat{\rho} = 2$ has a unique TIS state, with $\mathbf{n}(j) = 2$ for all j .

We will refer to the regions $0 < \rho < 1/2$, $1/2 < \rho < 2/3$, and $2/3 < \rho < 1$ as the low, intermediate, and high density regions, respectively. In these regions the dynamics of the F-TASEP takes, with probability one in any stationary state, a simple form: in the low density region, configurations do not change with time; at intermediate density, configurations translate two sites to the right at each time step; and at high density, configurations translate one site to the left at each time step. The corresponding result for the stack model is that configurations in the low or high density region are fixed under the dynamics, and those in the intermediate region translate one site to the right at each time step.

By considering sites occupied by particles as empty, and empty sites as occupied by cars, we can think of the F-TASEP as a traffic model in which cars advance synchronously whenever there are at least two empty spaces ahead of them. This is a special case of the family of traffic models considered in [10], obtained by taking $\alpha = \delta = 1$ and $\beta = \gamma = 0$ in the classification of table 1 of that reference. The three phases found above correspond then to three patterns of traffic flow; the phase boundaries for our case were determined, and the phases described, in [10]. Our high density region, $\rho > 2/3$, now becomes a low density car region, $\tilde{\rho} = 1 - \rho < 1/3$, in which traffic flows freely, with each car moving at the maximal possible velocity one. Our low density region, $\rho < 1/2$, becomes a jammed region, with $\tilde{\rho} > 1/2$, in which there is no motion. Finally, in our intermediate density region stretches of jammed traffic, in which each car sees only one empty site in front of it and therefore cannot move, alternate with stretches in which each car sees two empty spaces and hence moves with velocity one.

We now turn to a more detailed discussion of the behavior in each region of density. As indicated above, we are particularly interested in studying the distribution in the limit $t \rightarrow \infty$ when the initial configuration has a Bernoulli distribution. The existence and uniqueness of such a final distribution, for $\rho \notin \{1/2, 2/3\}$, is easy to show; the analysis for $\rho = 1/2$ and $\rho = 2/3$ is more complicated. See [9].

2. The low density region

We now consider the case in which the initial configuration η_0 is distributed according to the product Bernoulli measure $\mu^{(\rho)}$, with $0 < \rho < 1/2$, and want to find the final (stationary) distribution of η . We will denote expectation values in the final measure by $\langle \cdot \rangle$. We have already observed that this measure will be supported on trapped configurations with no adjacent occupied sites, and what we want to know is the probability distribution of these configurations. An important observation [9] is that almost every initial configuration η_0 in fact evolves to a final configuration η_∞ .

A little thought shows that any two consecutive sites which at some time during the evolution are not both empty will then be nonempty at all later times, so that two consecutive empty sites in any final configuration η_∞ must have been empty at all times. This means that no particle has departed from these sites during the evolution and so the distribution of η_∞ to the right of a double zero is independent of what preceded, i.e. what follows is the same as if we started with a semi-infinite Bernoulli measure at density ρ . The stationary state is thus given by a renewal process [14], and

all we need to know for its complete description is the probability that a double zero is followed by exactly n pairs 10 (and then an additional 0). This probability is given by $c_n \rho^n (1 - \rho)^{n+1}$, where

$$c_n = \frac{1}{n+1} \binom{2n}{n}, \quad n = 0, 1, 2, \dots, \tag{2.1}$$

is the n th Catalan number [15]; c_n counts the number of strings of n 0's and n 1's in which the number of 0's in any initial segment does not exceed the number of 1's. We remark that the density of double zeros is given by $\langle (1 - \eta(j))(1 - \eta(j+1)) \rangle = 1 - 2\rho$, since $\langle \eta(j)\eta(j+1) \rangle = 0$; this will be used in (2.3) below.

The structure of the TIS state for $\rho \leq 1/2$, that is, $\hat{\rho} \leq 1$, is particularly simple for the TASM: $\mathbf{n}(i)$ can take only values 0 and 1, and a double zero in the F-TASEP corresponds to a zero in the TASM. We may thus write $\zeta(i) = 1 - \mathbf{n}(i)$ and view $\zeta \in \{0, 1\}^{\mathbb{Z}}$ as a lattice gas configuration. Then given that $\zeta(j) = 1$, the probability that the next particle to the right of j will be at $j + k$ ($k > 0$) is independent of j and of $\zeta(j')$, $j' < j$, and is given by $c_{k-1} \rho^{k-1} (1 - \rho)^k$.

By calculating the generating function of the pair correlation function $g(k) = \langle \eta(j)\eta(j+k) \rangle$ in the low density TIS state of the F-TASEP, which may be obtained from the generating function of the distance between two successive double zeros, i.e. of the Catalan numbers, we discover an unexpected property of this state: for all $n \geq 0$,

$$g(2n+1) + g(2n+2) = 2\rho^2. \tag{2.2}$$

Alternatively, this can be obtained from another surprising property of the stationary state: in the distribution $\langle \cdot \rangle_{00}$ defined by conditioning on the event that $\eta(-1) = \eta(0) = 0$, i.e. in the stationary state for the semi-infinite system mentioned above, $\langle \eta(2n+1) \rangle_{00} = \rho$ for any $n \geq 0$, that is, the probability for site $2n+1$ to be occupied is always ρ . This is clearly the case for $n=0$, and a simple but rather long inductive analysis verifies it in general. But then for any $n \geq 0$,

$$\begin{aligned} \rho &= \langle \eta(2n+1) \rangle_{00} \\ &= \frac{1}{1-2\rho} \langle (1 - \eta(-1))(1 - \eta(0))\eta(2n+1) \rangle \\ &= \frac{1}{1-2\rho} (\rho - g(2n+1) - g(2n+2)), \end{aligned} \tag{2.3}$$

and this implies (2.2)

We can also show that the generating function of the truncated two point function $g^T(k) := g(k) - \rho^2$ has radius of convergence $\alpha = (4\rho(1 - \rho))^{-1/2} > 1$, so that $g^T(k)$ decays as α^{-k} , up to possible prefactors k^β . From (2.2) it now follows that, in the stationary state, the asymptotic value of V_L/L , where V_L is the variance of the number of particles in an interval of length L , has the same value as for the initial Bernoulli measure: by a simple argument,

$$\frac{V_L}{L} = \rho - \rho^2 + \sum_{k=1}^{\infty} g^T(k) + \frac{o(L)}{L}, \tag{2.4}$$

and (2.2) implies that

$$\sum_{k=1}^n g^T(k) = \begin{cases} g^T(n), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even,} \end{cases} \longrightarrow 0. \quad (2.5)$$

Hence $v := \lim_{L \rightarrow \infty} V_L/L = \rho(1 - \rho)$ for $\rho < 1/2$.

At $\rho = 1/2$, on the other hand, V_L is of order 1 in the TIS state, since the system is ordered (as $\dots 101010\dots$) in one of two possible configurations. We thus have a sharp transition in v . For a finite but large L there is a transition region, determined numerically to have width $L^{-1/2}$; see figure 1. A very similar phenomenon occurs when $\rho = 2/3$.

There is a different way to see that $v = \rho(1 - \rho)$ and that in fact this relation holds for all times, not just in the final state. For each particle lies initially in an interval between two double zeros which will persist into the stationary state, and the distance the particle can travel from $t = 0$ to $t = \infty$ is bounded by the length of this interval, which has expected value $1/(1 - 2\rho)$. Thus the number of particles in an interval of large length L will, at any time t , differ little from the number at $t = 0$. We can in fact compute exactly the distribution of the distance d that a typical particle travels from its initial to its final position:

$$\text{Prob}(d = k) = \frac{1 - 2\rho}{1 - \rho} \left(\frac{\rho}{1 - \rho} \right)^k. \quad (2.6)$$

The average distance moved is $\rho/(1 - 2\rho)$. Note that as $\rho \nearrow 1/2$ this expected distance diverges. This is so because at $\rho = 1/2$ the infinite system never becomes fixed, but rather oscillates (locally) between the two stationary configurations.

3. The high density region

As shown earlier, the only configurations possible in the stationary state at high densities are those in which there are no adjacent zeros or zeros separated by one occupied site. In terms of the stack model this corresponds to all sites being occupied by two or more particles: $\mathbf{n}(k) \geq 2$ for all k .

To bring out the parallels between the behaviors of the model in the low and high density regions it is convenient here to work in a moving frame. Thus in this section we consider a modified F-TASEP dynamics for which at each time step one first executes the F-TASEP rule, then adds a translation by one lattice site to the right, so that the possible limiting configurations described above are stationary under the new dynamics. No change is necessary for the TASM dynamics. Under the modified F-TASEP dynamics almost every initial configuration η_0 for the measure $\mu^{(\rho)}$ will in fact evolve to a final configuration η_∞ , and the same conclusion holds for the TASM.

The TIS state obtained from a high-density initial Bernoulli distribution has properties similar to those of the low density TIS state. In particular, the role of double zeros in defining a renewal process in the low density state is now played by blocks of three consecutive occupied sites. Since such a block may be followed by additional

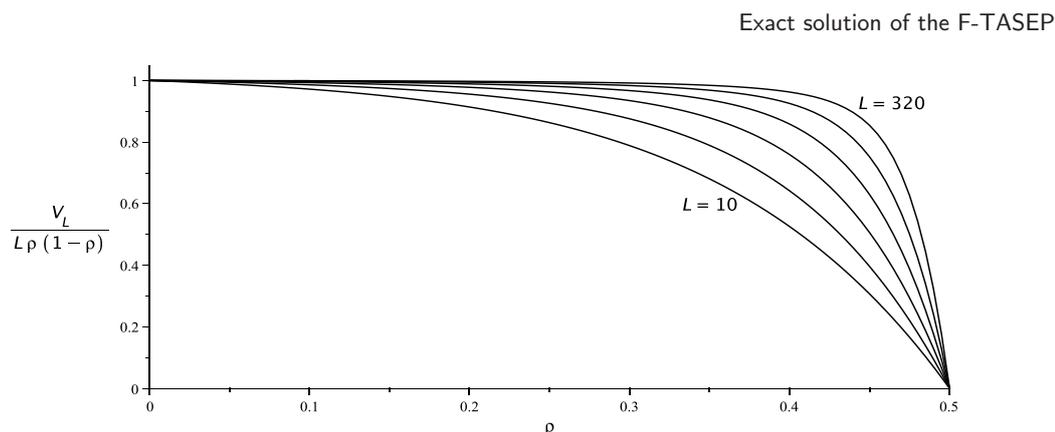


Figure 1. The normalized variance per unit length versus ρ , for $L = 10, 20, 40, 80, 160,$ and 320 .

occupied sites, it corresponds in the TASM to a site k with $\mathbf{n}(k) \geq 3$ in the final configuration, and it follows from (1.1) that such a site must have contained three or more particles at all times. Hence it must have sent one particle to its right at each time step, $t = 0, 1, 2, \dots$, so what happens to its right is independent of what is to its left (given that what happens to its left is such that it is such a site). This implies a similar statement in the F-TASEP with modified dynamics, and thus the final state is again obtained from a renewal process, as at low density, albeit a more complicated one: configurations in the final state have the form

$$\eta_\infty = \dots (011)^{n_1} 1 (011)^{n_2} 1 \dots, \tag{3.1}$$

where the n_k are independent with a computable distribution.

With this in hand we can obtain various properties of the stationary state which are analogous to those obtained in the low density region. For example: (a) if we condition on sites $-2, -1,$ and 0 being occupied then $\langle \eta(3n+1) \rangle = \rho$ for any $n \geq 0$, (b) $g(3n+1) + g(3n+2) + g(3n+3) = 3\rho^2$ for any $n \geq 1$ (compare (2.2)), and (c) $v = \rho(1 - \rho)$.

4. The intermediate density region

The TIS states in the intermediate region have a more complex structure than those in the high and low density regions. The configurations permitted in such states are those which do not contain any double zeros or any blocks of consecutively occupied sites of length greater than two; equivalently, these configurations are constructed from an ‘alphabet’ consisting of the strings 01 and 011 , that is, have the form

$$\dots (01)^{n_k} (011)^{m_k} (01)^{n_{k+1}} (011)^{m_{k+1}} \dots \tag{4.1}$$

In the TASM language these become configurations in which all sites are occupied by either one or two particles. To make configurations in the stationary measure actually be stationary we shift each configuration, after carrying out the prescribed forward jumps, to the left: by two sites in the F-TASEP and one site in the TASM. Under the modified F-TASEP dynamics almost every initial configuration η_0 for the measure $\mu^{(\rho)}$

will in fact evolve to a final configuration η_∞ , and the same conclusion holds for the TASM.

When the F-TASEP is started in an initial Bernoulli measure the final state is no longer a renewal process, at least so far as we can tell. In this case we cannot compute either the distribution of the variables n_k and m_k of (4.1) or the pair correlation function. However, numerical investigations show convincingly (but not rigorously) that in parallel with (2.2),

$$\sum_{k=1}^6 g(6n+k) = 6\rho^2. \quad (4.2)$$

This in turn implies, if supplemented by a mild assumption of decay of g (after truncation), that the variance of the number of particles in a large box will be, to leading order, the same as in the Bernoulli measure: $v = \rho(1 - \rho)$.

5. Further remarks

1. The proofs of many of the results stated here make use of a height or interface representation of the configurations [9]: for a configuration η the height $h(j)$ at site j is specified, up to an overall additive constant, by

$$h(j) - h(j-1) = (-1)^{\eta(j)}. \quad (5.1)$$

One can then define a dynamics for h which yields the F-TASEP dynamics for the corresponding particle configuration and is such that h increases (or stays the same) with time. When this dynamics is modified in the high and intermediate density regions, in parallel with the particle dynamics, $h(j)$ has a $t \rightarrow \infty$ limit in all density regions.

2. We can consider the continuous time F-TASEP in which particles jump, independently and at random times, to their right, provided they have an occupied site to their left and an empty site to their right. The TIS state for $\rho \leq 1/2$ (started from $\mu^{(\rho)}$) is the same as for the F-TASEP. We can also consider the continuous time facilitated *symmetric* exclusion, or CLG, model. Here too the lower density TIS state ($\rho < 1/2$) is a trapped one [13]. The same argument as before shows that the double zeros in the TIS state form a renewal process. Unlike the F-TASEP, however, these continuous time systems have no transition at $\rho = 2/3$, or indeed at any value $\rho > 1/2$.
3. Define the F_k -TASEP by requiring that a particle have k adjacent particles to its left before it can jump. Then from the analogue of (1.1),

$$0 \rightarrow 1 \rightarrow \dots \rightarrow k \rightleftharpoons k+1 \leftarrow k+2 \leftarrow k+3 \leftarrow \dots, \quad (5.2)$$

there will again be three classes of TIS states: (a) the low density phase $0 < \rho < k/(k+1)$, (b) the intermediate density phase $k/(k+1) < \rho < (k+1)/(k+2)$, and (c) the high density phase $(k+1)/(k+2) < \rho < 1$. The corresponding k -TASM will have $\mathbf{n}(j) \leq k$ in the low density region, $\mathbf{n}(j) = k$ or $k+1$ in the intermediate density region, and $\mathbf{n}(j) \geq k+1$ in the high density region. We expect that if started from a Bernoulli measure at density ρ the system will approach a stationary state.

4. One can show [9] that for any initial configuration η_0 with density $\rho < 1/2$ there is a final configuration η_∞ such that $\eta_t \rightarrow \eta_\infty$ as $t \rightarrow \infty$, and that the map $\eta_0 \mapsto \eta_\infty$ commutes with translations. The existence of such a deterministic mapping from initial to final configurations immediately implies that if the system is started in a TI initial state which is ergodic, or even mixing, then the final state will be ergodic or mixing, respectively. Moreover, if the initial state is the Bernoulli measure $\mu^{(\rho)}$ then the final state will in fact be isomorphic to a Bernoulli shift, though one of lower entropy than $\mu^{(\rho)}$ [16]. The same conclusions hold in the intermediate or high density regions (but not at $\rho = 1/2$ or $\rho = 2/3$) if one adopts the corresponding modified dynamics discussed above.
5. Our analysis in this note was confined mostly to translation invariant states. Suppose however that we start with an initial state which differs from the Bernoulli product state in some fixed region $[-M, M]$; e.g. we might use there a product measure with a different density. Then, contrary to what might be expected, at low density the resulting stationary state will remain non-translation invariant. At intermediate and higher densities the limiting states will be TI, with the localized perturbation transported away to infinity.

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References

- [1] Marro J and Dickman R 1999 *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge: Cambridge University Press)
- [2] Rossi M, Pastor-Satorras R and Vespignani A 2000 Universality class of absorbing phase transitions with a conserved field *Phys. Rev. Lett.* **85** 1803
- [3] Hexner D and Levine D 2015 Hyperuniformity of critical absorbing states *Phys. Rev. Lett.* **114** 110602
- [4] Martiniani S, Chaikin P M and Levine D 2019 Quantifying hidden order out of equilibrium *Phys. Rev. X* **9** 011031
- [5] Sidoravicius V and Teixeira A 2017 Absorbing-state transition for stochastic sandpiles and activated random walks *Electron. J. Probab.* **22** 1–35
- [6] Oliveira M J 2005 Conserved lattice gas model with infinitely many absorbing states in one dimension *Phys. Rev. E* **71** 016112
- [7] Gabel A, Krapivsky P L and Redner S 2010 Facilitated asymmetric exclusion *Phys. Rev. Lett.* **105** 210603
- [8] Baik J, Barraquand G, Corwin I and Suidan T 2016 Facilitated exclusion process *Proc. of the 2016 Abel Symp.*
- [9] Goldstein S, Lebowitz J L and Speer E R The discrete-time facilitated TASEP unpublished

- [10] Gray L and Griffeath D 2001 The ergodic theory of traffic jams *J. Stat. Phys.* **105** 413–52
- [11] Levine E, Ziv G, Gray L and Mukamel D 2004 Phase transitions in traffic models *J. Stat. Phys.* **117** 819–30
- [12] Evans M R and Hanney T 2005 Nonequilibrium statistical mechanics of the zero-range process and related models *J. Phys. A: Math. Gen.* **38** R195–240
- [13] Goldstein S, Lebowitz J L and Speer E R The one-dimensional conserved lattice gas unpublished
- [14] Feller W 1971 *An Introduction to Probability Theory and its Applications* 2nd edn (New York: Wiley)
- [15] Roman S 2015 *An Introduction to Catalan Numbers* (New York: Birkhäuser)
- [16] Ornstein D S 1974 *Ergodic Theory, Randomness, and Dynamical Systems* (New Haven, CT: Yale University Press)