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Aging in Spin Glass Models on Intermediate Time Scale: Universality of the Trap Model

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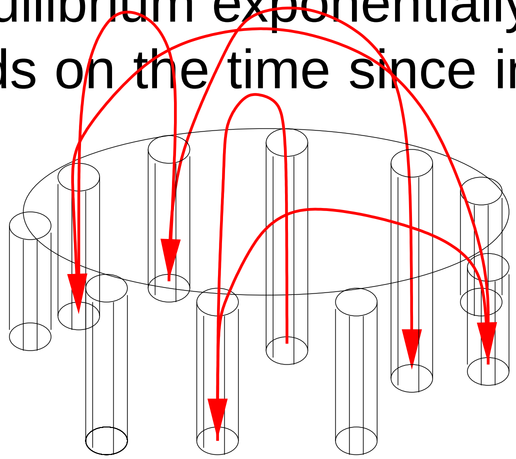
General remarks about ageing

Ageing refers to systems that do not converge to equilibrium exponentially fast. More specifically, the "speed" of the process depends on the time since initialisation.



Paradigmatic Example: REM-like trap model:
Continuous time Markov chain on $\mathcal{S} \equiv \{1, \dots, N\}$
transition rates $c_{i,j} = 1/\tau_i/N$, for $i \neq j$, x_i independent random variables with $\mathbb{P}[\tau_i > t] \sim t^{-\alpha}$, $\alpha < 1$.
Correlation function:

$$\Pi_N(t, t_w) \equiv \mathcal{P}_N\left(X_N(s) = X_N(t_w),\ \forall s \in [t_w, t_w + t]\right).$$

Ageing: (Bouchaud-Dean) For almost all $\underline{x}$, and for all $\theta > 0$,

$$\lim_{t_w \uparrow \infty} \lim_{N \uparrow \infty} \Pi_N(\theta t_w, t_w) = \frac{\sin(\pi\alpha)}{\pi} \int_{\frac{\theta}{1+\theta}}^{1} u^{-\alpha}(1-u)^{\alpha-1} du$$

This is a somewhat generic behaviour for many ageing systems.

Much of the current work on ageing is motivated by glasses and spin-glasses, where ageing is observed experimentally.

Systems we would like to understand involve:

- ▷ A random Hamiltonian (“energy landscape”), $H_N(\sigma)$, on configuration space Σ_N (e.g. $\{-1, 1\}^N$). E.g., H_N could be a Gaussian random process on Σ_N.
- ▷ A Markov process, $\sigma(t)$, on Σ_N with invariant measure $\mu_N(\sigma) \sim \exp(-\beta H_N(\sigma))$. (Usually with transition rates $p_N(\sigma, \sigma')$ that allow only changes of one coordinate of σ at a time).

There are essentially no rigorous results on ageing in these systems.

Trap models

One would like to describe the dynamics on that time scale by an effective dynamics on a reduced state space.

Key idea is that the process will spend most of its time in the deepest minima of H_N which it has explored by that time.

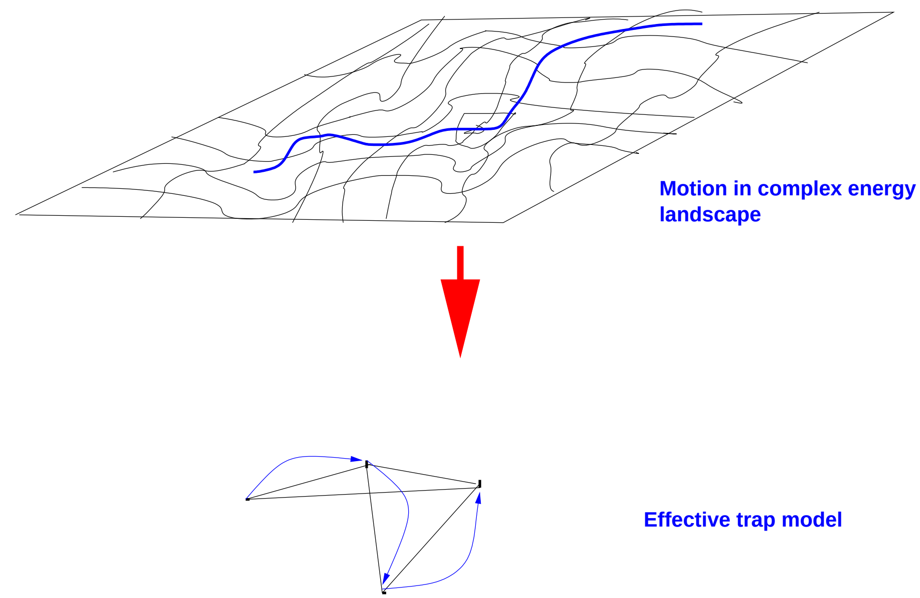
This leads to the heuristic approximation the dynamics in terms of

Trap Models (Bouchaud, Dean, ...):

$\mathcal{G} = (\mathcal{S}, \mathcal{E})$ graph.

$\underline{\tau} \equiv \{\tau_i, \, i \in \mathcal{S}\}$ iid α-stable r.v. with parameter $0 < \alpha < 1$.

$X(t)$ continuous–time random walk on $\mathcal{G}$ with transition rates, $c_{i,j}$, reversible with respect to $\underline{\tau}$.



Previous work

Most rigorous work on ageing can be put into three categories:

Analyse the behaviour of various trap models defined on different graphs:

- ▷ Complete graph (REM-like model): Bouchaud-Dean 92, B-Faggionato 05
- ▷ $\mathbb{Z}$: Fontes-Isopi-Newman 02, Ben Arous-Černý-Mountford 04
- ▷ $\mathbb{Z}^2$: Ben Arous-Černý-Mountford 04
- ▷ $\mathbb{Z}^d$: Černý 04, Černý-Ben Arous 05

Justify the trap model approximation from original Glauber dynamics in different models and at different time scales:

- ▷ REM (H_N iid Gaussian) almost equilibrium time scale: Ben Arous, B, Gayrard
- ▷ REM shorter time scales $O(\exp(\gamma N))$: Ben Arous-Černý 05
- ▷ p-spin models time scales $O(\exp(\gamma N))$: Ben Arous-B-Černý

Definition of our models

- ▷ **State space** $\Sigma_N \equiv \{-1,1\}^N$ equipped with **overlap** $R_N(\sigma,\tau) = N^{-1}\sum_{i=1}^N \sigma_i\tau_i$
- ▷ **Random environnment: A Gaussian process** H_N on Σ_N with covariance cov$(H_N(\sigma), H_N(\tau)) = NR_N(\sigma,\tau)^p$, $p \in \mathbb{N}$. Sherrington-Kirkpatrick model!
- ▷ The unbiased **Simple Random Walk (SRW)**, $Y_N(k)$, on Σ_N.
- ▷ A **clock process** $S_N(k)$

$$S_N(k) \equiv \sum_{i=1}^{k} e_i \exp\left(-\beta H_N(Y_N(k))\right)$$

where $e_i, i \in \mathbb{N}$ are iid exponential random variables with parameter 1.

Then the **process**

$$\sigma_N(t) \equiv Y_N\left(S_N^{-1}(t)\right)$$

is a continuous time Markov chain with unique reversible measure

$$\mu_{\beta,N}(\sigma) \equiv Z_{\beta,N}^{-1} \exp\left(-\beta H_N(\sigma)\right).$$

Main result

***Theorem* 1.** *There exists a function* $\zeta(p)$ *such that for all* $p \geq 3$ *and* γ *satisfying*

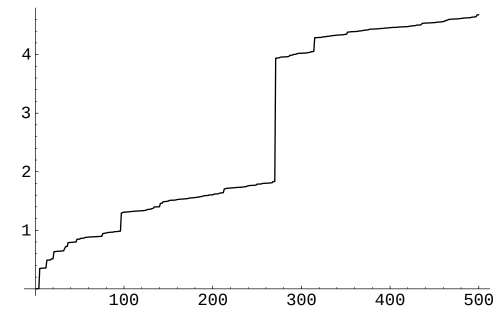
$$0 < \gamma < \min\left(\beta\zeta(p), \beta^2\right)$$

there exists $K > 0$ *such that*

$$\bar{S}_N(t) \equiv \frac{1}{e^{\gamma N}} S_N(tN^{1/2} e^{N\gamma^2/2\beta^2}) \xrightarrow{N\to\infty} V_{\gamma/\beta^2}(t) \qquad Y_N - \text{ a.s.,}$$

where V_α *is a* α*-stable subordinator.*
$\zeta(p)$ *is increasing and it satisfies*

$$\zeta(3) \simeq 1.0291 \qquad \text{and} \qquad \lim_{p\to\infty} \zeta(p) = \sqrt{2\log 2}.$$



Correlation functions

The convergence of the clock process together with the properties of SRW implies (with some extra work) ageing results for suitable correlation functions.

$$C^{\epsilon}_{N,\gamma}(t,t_w) \equiv \mathbb{P}\left[R_N\left(\sigma_N\left(t_w e^{\gamma N}\right), \sigma_N\left((t_w+t)e^{\gamma N}\right)\right) \geq 1-\epsilon\right)$$

Theorem 2. *There exists a function $\zeta(p)$ such that for all $p \geq 3$, γ satisfying*

$$0 < \gamma < \min\left(\beta\zeta(p), \beta^2\right),$$

and all $1 > \epsilon > 0$, with $\equiv \gamma/\beta^2$,

$$\begin{aligned} \lim_{N\uparrow\infty} C_{N,\gamma}(t_w, t_w+t) &= \frac{\sin(\pi\alpha)}{\pi}\int_{\frac{\theta}{1+\theta}}^{1} u^{-\alpha}(1-u)^{\alpha-1}du \\ &= \mathbb{P}\left((t_w, t_w+t) \cap \textit{range}(V_{\gamma/\beta^2}) = \emptyset\right] \end{aligned}$$

Heuristic picture

Time scale: $e^{N\gamma}$
Number of steps of SRW: $e^{N\gamma^2/2\beta^2} \ll \max\left(e^{\gamma N}, 2^N\right)$.
If H_N was iid, then

$$\max_{k \leq e^{N\gamma^2/2\beta^2}} H_N(\sigma) \sim N\gamma/\beta$$

Thus: *time spent in sites of max H_N: $O\left(e^{N\gamma}\right)$*
(i.e.: process spends almost all of its time in “few” sites).
The rescaled clock process records $e^{-\gamma N}\times$ time spend in those sites. Due to Poisson convergence of extremes, this yields precisely the γ/β^2-stable subordinator.

The assertion of the theorem says that in the correlated case, this picture essentially survives! Proof in correlated case requires

- ▷ Control on the properties of SRW
- ▷ Control of extremes on SRW trajectories

For a given SRW trajectory, consider the Gaussian process

$$X_N^0(k) \equiv H_N(Y_N(k))$$

as a Gaussian process on $\mathbb{N}$ with covariance

$$\text{cov}(X_N^0(k), X_N^0(\ell)) = \Lambda_{k\ell}^0 = \left(1 - 2N^{-1}\,\text{dist}(Y_N(k), Y_N(\ell))\right)^p$$

The key is to compare this process to the simpler process $X_N^1(k)$ defined as follows: Set $\nu = N^\omega$, $\omega \in (1/2, 1)$. Define

$$\text{cov}(X_N^1(k), X_N^1(\ell)) = \Lambda_{k\ell}^1 = \begin{cases} 1 - 2p|k-\ell|/N, & \text{if } [k/\nu] = [\ell/\nu], \\ 0, , & \text{else} \end{cases}$$

To characterise the law of process $\widetilde{S}_N(t)$, we use the Laplace transforms of the finite-dimensional marginals.
Define for any process $X(k)$, $k \in \mathbb{N}$, and $\boldsymbol{t} \equiv (t_1, \dots, t_n)$, $\boldsymbol{u} \equiv (u_1, \dots, u_n)$

$$F_N(X; \boldsymbol{t}, \boldsymbol{u}) \equiv \mathbb{E}\left[\exp\left(-\sum_{k=1}^{n} \frac{u_k}{e^{\gamma N}} \sum_{i=t_{k-1}r(N)}^{t_k r(N)-1} e_i e^{\beta\sqrt{N}X(i)}\right)\right]$$

(where the depence on X is only through the law of X).

Step 1: For the comparison process X^1, through explicite Gaussian computations and using indepedence over blocks of size ν,

$$\lim_{N\uparrow\infty} F_N(X^1; \boldsymbol{t}, \boldsymbol{u}) = \exp\left(-K \sum_{k=1}^{n} (t_k - t_{k-1}) u_k^{\gamma/\beta^2}\right)$$

which is the Laplace transform of a γ/β^2-stable subordinator, if $\gamma < \beta^2$.

Step 2: Show that, if $p \geq 3$,

$$\lim_{N\uparrow\infty} \Big| F_N(X^2; \boldsymbol{t}, \boldsymbol{u}) - F_N(X^1; \boldsymbol{t}, \boldsymbol{u}) \Big| = 0,$$

for almost every realisation of the SRW.

This uses the well-known interpolation formula

$$\mathbb{F}_N(X_N^0) - F_N(X_N^1) = \frac{1}{2} \int_0^1 dh \sum_{\substack{i,j=1 \\ i \neq j}}^{tr(N)} (\Lambda_{ij}^0 - \Lambda_{ij}^1) \mathbb{E}\Big[\frac{\partial^2 F_N(X_N^h)}{\partial X(i) \partial X(j)} \Big].$$

where $X_N^h = \sqrt{1-h} X_N^0 + \sqrt{h} X_N^1$.

Conclusions

Nice features:

- ▷ We have established ageing governed by convergence to α-stable subordinators in a wide range of spin glasses and for a wide range of time scales. This confirms the relevance of the REM like trap model.
- ▷ We could analyse the dynamics of our models without understanding equilibrium properties.
- ▷ These are the first results for seríously correlated spin glass models!

Drawbacks and future challenges:

- ▷ Can treat only the simple random time change dynamics (even in the REM!)
- ▷ $p = 2$ is open!
- ▷ What about longer time scales?

SRW properties

We must consider SRW on the hypercupe at time scales $e^{\rho N} \equiv r(N)$ with $\rho < \sqrt{2 \ln 2}$. On this scale, it is *extremely transient*.

- For time times $k \leq \epsilon N$, SRW is ballistic, i.e.

$$\operatorname{dist}(Y_N(j), Y_N(j+k)) \sim k$$

  with large probability.

- SRW never returns near its starting point, i.e. for all $\rho < \sqrt{2 \ln 2}$, there is $\delta > 0$, s.t.

$$\min_{\epsilon N \leq k \leq e^{\rho N}} \operatorname{dist}(Y_N(j), Y_N(j+k)) > \delta N$$

  with large probability.

- SRW is exponentially close to equilibrium after time $KN^2 \ln N$, i.e. for all $k \geq KN^2 \ln N$,

$$\left| \frac{P_y[Y_N(k) = x \vee Y_N(k+1) = x]}{2} - 2^{-N} \right| \leq 2^{-8N}$$