Geometric scale free graphs

Itai Benjamini, Ori Gurel-Gurevich and
Gady Kozma (speaker)

101^{st} statistical mechanics conference, 2009

Notion

There is no rigorous definition of a scale-free graph.

Notion

There is no rigorous definition of a scale-free graph.
People are interested in these properties

- The degree distribution has polynomial tail i.e.

$$\frac{\#\{\text{vertices } v \text{ with degree } k\}}{\#\text{vertices}} \sim k^{-\gamma}\,.$$

Notion

There is no rigorous definition of a scale-free graph.
People are interested in these properties

- The degree distribution has polynomial tail i.e.

$$\frac{\#\{\text{vertices } v \text{ with degree } k\}}{\#\text{vertices}} \sim k^{-\gamma} \,.$$

- The graph has small diameter with respect to its size (for example, polylogarithmic).

Notion

There is no rigorous definition of a scale-free graph.
People are interested in these properties

- The degree distribution has polynomial tail i.e.

$$\frac{\#\{\text{vertices } v \text{ with degree } k\}}{\#\text{vertices}} \sim k^{-\gamma}\,.$$

- The graph has small diameter with respect to its size (for example, polylogarithmic).
- The graph is random.

Notion

There is no rigorous definition of a scale-free graph. People are interested in these properties

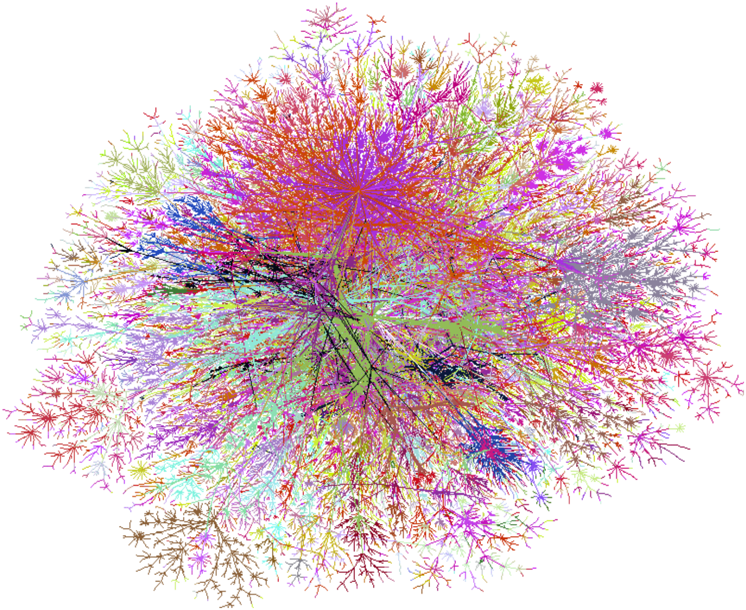
- The degree distribution has polynomial tail i.e.

$$\frac{\#\{\text{vertices } v \text{ with degree } k\}}{\#\text{vertices}} \sim k^{-\gamma}.$$

- The graph has small diameter with respect to its size (for example, polylogarithmic).
- The graph is random.

Such graphs are supposed to model various real-life graphs, such as the internet, telephone call graphs, social networks, protein interaction networks and predator-prey networks.

Internet topology, Cheswick & Burch



Small world papers citation graph, Lin Freeman 2004

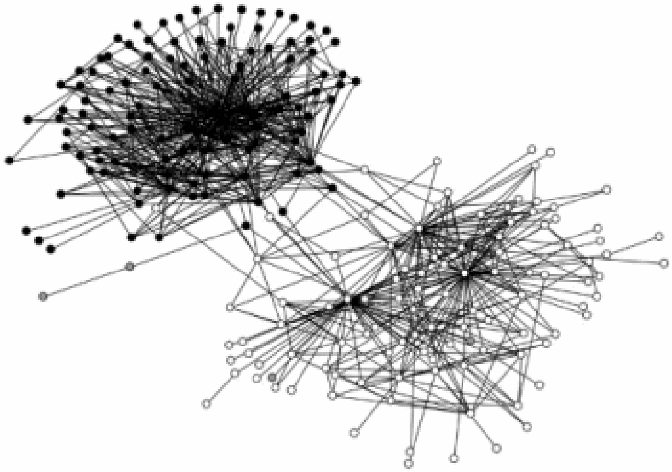
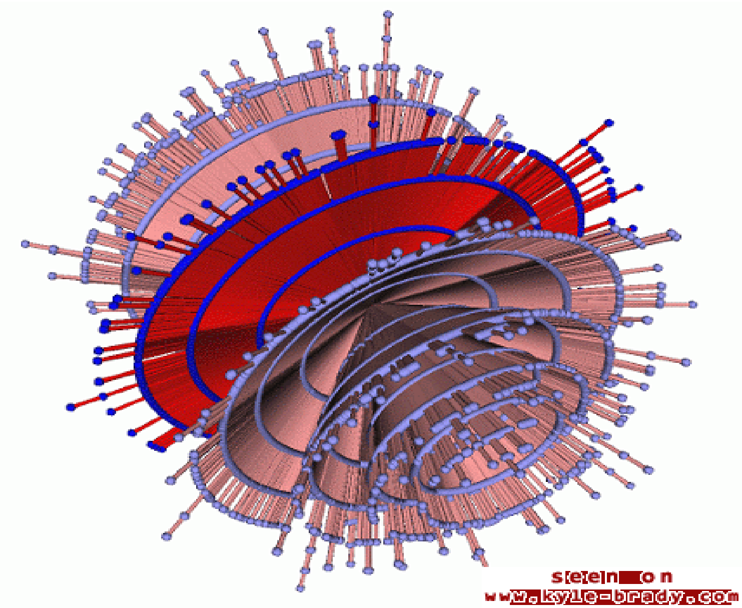
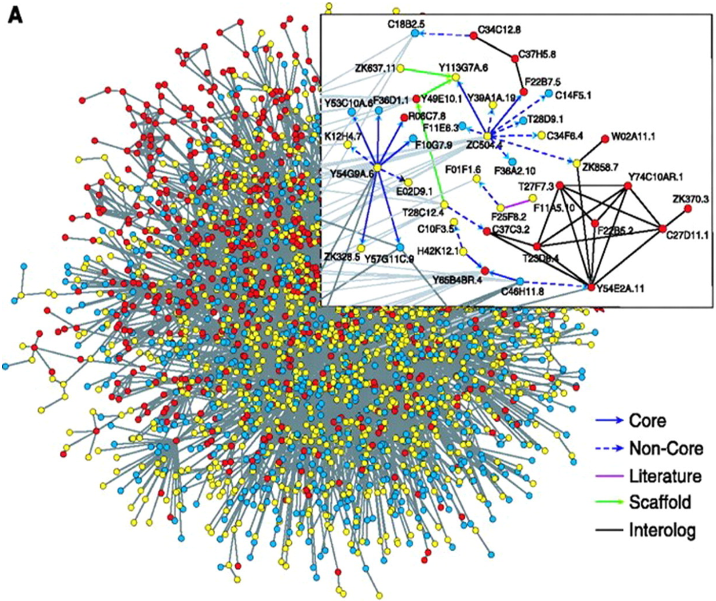


Figure 10.1. Citation patterns in the Small World literature

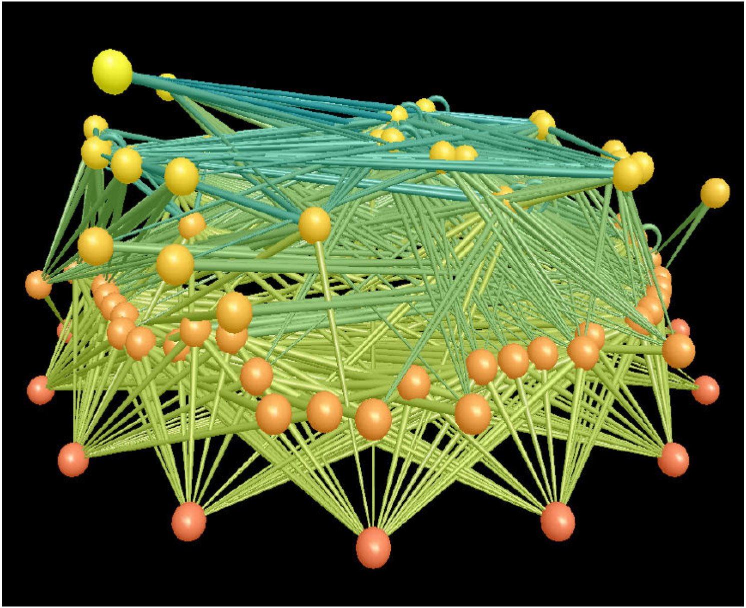
A social network



C. elegans protein interaction network, Li, et al., 2004



Predator-pray interactions, Martinez 1991.



In this talk we will discuss a new and unusual model for scale-free networks.

In this talk we will discuss a new and unusual model for scale-free networks. We do not claim it has any relevance to real life. It is meant to reflect on emerging notions of universality.

In this talk we will discuss a new and unusual model for scale-free networks. We do not claim it has any relevance to real life. It is meant to reflect on emerging notions of universality.

To understand what is "universality" we need to consider the main examples analyzed so far.

In this talk we will discuss a new and unusual model for scale-free networks. We do not claim it has any relevance to real life. It is meant to reflect on emerging notions of universality.

To understand what is “universality” we need to consider the main examples analyzed so far.

We will focus on cases which have been analyzed rigorously. We are most interested in the diameter, the spectral gap and the mixing time of random walk.

We are equally interested in finite and infinite graphs, mutatis mutandis.

Preferential attachment

- Every new vertex is connected to m random existing vertices v_i, with probability proportional to $\deg(v_i)$ (Barabási & Albert, 1999). For this model $\gamma = 3$ (Bollobás, Riordan, Spencer & Tusnády 2001).

Preferential attachment

- Every new vertex is connected to m random existing vertices v_i, with probability proportional to $\deg(v_i)$ (Barabási & Albert, 1999). For this model $\gamma = 3$ (Bollobás, Riordan, Spencer & Tusnády 2001).
- The diameter is almost surely $\approx \frac{\log n}{\log \log n}$ (Bollobás & Riordan 2004) except in the tree case $m = 1$ where it is $\log n$ (Pittel 1994).

Preferential attachment

- Every new vertex is connected to m random existing vertices v_i, with probability proportional to $\deg(v_i)$ (Barabási & Albert, 1999). For this model $\gamma = 3$ (Bollobás, Riordan, Spencer & Tusnády 2001).
- The diameter is almost surely $\approx \frac{\log n}{\log \log n}$ (Bollobás & Riordan 2004) except in the tree case $m = 1$ where it is $\log n$ (Pittel 1994).
- When $m \geq 2$ the spectral gap of the random walk is constant (Mihail, Papadimitriou & Saberi 2006).

Preferential attachment

- Every new vertex is connected to m random existing vertices v_i, with probability proportional to $\deg(v_i)$ (Barabási & Albert, 1999). For this model $\gamma = 3$ (Bollobás, Riordan, Spencer & Tusnády 2001).
- The diameter is almost surely $\approx \frac{\log n}{\log \log n}$ (Bollobás & Riordan 2004) except in the tree case $m = 1$ where it is $\log n$ (Pittel 1994).
- When $m \geq 2$ the spectral gap of the random walk is constant (Mihail, Papadimitriou & Saberi 2006).
- One can generalize and ask that the probabilities to proportional to $\deg(v_i) + \delta$ for some $\delta > -m$. In this case $\gamma = 3 + \frac{\delta}{m}$ (Cooper & Frieze 2003) so always $\gamma > 2$.

Preferential attachment

- Every new vertex is connected to m random existing vertices v_i, with probability proportional to $\deg(v_i)$ (Barabási & Albert, 1999). For this model $\gamma = 3$ (Bollobás, Riordan, Spencer & Tusnády 2001).
- The diameter is almost surely $\approx \frac{\log n}{\log \log n}$ (Bollobás & Riordan 2004) except in the tree case $m = 1$ where it is $\log n$ (Pittel 1994).
- When $m \geq 2$ the spectral gap of the random walk is constant (Mihail, Papadimitriou & Saberi 2006).
- One can generalize and ask that the probabilities to proportional to $\deg(v_i) + \delta$ for some $\delta > -m$. In this case $\gamma = 3 + \frac{\delta}{m}$ (Cooper & Frieze 2003) so always $\gamma > 2$.
- In this case, if $\gamma > 3$ then $\frac{\log n}{\log \log n} \leq \text{diam} \leq \log n$ and is conjectured to be $\log n$. If $\gamma < 3$, then $\text{diam} \leq \log \log n$ (van der Hofstad & Hooghiemstra, 2007 preprint).

The configuration model

- Pick your γ of choice. Choose degrees randomly independently with the required tail. Then choose a graph among all graphs with the desired degree sequence randomly uniformly.

The configuration model

- Pick your γ of choice. Choose degrees randomly independently with the required tail. Then choose a graph among all graphs with the desired degree sequence randomly uniformly.
- If $3 < \gamma$ then the diamter is $\log n$ (van der Hostad, Hooghiemstra & van Mieghem 2005).

The configuration model

- Pick your γ of choice. Choose degrees randomly independently with the required tail. Then choose a graph among all graphs with the desired degree sequence randomly uniformly.
- If $3 < \gamma$ then the diamter is $\log n$ (van der Hostad, Hooghiemstra & van Mieghem 2005).
- If $2 < \gamma < 3$ then the typical distance between two vertices is $\log \log n$ (van der Hofstad, Hooghiemstra & Znamenski 2007) though the diameter is $\log n$ at least when $\mathbb{P}(\deg = 1) > 0$ (Fernholz & Ramachandran 2007).

The configuration model

- Pick your γ of choice. Choose degrees randomly independently with the required tail. Then choose a graph among all graphs with the desired degree sequence randomly uniformly.
- If $3 < \gamma$ then the diamter is $\log n$ (van der Hostad, Hooghiemstra & van Mieghem 2005).
- If $2 < \gamma < 3$ then the typical distance between two vertices is $\log \log n$ (van der Hofstad, Hooghiemstra & Znamenski 2007) though the diameter is $\log n$ at least when $\mathbb{P}(\deg = 1) > 0$ (Fernholz & Ramachandran 2007).
- If $1 < \gamma < 2$ then the diameter is either 2 or 3 (van den Esker, van der Hofstad, Hooghiemstra & Znamenski 2005).

The configuration model

- Pick your γ of choice. Choose degrees randomly independently with the required tail. Then choose a graph among all graphs with the desired degree sequence randomly uniformly.
- If $3 < \gamma$ then the diamter is $\log n$ (van der Hostad, Hooghiemstra & van Mieghem 2005).
- If $2 < \gamma < 3$ then the typical distance between two vertices is $\log \log n$ (van der Hofstad, Hooghiemstra & Znamenski 2007) though the diameter is $\log n$ at least when $\mathbb{P}(\deg = 1) > 0$ (Fernholz & Ramachandran 2007).
- If $1 < \gamma < 2$ then the diameter is either 2 or 3 (van den Esker, van der Hofstad, Hooghiemstra & Znamenski 2005). This is due to having vertices with degree $>$ the total number of vertices.

Long-range percolation

- Take a grid in $\mathbb{Z}^d$. Unlike in usual percolation, $d = 1$ is very interesting.

Long-range percolation

- Take a grid in $\mathbb{Z}^d$. Unlike in usual percolation, $d = 1$ is very interesting.
- To avoid questions of connectivity and phase transition, do not remove the edges of the grid. Instead, add an edge between every x and every y with probability $(\beta + o(1))|x - y|^{-s}$, all independent.

Long-range percolation

- Take a grid in $\mathbb{Z}^d$. Unlike in usual percolation, $d = 1$ is very interesting.
- To avoid questions of connectivity and phase transition, do not remove the edges of the grid. Instead, add an edge between every x and every y with probability $(\beta + o(1))|x - y|^{-s}$, all independent.
- There is little point in talking about γ — if $s \leq d$ then all vertices have infinite degree. If $s > d$ then the degrees decay exponentially.

Long-range percolation

- Take a grid in $\mathbb{Z}^d$. Unlike in usual percolation, $d = 1$ is very interesting.
- To avoid questions of connectivity and phase transition, do not remove the edges of the grid. Instead, add an edge between every x and every y with probability $(\beta + o(1))|x - y|^{-s}$, all independent.
- There is little point in talking about γ — if $s \leq d$ then all vertices have infinite degree. If $s > d$ then the degrees decay exponentially.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004).

Long-range percolation

- Take a grid in $\mathbb{Z}^d$. Unlike in usual percolation, $d = 1$ is very interesting.
- To avoid questions of connectivity and phase transition, do not remove the edges of the grid. Instead, add an edge between every x and every y with probability $(\beta + o(1))|x - y|^{-s}$, all independent.
- There is little point in talking about γ — if $s \le d$ then all vertices have infinite degree. If $s > d$ then the degrees decay exponentially.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004). The reasons are similar to those of the configuration model case.

Long-range percolation

- Take a grid in $\mathbb{Z}^d$. Unlike in usual percolation, $d = 1$ is very interesting.
- To avoid questions of connectivity and phase transition, do not remove the edges of the grid. Instead, add an edge between every x and every y with probability $(\beta + o(1))|x - y|^{-s}$, all independent.
- There is little point in talking about γ — if $s \leq d$ then all vertices have infinite degree. If $s > d$ then the degrees decay exponentially.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004). The reasons are similar to those of the configuration model case.
- If $s = d$ then the diameter of the cube $\{1, \ldots, n\}^d$ is $\frac{\log n}{\log \log n}$ (Coppersmith, Gamarnik & Sviridenko 2002).

Long-range percolation contd.

- Put an edge between every x and every y with probability $(\beta + o(1))|x - y|^{-s}$, all independent.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004). The reasons are similar to those of the configuration model case.
- If $s = d$ then the diameter of the cube $1, \ldots, n^d$ is $\frac{\log n}{\log \log n}$ (Coppersmith, Gamarnik & Sviridenko 2002).

Long-range percolation contd.

- Put an edge between every x and every y with probability $(\beta + o(1))|x-y|^{-s}$, all independent.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004). The reasons are similar to those of the configuration model case.
- If $s = d$ then the diameter of the cube $1, \ldots, n^d$ is $\frac{\log n}{\log \log n}$ (Coppersmith, Gamarnik & Sviridenko 2002).
- If $d < s < 2d$ then the diameter is $\log^{K+o(1)} n$ where $K = \log_2(2d/s)$ (Biskup 2004). The mixing time is slow, $n^{s-1+o(1)}$ (Benjamini, Berger & Yadin 2008).

Long-range percolation contd.

- Put an edge between every x and every y with probability $(\beta + o(1))|x-y|^{-s}$, all independent.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004). The reasons are similar to those of the configuration model case.
- If $s = d$ then the diameter of the cube $1, \ldots, n^d$ is $\frac{\log n}{\log \log n}$ (Coppersmith, Gamarnik & Sviridenko 2002).
- If $d < s < 2d$ then the diameter is $\log^{K+o(1)} n$ where $K = \log_2(2d/s)$ (Biskup 2004). The mixing time is slow, $n^{s-1+o(1)}$ (Benjamini, Berger & Yadin 2008).
- In the case $s = 2d$ it is conjectured that the diameter is $n^{\theta(\beta)}$ — this is the only case where β matters. Partial results in the case $d = 1$ are in Benjamini & Berger 2001 and CGS02.

Long-range percolation contd.

- Put an edge between every x and every y with probability $(\beta + o(1))|x-y|^{-s}$, all independent.
- If $s < d$ then the diameter is $\left\lceil \frac{d}{d-s} \right\rceil$ (Benjamini, Kesten, Peres & Schramm 2004). The reasons are similar to those of the configuration model case.
- If $s = d$ then the diameter of the cube $1, \ldots, n^d$ is $\frac{\log n}{\log \log n}$ (Coppersmith, Gamarnik & Sviridenko 2002).
- If $d < s < 2d$ then the diameter is $\log^{K+o(1)} n$ where $K = \log_2(2d/s)$ (Biskup 2004). The mixing time is slow, $n^{s-1+o(1)}$ (Benjamini, Berger & Yadin 2008).
- In the case $s = 2d$ it is conjectured that the diameter is $n^{\theta(\beta)}$ — this is the only case where β matters. Partial results in the case $d = 1$ are in Benjamini & Berger 2001 and CGS02.
- If $s > 2d$ the diameter is n (BB01, $d = 1$) and the mixing time n^2 (BBY08).

Quick summary

γ	model	diam	Mixing
"1"	LRP[1], $s < d$	$\left\lceil \frac{d}{d-s} \right\rceil$	C???
"1"	LRP, $s = d$	$\frac{\log n}{\log\log n}$	
$(1, 2)$	conf.[2]	2 or 3	
$(2, 3)$	conf.	"$\log\log n$"	
$(2, 3)$	PA[3]	$\leq \log\log n$	
3	PA	$\frac{\log n}{\log\log n}$	$\leq \log n$
> 3	conf.	$\log n$	
> 3	PA	$\leq \log n$	
"∞"	LRP, $d < s < 2d$	$(\log n)^K$	$n^{s-1+o(1)}$

[1] Long-range percolation, $p_{xy} \approx |x - y|^{-s}$

[2] The configuration model

[3] Preferential attachment

Quick summary

γ	model	diam	Mixing
"1"	LRP[1], $s < d$	$\left\lceil \frac{d}{d-s} \right\rceil$	C???
"1"	LRP, $s = d$	$\frac{\log n}{\log \log n}$	
$(1, 2)$	conf.[2]	2 or 3	
$(2, 3)$	conf.	"$\log \log n$"	
$(2, 3)$	PA[3]	$\leq \log \log n$	
3	PA	$\frac{\log n}{\log \log n}$	$\leq \log n$
> 3	conf.	$\log n$	
> 3	PA	$\leq \log n$	
"∞"	LRP, $d < s < 2d$	$(\log n)^K$	$n^{s-1+o(1)}$

[1] Long-range percolation, $p_{xy} \approx |x - y|^{-s}$

[2] The configuration model

[3] Preferential attachment

It seems like some kind of weak universality is at play.

Percolation reminder

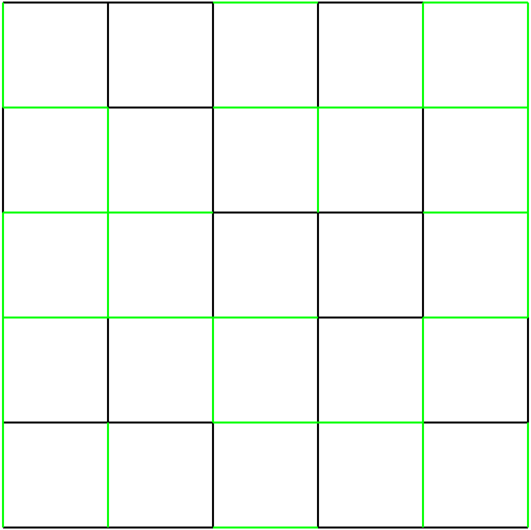
- Take the lattice $\mathbb{Z}^d$, and keep every edge with probability p, deleting it with probability $1-p$, independently. The critical p is defined by

$$p_c = \inf\{p : \exists \text{ an infinite cluster a.s.}\}\,.$$

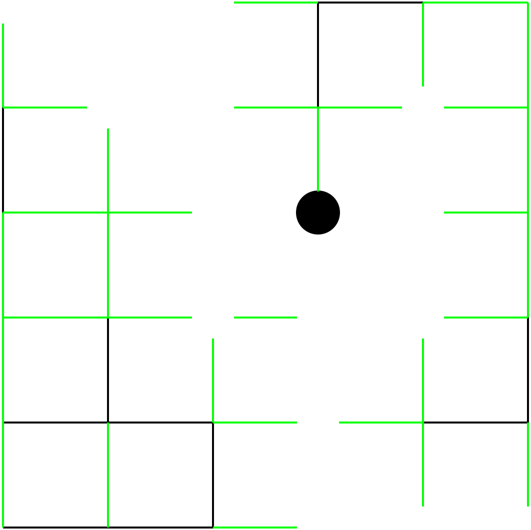
Percolation reminder

- Take the lattice $\mathbb{Z}^d$, and keep every edge with probability p, deleting it with probability $1-p$, independently. The critical p is defined by
$$p_c = \inf\{p : \exists \text{ an infinite cluster a.s.}\}\,.$$
- At p_c the model has polynomially decaying correlations. The most relevant exponent is
$$\mathbb{P}(|\mathcal{C}(\vec{0})| > n) \approx n^{-1/\delta}$$
which satisfies
$$\frac{91}{5} = \delta_2 > \delta_3 > \cdots > \delta_6 = \delta_7 = \cdots = 2\,.$$
Only the cases $d=2$ and $d>6$ have been proved rigorously, and only for some lattices. $d=2$ is due to Kesten 1987, Lawler Schramm & Werner 2001 and Smirnov 2001. $d>6$ is due to Hara & Slade 1990 and Barsky & Aizenman 1991.

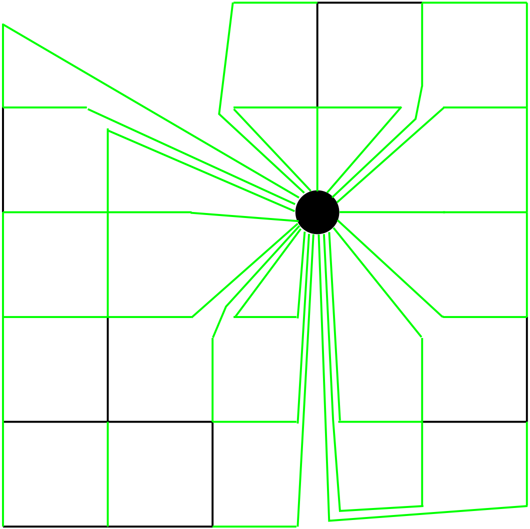
No edges are removed, edges are only colored in two colors.



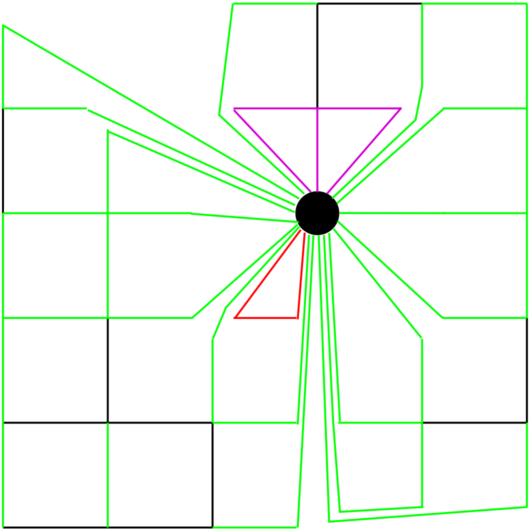
Take a black cluster and replace it with a single vertex.



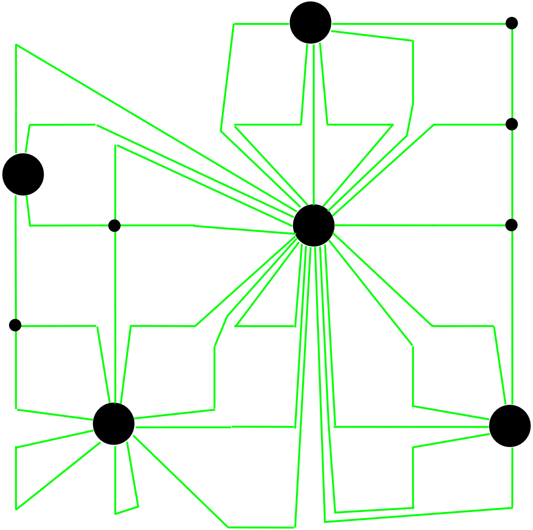
Connect it to all edges which used to connect to the cluster.



Note that this can create loops and multiple edges.



Repeat for all clusters.



Which p?

- Formally, for every edge, independently and with probability p, identify its two end points. Call the resulting graph CCP_p.

Which p?

- Formally, for every edge, independently and with probability p, identify its two end points. Call the resulting graph CCP_p.
- If $p > p_c$ then the infinite cluster becomes a point with infinite degree. This is similar to "$\gamma < 2$ models" in that the distance between typical points is constant and the random walk mixes in a constant number of steps.

Which p?

- Formally, for every edge, independently and with probability p, identify its two end points. Call the resulting graph CCP_p.
- If $p > p_c$ then the infinite cluster becomes a point with infinite degree. This is similar to "$\gamma < 2$ models" in that the distance between typical points is constant and the random walk mixes in a constant number of steps.
- If $p < p_c$ the contracted clusters are small and the graph would look like essentially like the original lattice. Rigorously, we believe this case would be amenable to the same techniques used to analyze random walk on the supercritical cluster.

Which p?

- Formally, for every edge, independently and with probability p, identify its two end points. Call the resulting graph CCP_p.
- If $p > p_c$ then the infinite cluster becomes a point with infinite degree. This is similar to "$\gamma < 2$ models" in that the distance between typical points is constant and the random walk mixes in a constant number of steps.
- If $p < p_c$ the contracted clusters are small and the graph would look like essentially like the original lattice. Rigorously, we believe this case would be amenable to the same techniques used to analyze random walk on the supercritical cluster.
- Hence we will focus on $p = p_c$, in which case we will call the graph CCCP.

Results

- We have $\gamma = 2 + \frac{1}{\delta}$ so $\gamma = 2\frac{5}{91}$ when $d = 2$ and $\gamma = \frac{5}{2}$ when $d > 6$.

Results

- We have $\gamma = 2 + \frac{1}{\delta}$ so $\gamma = 2\frac{5}{91}$ when $d = 2$ and $\gamma = \frac{5}{2}$ when $d > 6$.
- In $d = 2$ the typical distance between x and y is $\approx \log|x - y|$. In $d > 6$ it is $\approx \log\log|x - y|$.

Results

- We have $\gamma = 2 + \frac{1}{\delta}$ so $\gamma = 2\frac{5}{91}$ when $d = 2$ and $\gamma = \frac{5}{2}$ when $d > 6$.
- In $d = 2$ the typical distance between x and y is $\approx \log|x - y|$. In $d > 6$ it is $\approx \log\log|x - y|$. Note that universality would suggest that both should be $\log\log$.

Results

- We have $\gamma = 2 + \frac{1}{\delta}$ so $\gamma = 2\frac{5}{91}$ when $d = 2$ and $\gamma = \frac{5}{2}$ when $d > 6$.
- In $d = 2$ the typical distance between x and y is $\approx \log|x-y|$. In $d > 6$ it is $\approx \log\log|x-y|$. Note that universality would suggest that both should be $\log\log$.
- In $d = 2$ we show that the speed is polynomial in the sense that
$$\mathbb{E}(|R(t)|) \le t^K .$$
In the graph metric this translate to
$$\mathbb{E}(\mathrm{dist}(\vec{0}, R(n))) \approx \log t .$$
The exact value of K is related to other exponents which are known only numerically.

Results cntd.

	$d=2$	$d>6$	Universality
$\mathrm{dist}(x,y)$	$\log\lvert x-y\rvert$	$\log\log\lvert x-y\rvert$	$\log\log\lvert x-y\rvert$
$\mathrm{dist}(0,R(t))$	$\log t$		t

Results cntd.

	$d = 2$	$d > 6$	Universality
$\text{dist}(x, y)$	$\log \lvert x - y \rvert$	$\log \log \lvert x - y \rvert$	$\log \log \lvert x - y \rvert$
$\text{dist}(0, R(t))$	$\log t$		t

- In $d > 6$ we show that

$$\mathbb{E}(|R(t)|) \approx \sqrt{t \log t}.$$

Again, this means that

$$\mathbb{E}(\text{dist}(0, R(t)) \approx \log \log t$$

Results cntd.

	$d = 2$	$d > 6$	Universality
$\text{dist}(x, y)$	$\log \lvert x-y \rvert$	$\log\log \lvert x-y \rvert$	$\log\log \lvert x-y \rvert$
$\text{dist}(0, R(t))$	$\log t$		t

- In $d > 6$ we show that

$$\mathbb{E}(|R(t)|) \approx \sqrt{t \log t}.$$

Again, this means that

$$\mathbb{E}(\text{dist}(0, R(t)) \approx \log\log t$$

Despite the high degrees and the hyperfast connectivity, the random walk is slow.

More geometry

	$d = 2$	$d > 6$	Universality
$\mathrm{dist}(x, y)$	$\log\|x-y\|$	$\log\log\|x-y\|$	$\log\log\|x-y\|$
$\mathrm{dist}(0, R(t))$	$\log t$	$\log\log t$	t

- CCCP in $d > 6$ satisfies the same isoperimetric inequality as $\mathbb{Z}^d$,

$$|\partial A| \geq c|A|^{(d-1)/d} \quad \forall A \text{ finite},$$

and no better.

More geometry

	$d=2$	$d>6$	Universality
$\mathrm{dist}(x,y)$	$\log\lvert x-y\rvert$	$\log\log\lvert x-y\rvert$	$\log\log\lvert x-y\rvert$
$\mathrm{dist}(0,R(t))$	$\log t$	$\log\log t$	t

- CCCP in $d>6$ satisfies the same isoperimetric inequality as $\mathbb{Z}^d$,

$$|\partial A| \geq c|A|^{(d-1)/d} \quad \forall A \text{ finite},$$

  and no better.
- The spectral gap of the Laplacian on a ball of radius R is between $\frac{1}{R^2}$ and $\frac{\log R}{R^2}$.

More geometry

	$d=2$	$d>6$	Universality
$\mathrm{dist}(x,y)$	$\log\lvert x-y\rvert$	$\log\log\lvert x-y\rvert$	$\log\log\lvert x-y\rvert$
$\mathrm{dist}(0,R(t))$	$\log t$	$\log\log t$	t

- CCCP in $d>6$ satisfies the same isoperimetric inequality as $\mathbb{Z}^d$,

$$|\partial A| \geq c|A|^{(d-1)/d} \quad \forall A \text{ finite},$$

  and no better.

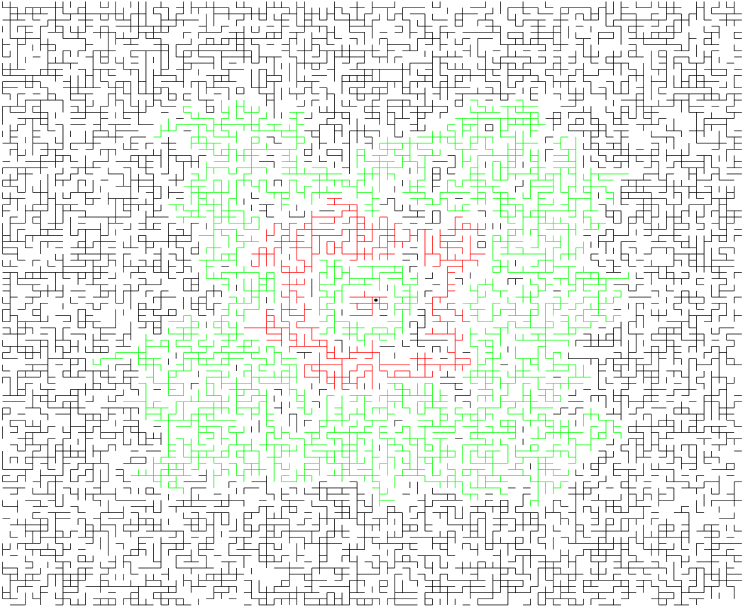
- The spectral gap of the Laplacian on a ball of radius R is between $\frac{1}{R^2}$ and $\frac{\log R}{R^2}$. This precision is not enough to determine whether CCCP is Liouville or not!

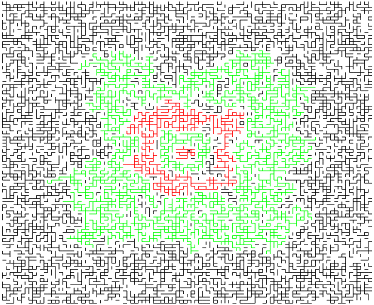
We will now discuss the geometric picture more and give some heuristic arguments and proof sketches.

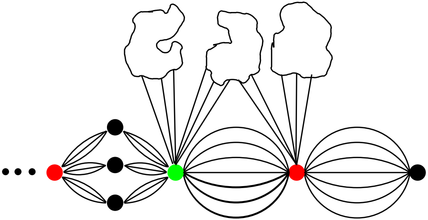
Surrounding clusters, $d = 2$

- In $d = 2$ there are clusters surrounding $\vec{0}$ in every scale.

Faux-simulation

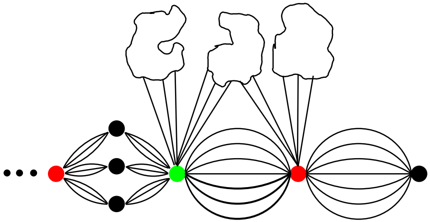






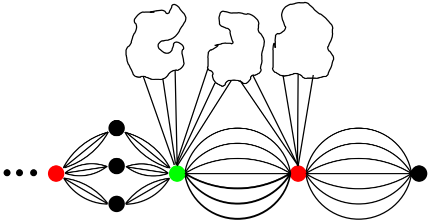
Surrounding clusters, $d = 2$

- In $d = 2$ there are clusters surrounding $\vec{0}$ in every scale.



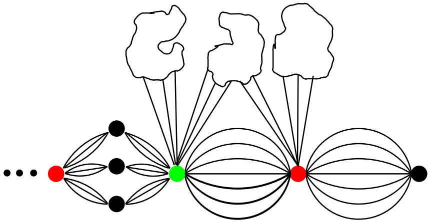
Surrounding clusters, $d = 2$

- In $d = 2$ there are clusters surrounding $\vec{0}$ in every scale. Typically the cluster at scale r will touch the cluster at scale $2r$.



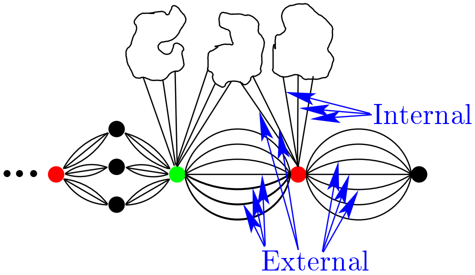
Surrounding clusters, $d = 2$

- In $d = 2$ there are clusters surrounding $\vec{0}$ in every scale. Typically the cluster at scale r will touch the cluster at scale $2r$.
- This shows that $\text{dist}(x, y) \approx \log |x - y|$ since you need to traverse each such cluster, and that's all you need to do.



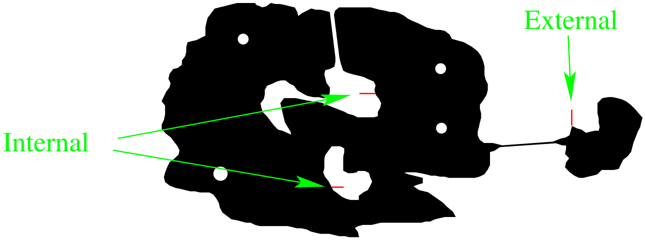
Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r.



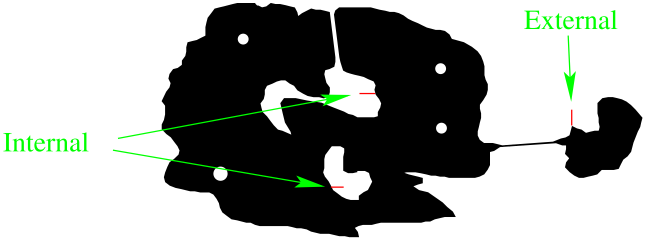
Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r. We return to the geometric picture and we see this is related to the arm exponents.



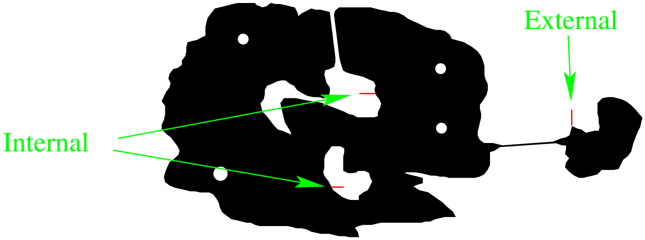
Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r. We return to the geometric picture and we see this is related to the arm exponents. An internal edge requires one black arm and therefore there are $\approx r^{91/48}$ of them.



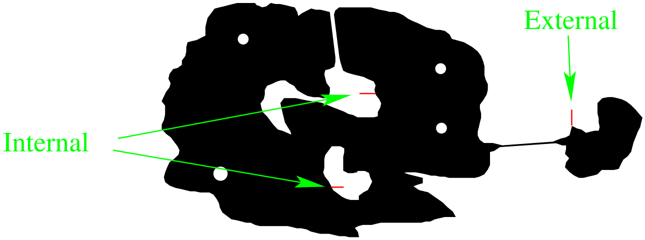
Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r. We return to the geometric picture and we see this is related to the arm exponents. An internal edge requires one black arm and therefore there are $\approx r^{91/48}$ of them. An external edge requires one black arm and two white arms and hence there are $\approx r^{4/3}$.



Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r. We return to the geometric picture and we see this is related to the arm exponents. An internal edge requires one black arm and therefore there are $\approx r^{91/48}$ of them. An external edge requires one black arm and two white arms and hence there are $\approx r^{4/3}$. This is enough to show that the Euclidean speed is polynomial.



Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r. We return to the geometric picture and we see this is related to the arm exponents. An internal edge requires one black arm and therefore there are $\approx r^{91/48}$ of them. An external edge requires one black arm and two white arms and hence there are $\approx r^{4/3}$. This is enough to show that the Euclidean speed is polynomial.
- We remark that *globally* the random walk is transient (even though it gets stuck at heavy vertices for a long time).

Random walk, $d = 2$

- Let's compare the number of internal and external edges for a surrounding cluster in scale r. We return to the geometric picture and we see this is related to the arm exponents. An internal edge requires one black arm and therefore there are $\approx r^{91/48}$ of them. An external edge requires one black arm and two white arms and hence there are $\approx r^{4/3}$. This is enough to show that the Euclidean speed is polynomial.
- We remark that *globally* the random walk is transient (even though it gets stuck at heavy vertices for a long time). This can be demonstrated by constructing an explicit flow with finite energy.

Connectivity, $d > 6$

- When $d > 6$, two typical clusters in scale r are connected by $\left\lceil \frac{d}{2} \right\rceil - 3$ hops.

Connectivity, $d > 6$

- When $d > 6$, two typical clusters in scale r are connected by $\lceil \frac{d}{2} \rceil - 3$ hops. Heuristically this can be seen because a cluster is 4 dimensional.

Connectivity, $d > 6$

- When $d > 6$, two typical clusters in scale r are connected by $\lceil \frac{d}{2} \rceil - 3$ hops. Heuristically this can be seen because a cluster is 4 dimensional. Each point on its boundary (which is also 4 dimensional) has probability $\approx r^{-2}$ to belong to a large cluster so the clusters reachable by one hop form a 6 dimensional object. And by k hops a $4 + 2k$ dimensional object.

Connectivity, $d > 6$

- When $d > 6$, two typical clusters in scale r are connected by $\lceil \frac{d}{2} \rceil - 3$ hops. Heuristically this can be seen because a cluster is 4 dimensional. Each point on its boundary (which is also 4 dimensional) has probability $\approx r^{-2}$ to belong to a large cluster so the clusters reachable by one hop form a 6 dimensional object. And by k hops a $4 + 2k$ dimensional object. When the object becomes $d - 4$ dimensional or more, it will intersect any other cluster.

Connectivity, $d > 6$

- When $d > 6$, two typical clusters in scale r are connected by $\lceil \frac{d}{2} \rceil - 3$ hops. Heuristically this can be seen because a cluster is 4 dimensional. Each point on its boundary (which is also 4 dimensional) has probability $\approx r^{-2}$ to belong to a large cluster so the clusters reachable by one hop form a 6 dimensional object. And by k hops a $4 + 2k$ dimensional object. When the object becomes $d - 4$ dimensional or more, it will intersect any other cluster.
- The formal proof used the 2-point function and diagrammatic bounds.

Connectivity, $d > 6$

- When $d > 6$, two typical clusters in scale r are connected by $\lceil \frac{d}{2} \rceil - 3$ hops. Heuristically this can be seen because a cluster is 4 dimensional. Each point on its boundary (which is also 4 dimensional) has probability $\approx r^{-2}$ to belong to a large cluster so the clusters reachable by one hop form a 6 dimensional object. And by k hops a $4 + 2k$ dimensional object. When the object becomes $d - 4$ dimensional or more, it will intersect any other cluster.
- The formal proof used the 2-point function and diagrammatic bounds.
- Since a ball of radius r typically intersects a much larger cluster — of scale $r^{d/2-2}$ — we get a doubly-exponential increasing sequence, so $\mathrm{dist}(x, y) \approx \log\log|x - y|$.

Random walk, $d > 6$

- The argument used in $d = 2$ no longer holds because even large clusters are *uniformly transient*

Random walk, $d > 6$

- Consider the environment viewed from the particle.

Random walk, $d > 6$

- Consider the environment viewed from the particle. From $\delta = 1/2$ we see that the second moment of a single step only grows logarithmically.

Random walk, $d > 6$

- Consider the environment viewed from the particle. From $\delta = 1/2$ we see that the second moment of a single step only grows logarithmically. The techniques of De Masi, Ferrari, Goldstein & Wick 1989 should be applicable to this case as well.

Random walk, $d > 6$

- Consider the environment viewed from the particle. From $\delta = 1/2$ we see that the second moment of a single step only grows logarithmically. The techniques of De Masi, Ferrari, Goldstein & Wick 1989 should be applicable to this case as well.
- Because CCCP is a quotient of $\mathbb{Z}^d$, it must satisfy

$$|\partial A| \geq c|A|^{(d-1)/d} \quad \forall A \subset \text{CCCP} \ .$$

Random walk, $d > 6$

- Consider the environment viewed from the particle. From $\delta = 1/2$ we see that the second moment of a single step only grows logarithmically. The techniques of De Masi, Ferrari, Goldstein & Wick 1989 should be applicable to this case as well.
- Because CCCP is a quotient of $\mathbb{Z}^d$, it must satisfy

$$|\partial A| \geq c|A|^{(d-1)/d} \quad \forall A \subset \text{CCCP} \ .$$

To see that this cannot be improved, we need to demonstrate Fölner sets. Examine the set

$$\{x \in B(r) : x \not\leftrightarrow \partial B(r)\} \, .$$

Random walk, $d > 6$

- Consider the environment viewed from the particle. From $\delta = 1/2$ we see that the second moment of a single step only grows logarithmically. The techniques of De Masi, Ferrari, Goldstein & Wick 1989 should be applicable to this case as well.
- Because CCCP is a quotient of $\mathbb{Z}^d$, it must satisfy

$$|\partial A| \geq c|A|^{(d-1)/d} \quad \forall A \subset \text{CCCP} .$$

To see that this cannot be improved, we need to demonstrate Fölner sets. Examine the set

$$\{x \in B(r) : x \not\leftrightarrow \partial B(r)\} .$$

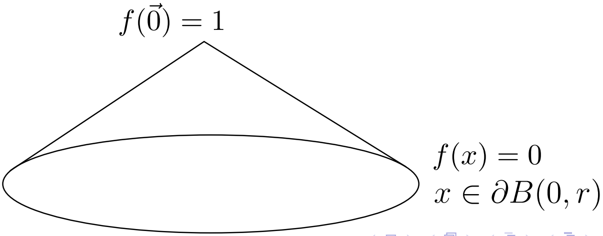
Because $\mathbb{P}(x \leftrightarrow \partial B(r)) \approx r^{-2}$ (K & Nachmias, in preparations), we see that the number of points removed is surface order, r^{d-1}.

Spectral gap, $d > 6$

- The isoperimetric inequality shows that the spectral gap is $\geq 1/r^2$ (Cheeger's inequality).

Spectral gap, $d > 6$

- The isoperimetric inequality shows that the spectral gap is $\geq 1/r^2$ (Cheeger's inequality).
- To see that the spectral gap is $\leq \frac{\log r}{r^2}$, examine the function f below.

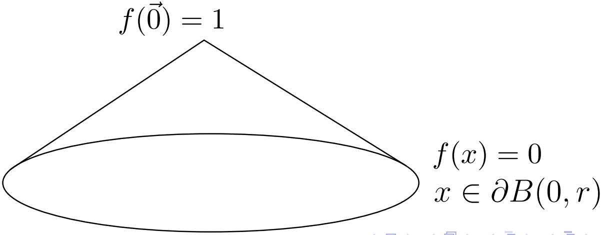


Spectral gap, $d > 6$

- The isoperimetric inequality shows that the spectral gap is $\geq 1/r^2$ (Cheeger's inequality).
- To see that the spectral gap is $\leq \frac{\log r}{r^2}$, examine the function f below. This is not well defined on CCCP (because it is not constant on the clusters), so define

$$g(\mathcal{C}) = \text{the average of } f \text{ on } \mathcal{C}$$

for every cluster $\mathcal{C}$.

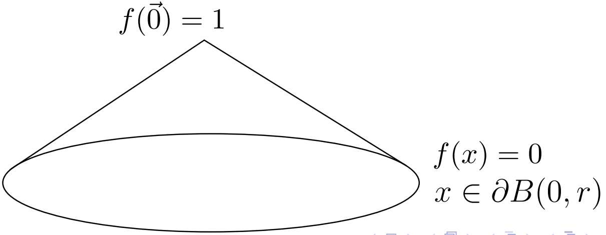


Spectral gap, $d > 6$

- The isoperimetric inequality shows that the spectral gap is $\geq 1/r^2$ (Cheeger's inequality).
- To see that the spectral gap is $\leq \frac{\log r}{r^2}$, examine the function f below. This is not well defined on CCCP (because it is not constant on the clusters), so define

$$g(\mathcal{C}) = \text{the average of } f \text{ on } \mathcal{C}$$

for every cluster $\mathcal{C}$. A similar calculation to the above shows that the energy of ∇f is $\approx \frac{\log r}{r^2}$ giving an upper bound on the spectral gap.

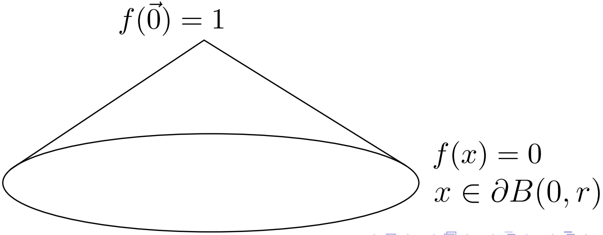


Spectral gap, $d > 6$

- The isoperimetric inequality shows that the spectral gap is $\geq 1/r^2$ (Cheeger's inequality).
- To see that the spectral gap is $\leq \frac{\log r}{r^2}$, examine the function f below. This is not well defined on CCCP (because it is not constant on the clusters), so define

$$g(\mathcal{C}) = \text{the average of } f \text{ on } \mathcal{C}$$

for every cluster $\mathcal{C}$. A similar calculation to the above shows that the energy of ∇f is $\approx \frac{\log r}{r^2}$ giving an upper bound on the spectral gap. We believe this is the right answer.



Thank you