



Scaling Limit of a Generalized Contact Process

Logan Chariker¹ · Anna De Masi² · Joel L. Lebowitz³ · Errico Presutti⁴

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Abstract

We derive macroscopic equations for a generalized contact process that is inspired by a neuronal integrate and fire model on the lattice \mathbb{Z}^d . The states at each lattice site can take values in $0, \dots, k$. These can be interpreted as neuronal membrane potential, with the state k corresponding to a firing threshold. In the terminology of the contact processes, which we shall use in this paper, the state k corresponds to the individual being infectious (all other states are noninfectious). In order to reach the firing threshold, or to become infectious, the site must progress sequentially from 0 to k . The rate at which it climbs is determined by other neurons at state k , coupled to it through a Kac-type potential, of range γ^{-1} . The hydrodynamic equations are obtained in the limit $\gamma \rightarrow 0$. Extensions of the microscopic model to include excitatory and inhibitory neuron types, as well as other biophysical mechanisms, are also considered.

Keywords Neurons with discrete voltage · Integrate and fire · Generalized contact process · Mean field · Spatial dependence · Hydrodynamic limit

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✉ Logan Chariker
chariker@ias.edu

Anna De Masi
demasi@univaq.it

Joel L. Lebowitz
lebowitz@math.rutgers.edu

Errico Presutti
errico.presutti@gmail.com

¹ School of Natural Sciences, Institute for Advanced Study, Princeton, NJ, USA

² Università degli Studi dell'Aquila, 67100 L'Aquila, Italy

³ Departments of Mathematics and Physics, Rutgers University, New Brunswick, NJ, USA

⁴ Gran Sasso Science Institute, 67100 L'Aquila, Italy

1 Introduction

The derivation of macroscopic deterministic time evolution equations from underlying microscopic dynamics is one of the central problems of non-equilibrium statistical mechanics. This micro-to-macro transition is a very difficult mathematical problem with only limited progress so far [3, 11, 14]. This can be overcome to some extent when the underlying microscopic dynamics is stochastic with very strong ergodic properties. Examples are the time evolution of the stochastic Ising model via Glauber or Kawasaki dynamics. There one has derived rigorously macroscopic equations in a space-time scaling limit [2, 4, 7, 8]. These equations are of the mean field type using long range Kac type interactions on the microscopic scale.

In this note we derive macroscopic equations presented and partially solved in [1]. The microscopic model system described here is inspired by neuronal integrate-and-fire models [6]. In the simple version of this model the membrane voltage increases until it reaches a maximum threshold value at which time it fires (spikes). When it fires that neuron's membrane voltage gets reset to its minimum value. At the same time other neurons connected to it, whose potential is below threshold, increase their potential at a rate depending on the strength of their connectivity to the neuron which has just spiked. In the macroscopic equations we considered in [1] we discretized the values which the membrane potential can take restricting it to the integer set $\{0, 1, \dots, k\}$. When and only when a neuron is in state k , its maximum value, it causes other neurons connected to it with potential $j < k$ to transit to the next level $j + 1$. Independently, neurons with potential values k spike and assume the value 0. The neurons in the microscopic model live on the d -dimensional lattice \mathbb{Z}^d with spacing γ and their interaction is given by a Kac type function, $J(\gamma|x - y|)$ [8]. In the limit $\gamma \rightarrow 0$ one obtains the macroscopic equations.

It turns out that for $k = 1$ the model is equivalent to the well known contact process with the state $j = 0$ corresponding to the healthy state and the state $j = 1$ the infected one [12, 13]. For $k > 1$ the model can be thought of as a generalized contact process with only the state $j = k$ being infectious. In terms of neural models the case $k = 1$ corresponds to the stochastic Wilson-Cowan model [9, 15], which is a popular simplified model of neural systems. Setting $k > 1$ introduces inactive states which behave like subthreshold neuron potentials and leads to more complicated behavior.

The analysis in this note will be done entirely in the context of the generalized contact process. We consider an extension of the classical contact process where the state of an individual is described by a potential U : when $U = 0$ the individual is healthy, when $0 < U < k$ is sick but not contagious and when $U = k$ is both sick and contagious (in the classical contact process $k = 1$). Infections are long range and described by a Kac potential with range γ^{-1} . We study the system in the macroscopic limit $\gamma \rightarrow 0$.

In more realistic models there are two types of neurons, excitatory ones which act as those described above and inhibitory ones [6, 10]. The latter ones also have a threshold for firing but instead of increasing the potential of other neurons when firing it decreases them. This, as well as other generalizations can be incorporated in the microscopic model studied here. They lead to more complicated macroscopic equations, which we are currently exploring. Their derivation uses the same formalism as the derivation given here. These will be discussed briefly at the end of this note.

The outline of the rest of the paper is as follows. In Sect. 2 we give a precise definition of the microscopic model and present the hydrodynamic limit equations. In Sects. 3–7 we prove the hydrodynamic limit. In Sect. 8 we describe some generalizations of the model.

2 The Model

2.1 The Macroscopic Region

The macroscopic region Ω is a torus in \mathbb{R}^d of side length L ; for simplicity L is a large positive integer.

2.2 The Microscopic Region

Let $\gamma = 2^{-n_1}$, n_1 a positive integer. The microscopic region is the torus $\Omega_\gamma = \gamma^{-1}\Omega \cap \mathbb{Z}^d$.

2.3 The Kac Potential

Dynamics are defined in terms of the Kac potential $J_\gamma(x, y) = a_\gamma \gamma^d J(\gamma x, \gamma y)$, $x, y \in \Omega_\gamma$. a_γ is the normalization coefficient which makes $J_\gamma(x, y)$ a probability; $J(r, r')$ is a smooth, non negative, symmetric probability kernel with finite range $R < L/2$. Call

$$R_\gamma = \frac{1}{2} \sup\{|x - y| : J_\gamma(x, y) > 0\}, \quad R_\gamma = \gamma^{-1}R \tag{2.1}$$

the range of the interaction $J_\gamma(x, y)$.

The macroscopic limit is defined by letting $\gamma \rightarrow 0$.

2.4 The Time Evolution

Time evolution is described by a jump Markov process where there are two types of jumps related to infection and recovery. The individual at site x with $U(x) = k$ recovers at rate 1 and the potential after recovery becomes $U(x) = 0$. Moreover the individual at site x with $U(x) = k$ infects the one at site y if $U(y) < k$ at rate $\lambda^* J_\gamma(x, y)$ and the effect of the infection is that $U(y) \rightarrow U(y) + 1$.

We denote by $U_t(x)$, $x \in \Omega_\gamma$ the potentials at time t and we denote by P^γ the law of this process in Ω_γ .

Definition 2.1 The Initial Condition For any fixed γ the potentials $U_0(x)$, $x \in \Omega_\gamma$, at time 0 are mutually independent and the following holds:

- $P^\gamma[U_0(x) = i] = \rho_0(\gamma x, i)$, $i = 0, \dots, k$ where $\rho_0(r, i) > 0$, $i = 0, \dots, k$ is such that $\sum_i \rho_0(r, i) = 1$ for any $r \in \Omega$. The variables $\rho_0(r, i)$ are the initial densities.
- The densities $\rho_0(r, i)$, $r \in \Omega$, $i = 0, \dots, k$, have values in $[\epsilon, 1 - \epsilon]$, $\epsilon > 0$ and are smooth.

Assuming propagation of chaos, namely that the potentials $U_t(x)$, $x \in \Omega_\gamma$ at any $t > 0$ are mutually independent and $P^\gamma[U_t(x) = i] = \rho_t(\gamma x, i)$, $i = 0, \dots, k$, then ρ_t satisfies

$$\frac{d}{dt} \rho_t(r, i) = \{\rho_t(r, i - 1) - \rho_t(r, i)\} \int_\Omega dr' \lambda^* \rho_t(r', k) J(r', r) \tag{2.2}$$

when $1 \leq i \leq k - 1$; for $i = k$

$$\frac{d}{dt} \rho_t(r, k) = -\rho_t(r, k) + \rho_t(r, k - 1) \int_\Omega dr' \lambda^* \rho_t(r', k) J(r', r) \tag{2.3}$$

and for $i = 0$:

$$\frac{d}{dt} \rho_t(r, 0) = \rho_t(r, k) - \rho_t(r, 0) \int_{\Omega} dr' \lambda^* \rho_t(r', k) J(r', r) \tag{2.4}$$

the initial condition being $\rho_0(r, i)$, ρ_0 as in Definition 2.1. These are the macroscopic equations analyzed in [1]. There, for $\lambda^* \leq k$, it is shown that the only steady state solutions to the equations are the trivial ones with $\rho_t(r, k) = 0$, where nothing is in the active state k . For $\lambda^* > k$, the nontrivial steady state solutions are derived, and they are shown to be constant in x and linearly stable. From the examination of numerically computed trajectories it is conjectured that all solutions tend towards a steady state. The complete solution is given in the spatially uniform case (where the initial condition is constant in x), and a partial solution is given in the general case. Traveling wave solutions are also investigated numerically.

We will use three scaling parameters $\gamma = 2^{-n_1}$, $n_1 \in \mathbb{N}$, $\delta = 2^{-n_2}$, $n_2 \in \mathbb{N}$ and $\xi = 2^{-n_3}$, $n_3 \in \mathbb{N}$. The macroscopic limit, namely the solution of Eqs. (2.2)–(2.4), is obtained by first letting $\gamma \rightarrow 0$, then $\delta \rightarrow 0$ and finally $\xi \rightarrow 0$.

To prove this limiting behavior we first prove the hydrodynamic limit for a modified dynamics, called the auxiliary process, with a Kac potential A_γ , see (3.2), which is a coarse grained version of J_γ . We then obtain in the limit $\gamma \rightarrow 0$ a macroscopic equation with a kernel A_ξ , see (3.6), which is a coarse graining version of J , see Theorem 3.2. In the limit $\xi \rightarrow 0$ we get (2.2)–(2.4), see Theorem 3.3.

3 The Auxiliary Process

The auxiliary process is defined as the previous one but with a piecewise constant kernel $A_\gamma(x, y)$ in the place of $J_\gamma(x, y)$.

In order to define A_γ we need the following definition.

Definition 3.1 *The basic partition.* We call

$$\Omega_\gamma^* = \{r \in \Omega : r = \gamma x, x \in \mathbb{Z}^d\}, \quad \Omega^* = \bigcup_{\gamma} \Omega_\gamma^*, \quad \Omega_\gamma = \gamma^{-1} \Omega_\gamma^* \tag{3.1}$$

The basic partition of Ω is denoted by π_ξ , $\xi = L2^{-n_3}$; its atoms $C^\xi \in \pi_\xi$ are cubes of side ξ and $C^\xi(r)$, $r \in \Omega$, denotes the atom which contains r .

The microscopic basic partition $\pi_{\gamma, \xi}$ is made by atoms $C^{\gamma, \xi} = \gamma^{-1} C^\xi$, $C^\xi \in \pi_\xi$, $C^{\gamma, \xi}$ has $N = (\gamma^{-1} \xi)^d$ elements. We write $C^{\gamma, \xi}(x) = \gamma^{-1} C^\xi(\gamma x)$, $C^\xi \in \pi_\xi$.

We say that two atoms $C^{\gamma, \xi}$ and $D^{\gamma, \xi}$ of the basic partition interact with each other if there are $x \in C^{\gamma, \xi}$ and $y \in D^{\gamma, \xi}$ such that $J_\gamma(x, y) > 0$.

To simplify the notation we will drop the superscript (γ, ξ) from the cubes $C^{\gamma, \xi}$ unless confusion may arise.

3.1 The New Piecewise Constant Kernel

In the new process the rate at which x infects $y \neq x$ is $\lambda^* A_\gamma(x, y)$. The new kernel $A_\gamma(x, y)$ is defined by averaging $J_\gamma(x, y)$ over the atoms of the basic partition $\pi_{\gamma, \xi}$, more precisely

$$A_\gamma(x, y) = \frac{1}{N^2} \sum_{x' \in C^{\gamma, \xi}(x)} \sum_{y' \in D^{\gamma, \xi}(y)} J_\gamma(x', y') \tag{3.2}$$

where $J_\gamma(x, x) = 0$ and $N = |C^{\gamma, \xi}|$.

3.2 Properties of A_γ

- For any x , $A_\gamma(x, \cdot)$ is a probability, in fact

$$\begin{aligned} \sum_y A_\gamma(x, y) &= \sum_{D \in \pi_{\gamma, \xi}} \sum_{y \in D} A_\gamma(x, y) = \sum_{D \in \pi_{\gamma, \xi}} A_\gamma(x, y_D) |D| \\ &= \frac{1}{N} \sum_{x' \in C(x)} \sum_{D \in \pi_{\gamma, \xi}} \sum_{y' \in D} J_\gamma(x', y') = \sum_{D \in \pi_{\gamma, \xi}} \sum_{y' \in D} J_\gamma(x, y') \\ &= \sum_z J_\gamma(x, z) = 1 \end{aligned} \tag{3.3}$$

where y_D is any point in D . In the third equality we have used that $|D| = N$.

- $A_\gamma(x, y) = 0$ if $C(x)$ and $D(y)$ do not interact.
- If instead $C(x)$ and $D(y)$ interact

$$A_\gamma(x, y) \leq \gamma^d \|J(r, r')\|_\infty =: c \xi^d \frac{1}{N} \tag{3.4}$$

We denote by $\mathcal{P}^{A_\gamma, \gamma}$ the law of the auxiliary process with initial conditions unchanged, see Definition 2.1 and by $\mathcal{P}^{A_\gamma, \gamma}$ the corresponding law of the density variables

$$v_{i,t}^{\gamma, \xi}(x) = \frac{1}{N} |C_{i,t}^{\gamma, \xi}(x)|, \quad C_{i,t}^{\gamma, \xi} = \{x' \in C^{\gamma, \xi}(x) : U_t(x') = i\} \tag{3.5}$$

The following Theorem will be proved in Sect. 6.

Theorem 3.2 Fix $T > 0$ and $t \in [0, T]$, we denote by $\mathcal{P}_t^{A_\gamma, \gamma}$ the restriction of $\mathcal{P}^{A_\gamma, \gamma}$ to time t . In analogy with (3.2) we define the kernel $A_\xi(r, r'), r, r' \in \Omega^*$ as

$$A_\xi(r, r') = \frac{1}{\xi^{2d}} \int_{C^\xi(r)} dr_1 \int_{C^\xi(r')} dr'_1 J(r_1, r'_1) \tag{3.6}$$

Then $\mathcal{P}_t^{A_\gamma, \gamma}$ converges weakly as $\gamma \rightarrow 0$ to a probability $\mathcal{P}_t^{A_\xi}$ which is supported by $\varphi_\xi(r, i; t)$ $r \in \Omega$ which is the solution at time t of the equations

$$\frac{d}{dt} \varphi_\xi(r, i; t) = \{\varphi_\xi(r, i - 1; t) - \varphi_\xi(r, i; t)\} \int_\Omega dr' \lambda^* \varphi_\xi(r', k; t) A_\xi(r', r) \tag{3.7}$$

when $1 \leq i \leq k - 1$; for $i = k$

$$\frac{d}{dt} \varphi_\xi(r, k; t) = -\varphi_\xi(r, k; t) + \varphi_\xi(r, k - 1; t) \int_\Omega dr' \lambda^* \varphi_\xi(r', k; t) A_\xi(r', r) \tag{3.8}$$

and for $i = 0$:

$$\frac{d}{dt} \varphi_\xi(r, 0; t) = \varphi_\xi(r, k; t) - \varphi_\xi(r, 0; t) \int_\Omega dr' \lambda^* \varphi_\xi(r', k; t) A_\xi(r', r) \tag{3.9}$$

the initial condition being $\varphi_\xi(r, i; 0) = \frac{1}{|C^\xi(r)|} \int_{C^\xi(r)} dr' \rho_0(r', i)$, ρ_0 as in Definition 2.1.

Observe that $\varphi_\xi(r, i; t)$, $r \in \Omega$ is constant on the cubes C^ξ . Convergence of the space-time joint distribution of the densities will be proved in Sect. 8 together with the following Theorem.

Theorem 3.3 *The function $\varphi_\xi(r, i; t)$, $r \in \Omega$, $t \in [0, T]$ converges as $\xi \rightarrow 0$ to $\rho_t(r, i)$ solution of the Eqs. (2.2)–(2.4).*

Sketch of the proof of Theorem 3.2

In the theory of hydrodynamic limit for stochastic interacting particle systems a typical procedure is to use the martingale decomposition for the variables of interest, see for instance the book [11]. Applied to our case we have

$$v_{i,t}^{\gamma,\xi}(x) - v_{i,0}^{\gamma,\xi}(x) = \int_0^t ds L_\gamma v_{i,s}^{\gamma,\xi}(x) + M_{i,t}^{\gamma,\xi}(x) \tag{3.10}$$

where L_γ is the generator of the process and $M_{i,t}^{\gamma,\xi}(x)$ is a martingale. $M_{i,t}^{\gamma,\xi}(x)$ is a “fluctuation term” and one can often prove that in the hydrodynamic limit $N \rightarrow \infty$, $M_{i,t}^{\gamma,\xi}(x)$ vanishes with probability going to 1. The hardest problem is to control $L_\gamma v_{i,t}^{\gamma,\xi}(x)$, whose explicit expression for $1 \leq i \leq k - 1$ in our case is

$$L_\gamma v_{i,t}^{\gamma,\xi}(x) = \left(v_{i-1,t}^{\gamma,\xi}(x) - v_{i,t}^{\gamma,\xi}(x) \right) \lambda^* \sum_y A_\gamma(y, x) v_{k,t}^{\gamma,\xi}(y) \tag{3.11}$$

By compactness $v_{i,t}^{\gamma,\xi}(x)$ converges (by subsequences) weakly in probability to some limit density but the problem is that in (3.11) the functions $v^{\gamma,\xi}$ appear quadratically and in general the weak limit of a product is not the product of the weak limits of the factors.

To close the equations one then needs to prove a factorization property for the $v_{i,t}^{\gamma,\xi}$, i.e. propagation of chaos or local equilibrium. We overcome this difficulty by using the same method as in [2, 5]. We discretize time, see Sect. 4: we use a mesh δ which will vanish after taking the limit $\gamma \rightarrow 0$ and study the process in the generic time interval $[n\delta, (n + 1)\delta]$ with $n \leq \delta^{-1}T$ having conditioned on the values of the potential $U_t(x)$ at time $t = n\delta$.

The increments of the densities in a time interval $[n\delta, (n + 1)\delta]$ (having fixed the potentials at time $n\delta$) are given in (4.7)–(4.9) in terms of variables $M_{C,D;i}$, C, D cubes of the basic partition, $i \in \{0, \dots, k\}$, and variables M_D , see (4.6).

Probability estimates on $\mathcal{M}_{C,D;i}$ are obtained in Theorem 5.1, those for M_D in Theorem 5.4. The proof of Theorem 3.2 is then given in Sect. 6.

The crucial point is to prove the probability estimates stated in Theorem 5.1 and in Theorem 5.4. We use a graphical representation of the process where we represent by an arrow (x, y) the infection to the individual at y due to the individual at x ; the recovery of an individual at x is described by a “marked point”.

The collection of arrows and marked points define a natural graph structure, see the paragraph *A graph structure* in the next section. To reconstruct the true process we introduce time variables $\underline{t}(x, y)$ and $\underline{t}(x)$, $\underline{t}(x, y)$ is a finite sequence of times $t_m(x, y)$ and $\underline{t}(x)$ of times $t_m(x)$. $\underline{t}(x, y)$ and $\underline{t}(x)$ are mutually independent Poisson processes with mean $\lambda^* A_\gamma(x, y)$ and respectively mean 1.

The above graph structure is realized by drawing an arrow (x, y) at a time $t \in \underline{t}(x, y)$ and a marked point at x at $t \in \underline{t}(x)$. Knowledge of all $\underline{t}(x, y)$ and $\underline{t}(x)$ allows to reconstruct the true process, see the paragraph *A realization of the process: the clock process* in Sect. 4. However to know whether at $t = t(x, y)$ there is an infection we need to know all the values $t(x', y'; s)$ and $t(x'; s)$ for all $s \leq t$ as well as the values of the initial potentials.

The analysis of the graph structure of arrows and marked points ignoring the times when they are drawn is quite simple because the variables $\underline{t}(x, y)$ and $\underline{t}(x)$ are mutually independent. The first crucial point is that an arrow (x, y) corresponds to an infection if at the initial time $U(x) = k$ and $U(y) = i$, $i < k$, provided that the cluster containing (x, y) is made only by the arrow (x, y) , see Lemma 4.1 and the paragraph *A graph structure* in the next section for the definition of clusters. Analogous property holds for marked points. Thus when clusters have only one element the time when the event occurs is not relevant.

The second crucial point is that clusters with more than one element are probabilistically negligible. An estimate is proved in Corollary 4.4. As argued after (5.8) this is good enough for clusters with at least 3 elements, for clusters with only two elements we have a more refined argument proved in Lemma 5.3.

The crucial step in the proof of Corollary 4.4 is to reduce to a branching process which is studied in Appendix A.

4 Time Discretization and a Realization of the Process

4.1 Time Discretization

We discretize time with mesh $\delta = 2^{-n_2}$, $n_2 \geq 1$. We fix δ and a time interval $[n\delta, (n+1)\delta]$, for a while we will study the process in such a time interval having conditioned on the values $U_{n\delta}$ of the potentials at time $n\delta$. By choosing δ small enough the process becomes considerably simpler and we will exploit the following realization of the process.

4.2 A Realization of the Process: The Clock Process

We attach to any ordered pair (x, y) , $x \neq y$, independent clocks called (x, y) -clocks which ring at exponential rate $\lambda^* A_y(x, y)$. The clocks start at time $n\delta$ and are stopped at time $(n+1)\delta$, recall that we are studying the process restricted to the time interval $[n\delta, (n+1)\delta]$. We denote by $\underline{t}(x, y)$ the times when the (x, y) -clock rings. We introduce also x -clocks which ring at rate 1, $\underline{t}(x)$ being the times when the x -clock rings. All the above clocks are independent of each other.

Our process is recovered as follows. If the (x, y) -clock rings and at the time of the ring $U(y) < k$ and $U(x) = k$ then $U(y) \rightarrow U(y) + 1$. Moreover if the x -clock rings at a time when $U(x) = k$ then $U(x) \rightarrow 0$. All these rings are *effective* while the other rings where the above conditions are not fulfilled are *ineffective*, the potentials are unchanged and they can be ignored. However it is a very complicated task to understand whether a clock ring is or is not effective, it depends on all the clock rings $\{\underline{t}(x, y); \underline{t}(x)\}$. As already mentioned it is convenient to introduce a graph structure.

4.3 A Graph Structure

When the (x, y) -clock rings we draw an oriented arrow (x, y) , when the x -clock rings we draw a marked point at x . Two arrows are connected if they have a point in common, a marked point is connected to an arrow if it is one of the two points of the arrow. Clusters are the maximal connected sets of marked points and arrows. Notice that a same arrow may appear several times in a cluster as well as a same marked point. In the Appendix we compute the probability of a cluster.

We denote by C_1 the clusters made by a single element, i.e. either a marked point or an arrow. C_j are the clusters with j elements. We will see that if the time mesh δ is small the relevant clusters are the single clusters C_1 . In such a case we have:

Lemma 4.1 *Let $U_{n\delta}(x) = k, U_{n\delta}(y) = i, i < k$, and let $C_1 = (x, y)$ then $U_{(n+1)\delta}(x) = k$ and $U_{(n+1)\delta}(y) = i + 1$. Analogously if x is a marked point with $U_{n\delta}(x) = k$ and $C_1 = x$ then $U_{(n+1)\delta}(x) = 0$*

Proof The potentials $U_{n\delta}(x)$ and $U_{n\delta}(y)$ can only change when the (x, y) -clock rings because $(x, y) = C_1$ and all the other arrows are not connected to (x, y) nor the marked points. Then by the assumption $U_{n\delta}(x) = k$ and $U_{n\delta}(y) = i$ the (x, y) -ring is effective hence the statement in the lemma. The case of $C_1 = x$ is proved similarly. \square

Definition 4.2 In the sequel we will denote by c constants which do not depend on N and δ .

The following Theorem is proved in the Appendix.

Theorem 4.3 *For any $a \in (0, 1)$ and any $\epsilon > 0$ such that $1 - a - 2\epsilon > 0$, there is a constant c so that for any γ and δ small enough the following holds. Let (x, y) be an arrow then for any two atoms C and D of the basic partition*

$$N^{-1} \sum_{x \in C} \sum_{y \in D} \sum_{j \geq 1} \delta^{-aj} \sum_{C_j \ni (x,y)} P^{A\gamma,\gamma}[C_j] \leq c\delta^{1-a-2\epsilon} \tag{4.1}$$

Moreover

$$N^{-1} \sum_{x \in C} \sum_{j \geq 1} \delta^{-aj} \sum_{C_j \ni x} P^{A\gamma,\gamma}[C_j] \leq c\delta^{1-a-2\epsilon} \tag{4.2}$$

We will use the following consequence of Theorem 4.3:

Corollary 4.4 *Let C and D be two atoms of the basic partition, then for any $j^* \geq 1$*

$$P^{A\gamma,\gamma} \left[N^{-1} \sum_{x \in C} \sum_{y \in D} \sum_{j \geq j^*} \mathbf{1}_{C_j \ni (x,y)} > \delta^{bj^*} \right] \leq c\delta^{(a-b)j^*+1-a-2\epsilon}, \tag{4.3}$$

where $a \in (0, 1), b \in (0, 1)$, for ϵ as in Theorem 4.3. Analogously

$$P^{A\gamma,\gamma} \left[N^{-1} \sum_{x \in C} \sum_{j \geq j^*} \mathbf{1}_{C_j \ni x} > \delta^{bj^*} \right] \leq c\delta^{(a-b)j^*+1-a-2\epsilon}, \tag{4.4}$$

where $a \in (0, 1), b \in (0, 1)$.

Proof We take $a > 0$ and write

$$\begin{aligned} \sum_{j \geq j^*} \sum_{C_j \ni (x,y)} P^{A\gamma,\gamma}[C_j] &= \sum_{j \geq j^*} \delta^{ja} \delta^{-ja} \sum_{C_j \ni (x,y)} P^{A\gamma,\gamma}[C_j] \\ &\leq \delta^{j^*a} \sum_j \delta^{-ja} \sum_{C_j \ni (x,y)} P^{A\gamma,\gamma}[C_j] \end{aligned}$$

Then using Theorem 4.3 we have

$$N^{-1} \sum_{x \in C} \sum_{y \in D} \sum_{j \geq j^*} \sum_{C_j \ni (x,y)} P^{A\gamma,\gamma}[C_j] \leq c\delta^{j^*a} \delta^{1-a-2\epsilon} \tag{4.5}$$

By the Markov inequality we then have (4.3). The proof of (4.4) is similar and omitted. \square

For simplicity, in the sequel we write P^γ instead of $P^{A\gamma,\gamma}$.

Definition 4.5 We denote by $\kappa_{x,y,i}(n)$, $i < k$, the number of effective (x, y) -rings in the time interval $[n\delta, (n + 1)\delta)$, namely those such that when the clock rings $U(x) = k$ and $U(y) = i$; we denote by $\kappa_x(n)$ the number of effective x -rings, namely the times t in $\underline{t}(x)$ when $U_t(x) = k$. We then define for two cubes C and D

$$\mathcal{M}_{C,D;i}(n) = \sum_{x \in C} \sum_{y \in D} \kappa_{x,y,i}(n), \quad i < k; \quad \mathcal{M}_D(n) = \sum_{x \in D} \kappa_x(n) \tag{4.6}$$

Since in the following n is fixed we drop the dependence on n in (4.6).

Recalling (3.5) for notation we have for $0 < i < k$

$$v_{i,(n+1)\delta}(y) - v_{i,n\delta}(y) = \frac{1}{N} \sum_C [\mathcal{M}_{C,D(y);i-1} - \mathcal{M}_{C,D(y);i}] \tag{4.7}$$

$$v_{0,(n+1)\delta}(y) - v_{0,n\delta}(y) = \frac{1}{N} \left(\mathcal{M}_{D(y)} - \sum_C \mathcal{M}_{C,D(y);0} \right) \tag{4.8}$$

$$v_{k,(n+1)\delta}(y) - v_{k,n\delta}(y) = \frac{1}{N} \left(-\mathcal{M}_{D(y)} + \sum_C \mathcal{M}_{C,D(y);k-1} \right) \tag{4.9}$$

5 Probability Estimates

In this Section we estimate \mathcal{M}_D and $\mathcal{M}_{C,D;i}$, $i < k$, see Definition 4.5. They are in general very complicated functions in the space of the clock rings $\{\underline{t}(x, y); \underline{t}(x)\}$, we shall see however that only cases with few rings are important, the others give a small contribution. This will be the crucial point in the proof of the following theorem, Theorem 5.1, which concerns the number of events where an individual in a cube C with potential k infects an individual in the cube D which has potential $i < k$.

Probability estimates on $\mathcal{M}_{C,D;i}$

Recall that we have fixed a time interval $[n\delta, (n + 1)\delta]$ and we will not make explicit the dependence on such interval unless confusion may arise.

Theorem 5.1 *For any $\theta \in (0, 1)$, $a \in (\frac{1}{2}, 1)$, $b \in (\frac{1}{3}, \frac{2}{3}a)$ and $\epsilon < \frac{1-a}{2}$ there is a constant c so that for all $i < k$ and all x such that $U_{n\delta}(x) = k$ and all y so that $U_{n\delta}(y) = i$*

$$\begin{aligned} &P^\gamma \left[|N^{-1} \mathcal{M}_{C(x),D(y);i} - \lambda^* \delta N A_\gamma(x, y) v_{k,n\delta}(x) v_{i,n\delta}(y)| \right. \\ &\quad \left. \leq c(N^{-\epsilon} + N^{-\theta} + \delta^{3b} + \delta^{2-a}) \right] \\ &\geq 1 - c \left(\frac{\delta^2}{N^{1-\epsilon}} + \frac{\delta}{N} + \frac{\delta \lambda^*}{N^{1-2\theta}} + \delta^{1+2a-3b-2\epsilon} + \frac{\delta^a}{N} \right) \end{aligned} \tag{5.1}$$

Notice that the conditions $a \in (\frac{1}{2}, 1)$, $b \in (\frac{1}{3}, \frac{2}{3}a)$ and $\epsilon < \frac{1-a}{2}$ imply that $2a - 3b - 2\epsilon > 0$.

The proof will be obtained in several steps.

The first step is to reduce to cases where $|\underline{t}(x, y)| = 1$. To this end, recalling Definition 4.5, we consider two cubes C and D and for $i < k$ we write

$$\mathcal{M}_{C,D;i} = \sum_{x \in C} \sum_{y \in D} \kappa_{x,y,i} \mathbf{1}_{|\underline{t}(x,y)|=1} + R_{C,D;i} \tag{5.2}$$

where

$$R_{C,D;i} = \sum_{x \in C} \sum_{y \in D} \kappa_{x,y,i} \mathbf{1}_{|t(x,y)| > 1}$$

Lemma 5.2 *There is a constant c (independent of N and δ) so that for any $\epsilon > 0$:*

$$P^\gamma \left[\frac{1}{N} R_{C,D;i} > N^{-\epsilon} \right] \leq c \frac{\delta^2}{N^{1-\epsilon}} \tag{5.3}$$

Proof Recall that by definition

$$P^\gamma \left[|t(x,y)| = n \right] = e^{-\lambda^* \delta A_\gamma(x,y)} \frac{[\lambda^* \delta A_\gamma(x,y)]^n}{n!} \tag{5.4}$$

We then bound

$$\frac{1}{N} R_{C,D;i} \leq \frac{1}{N} \sum_{x \in C} \sum_{y \in D} |t(x,y)| \mathbf{1}_{|t(x,y)| > 1}$$

and using (3.4) we get

$$E^\gamma \left[\frac{1}{N} R_{C,D;i} \right] = \frac{1}{N} \sum_{x \in C} \sum_{y \in D} \sum_{k=2}^\infty k e^{-\lambda^* \delta A_\gamma(x,y)} \frac{[\lambda^* \delta A_\gamma(x,y)]^k}{k!} \leq c \frac{\delta^2}{N}$$

By the Markov inequality we then get (5.3). □

Let x be such that $U_{n\delta}(x) = k$ and y be such that $U_{n\delta}(y) = i$, then

$$\begin{aligned} \kappa_{x,y,i} \mathbf{1}_{|t(x,y)|=1} &= \mathbf{1}_{|t(x,y)|=1} \mathbf{1}_{\{(x,y) \in C_1\}} + \kappa_{x,y,i} \mathbf{1}_{|t(x,y)|=1} \sum_{j \geq 2} \sum_{C_j} \mathbf{1}_{\{(x,y) \in C_j\}} \\ &= \mathbf{1}_{|t(x,y)|=1} + \mathbf{1}_{|t(x,y)|=1} [\kappa_{x,y,i} - 1] \sum_{j \geq 2} \sum_{C_j} \mathbf{1}_{\{(x,y) \in C_j\}} \end{aligned} \tag{5.5}$$

Thus from (5.2) we get

$$\left| \frac{1}{N} \mathcal{M}_{C,D;i} - \frac{1}{N} \sum_{x \in C} \sum_{y \in D} \mathbf{1}_{|t(x,y)|=1} \right| \leq \frac{1}{N} R_{C,D;i} + \frac{1}{N} T_{C,D} \tag{5.6}$$

where

$$T_{C,D} = \sum_{j \geq 2} T_{C,D}^{(j)}, \quad T_{C,D}^{(j)} = \sum_{x \in C} \sum_{y \in D} \mathbf{1}_{\{(x,y) \in C_j\}} \tag{5.7}$$

having used that $0 \leq \kappa_{x,y,i} \leq 1$ if $\mathbf{1}_{|t(x,y)|=1}$.

By Corollary 4.4 we have:

$$P^\gamma \left[\frac{1}{N} \sum_{j' \geq j} T_{C,D}^{(j')} > \delta^{jb} \right] \leq c \delta^{(a-b)j+1-a-2\epsilon} \quad a \in (0, 1), \quad b \in (0, 1) \tag{5.8}$$

We will eventually need to iterate the estimate over all the time intervals $[\delta n, \delta(n+1)]$, i.e. δ^{-1} times, so that we want $\delta^{-1} \delta^{jb}$ and $\delta^{-1} \delta^{j(a-b)+1-a-2\epsilon}$ to vanish when $\delta \rightarrow 0$. When applied to the case $j = 2$, the above requires that $b > 1/2$ and also that $b < a/2 < 1/2$, so that for $j = 2$ the conditions cannot be fulfilled.

Instead if $j \geq 3$ by choosing

$$a \in \left(\frac{1}{2}, 1\right) \text{ and } b \in \left(\frac{1}{3}, \frac{2}{3}a\right) \text{ so that } 2a - 3b > 0$$

we have for ϵ sufficiently small

$$-1 + jb > 0, \quad -1 + j(a - b) + 1 - a - 2\epsilon > 0, \quad \forall j \geq 3 \tag{5.9}$$

The analysis of $T_{C,D}^{(2)}$ requires a more refined estimate which is the content of the next lemma.

Lemma 5.3 *For any $a > 0$*

$$P^\gamma \left[\frac{1}{N} T_{C,D}^{(2)} > \delta^{2-a} \right] \leq c \frac{\delta^a}{N} \tag{5.10}$$

Proof A cluster C_2 which contains the arrow (x, y) is equal to

$$\begin{aligned} &\{(x, y), (y, z)\} \cup \{(x, y), (z, y)\} \cup \{(x, y), (x, z)\} \\ &\cup \{(x, y), (z, x)\} \cup \{(x, y), x\} \cup \{(x, y), y\} \end{aligned}$$

in the last two terms besides the arrow (x, y) the single points x and y are marked points and in the previous terms z is any point different from x and y .

Since the estimates are similar for simplicity we just examine the case with two arrows, $(x, y), (y, z)$. We denote by $\eta_{x,y,z} \in \{0, 1\}$ the indicator of this set, thus

$$\eta_{x,y,z} = \mathbf{1}_{|L(x,y)|=1} \mathbf{1}_{|L(y,z)|=1} \mathbf{1}_{C_2=\{(x,y),(y,z)\}}$$

We call

$$F = \frac{1}{N} \sum_{x \in C} \sum_{y \in D} \sum_z \eta_{x,y,z}$$

We first compute the expectation recalling that the clocks are independent (see the paragraph *A realization of the process: the clock process*) and using (3.4) we get

$$\begin{aligned} \mathbb{E}[F] &= \frac{1}{N^3} \sum_{x \in C} \sum_{y \in D} \sum_z e^{-\lambda^* \delta (A_\gamma(x,y) + A_\gamma(y,z))} (\lambda^* \delta)^2 N A_\gamma(x, y) N A_\gamma(y, z) \\ &\leq c \xi^d \delta^2 \end{aligned}$$

We next compute the variance and using independence we get

$$\begin{aligned} &E^\gamma \left[(F - E^\gamma[F])^2 \right] \\ &= \frac{1}{N^4} \sum_{x \in C, y \in D} \sum_z e^{-\lambda^* \delta (A_\gamma(x,y) + A_\gamma(y,z))} (\lambda^* \delta)^2 N A_\gamma(x, y) N A_\gamma(y, z) \\ &\quad - E^\gamma[F]^2 \\ &\leq \frac{1}{N} c \xi^d \delta^2 \end{aligned}$$

Thus (5.10) follows from Chebishev inequality. □

Proof of Theorem 5.1 We fix \bar{x} and \bar{y} as in the hypothesis of the Theorem and we call $\bar{C} = C(\bar{x})$ and $\bar{D} = D(\bar{y})$.

We write (recall (5.6))

$$\begin{aligned} & |N^{-1} \mathcal{M}_{\bar{C}, \bar{D}; i} - \lambda^* \delta N A_\gamma(\bar{x}, \bar{y}) v_{k, n\delta}(\bar{x}) v_{i, n\delta}(\bar{y})| \\ & \leq |S_N - \lambda^* \delta N A_\gamma(\bar{x}, \bar{y}) v_{k, n\delta}(\bar{x}) v_{i, n\delta}(\bar{y})| + \frac{1}{N} R_{\bar{C}, \bar{D}; i} + \frac{1}{N} T_{\bar{C}, \bar{D}} \end{aligned} \tag{5.11}$$

where

$$S_N = \frac{1}{N} \sum_{x \in \bar{C}} \sum_{y \in \bar{D}} \mathbf{1}_{|\underline{t}(x, y)|=1}, \tag{5.12}$$

The term $R_{\bar{C}, \bar{D}; i}$ has been treated in Lemma 5.2 and the one with $T_{\bar{C}, \bar{D}}$ is estimated in (5.8) for $j = 3$ and Lemma 5.3 for $j = 2$.

We first compute $E^\gamma[S_N]$, from (5.4) we get

$$\begin{aligned} E^\gamma[S_N] &= e^{-\lambda^* \delta A_\gamma(\bar{x}, \bar{y})} \lambda^* \delta N A_\gamma(\bar{x}, \bar{y}) \frac{|\bar{C}|}{N} \frac{|\bar{D}|}{N} \\ &= e^{-\lambda^* \delta A_\gamma(\bar{x}, \bar{y})} \lambda^* \delta N A_\gamma(\bar{x}, \bar{y}) v_{k, n\delta}(\bar{x}) v_{i, n\delta}(\bar{y}) \end{aligned} \tag{5.13}$$

By (3.4) $N A_\gamma(\bar{x}, \bar{y}) \leq c \lambda^* \xi^d$ so that the right hand side of (5.13) is bounded by $\delta \lambda^* c \xi^d$. Since the clocks are independent we get

$$\begin{aligned} \mathbb{E}^\gamma \left[(S_N - \mathbb{E}^\gamma[S_N])^2 \right] &= \frac{1}{N^2} \sum_{x \in \bar{C}} \sum_{y \in \bar{D}} \{ P^\gamma(|\underline{t}(x, y)| = 1) - P^\gamma(|\underline{t}(x, y)| = 1)^2 \} \\ &\leq c \frac{\delta \lambda^* \xi^d}{N} \end{aligned} \tag{5.14}$$

By (3.4)

$$1 - e^{-\lambda^* \delta A_\gamma(\bar{x}, \bar{y})} \leq c \frac{\delta \xi^d}{N} \tag{5.15}$$

Thus, given any $\theta > 0$ by Chebishev inequality and using (5.14) and (5.15) we get

$$P^\gamma \left[|S_N - \lambda^* \delta N A_\gamma(\bar{x}, \bar{y}) v_{k, n\delta}(\bar{x}) v_{i, n\delta}(\bar{y})| > N^{-\theta} \right] \leq c \frac{\delta \xi^d}{N} + c \frac{\delta \lambda^* \xi^d}{N^{1-2\theta}} \tag{5.16}$$

We thus get the theorem. □

Probability estimates of \mathcal{M}_D

The analysis of \mathcal{M}_D defined in (4.6) is very similar to the one we did for $\mathcal{M}_{C, D; i}$ and sketched below. The analogue of Theorem 5.1 is:

Theorem 5.4 For any $\theta \in (0, 1)$, $a \in (\frac{1}{2}, 1)$, $b \in (\frac{1}{3}, \frac{2}{3}a)$ and $\epsilon < \frac{1-a}{2}$ there is a constant c so that for all x such that $U_{n\delta}(x) = k$

$$\begin{aligned} & P^\gamma \left[|N^{-1} \mathcal{M}_{D(x)} - \delta v_{k, n\delta}(x)| \leq c(N^{-\epsilon} + \delta N^{-1} + N^{-\theta} + \delta^{3b} + \delta^{-2a}) \right] \\ & \geq 1 - c \left(\frac{\delta^2}{N^{1-\epsilon}} + \frac{\delta}{N} + \frac{\delta \lambda^*}{N^{1-2\theta}} + \delta^{1+2a-3b-2\epsilon} + \frac{\delta^a}{N} \right) \end{aligned} \tag{5.17}$$

We write

$$\mathcal{M}_D = \sum_{x \in D} \kappa_x \mathbf{1}_{|\underline{l}(x)|=1} + R_D, \quad R_D = \sum_{x \in D} \kappa_x \mathbf{1}_{|\underline{l}(x)|>1} \tag{5.18}$$

Proceeding as in the proof of Lemma 5.2 we have (proof is omitted):

Lemma 5.5 *For all $\epsilon > 0$ we have*

$$P^\gamma \left[\frac{1}{N} R_D > N^{-\epsilon} \right] \leq c \frac{\delta^2}{N^{1-\epsilon}} \tag{5.19}$$

Analogously to (5.5) we write

$$\begin{aligned} \kappa_x \mathbf{1}_{|\underline{l}(x)|=1} &= \mathbf{1}_{|\underline{l}(x)|=1} \mathbf{1}_{\{x=C_1\}} + \kappa_x \mathbf{1}_{|\underline{l}(x)|=1} \sum_{j \geq 2} \sum_{\mathbf{C}_j} \mathbf{1}_{\{x \in \mathbf{C}_j\}} \\ &= \mathbf{1}_{|\underline{l}(x)|=1} + \mathbf{1}_{|\underline{l}(x)|=1} [\kappa_x - 1] \sum_{j \geq 2} \sum_{\mathbf{C}_j} \mathbf{1}_{\{x \in \mathbf{C}_j\}} \end{aligned} \tag{5.20}$$

Thus

$$\left| \frac{1}{N} \mathcal{M}_D - \frac{1}{N} \sum_{x \in D} \mathbf{1}_{|\underline{l}(x)|=1} \right| \leq \frac{1}{N} R_D + \frac{1}{N} T_D \tag{5.21}$$

where

$$T_D = T_{C,D}^{(2)} + \sum_{x \in D} \sum_{j \geq 3} \sum_{\mathbf{C}_j} \mathbf{1}_{\{x \in \mathbf{C}_j\}} \tag{5.22}$$

The following lemma is a consequence of Corollary 4.4 and an argument very similar to the one of Lemma 5.3:

Lemma 5.6 *For any $a \in (\frac{1}{2}, 1)$ and $b \in (\frac{1}{3}, \frac{2}{3}a)$ we have*

$$P^\gamma \left[\frac{1}{N} \sum_{x \in D} \sum_{j \geq 3} \sum_{\mathbf{C}_j} \mathbf{1}_{\{x \in \mathbf{C}_j\}} > \delta^{3b} \right] \leq c \delta^{1+2a-3b-2\epsilon}, \quad P^\gamma \left[\frac{1}{N} T_D^{(2)} > \delta^{2-a} \right] \leq c \frac{\delta^a}{N} \tag{5.23}$$

Proof The first inequality is the estimate in (4.4) for $j^* = 3$. We thus prove only the second one. A cluster \mathbf{C}_2 which contains the marked point x is equal to

$$\{x, (x, y)\} \cup \{x, (y, x)\}$$

We only examine the case $\{x, (x, y)\}$ the other being similar. We call

$$F = \frac{1}{N} \sum_{x \in D} \sum_{y \in D} \eta_{x,y}, \quad \eta_{x,y} = \mathbf{1}_{|\underline{l}(x,y)|=1} \mathbf{1}_{|\underline{l}(x)|=1}$$

Using that the clocks are independent and (3.4) we get

$$E^\gamma [F] = \frac{1}{N} \sum_{x \in D} \sum_{y \in D} \left(e^{-\delta} \delta \{ e^{-\lambda^* \delta A_\gamma(x,y)} \lambda^* \delta A_\gamma(x, y) \} \right) \leq c \xi^d \delta^2$$

We next compute the variance and using independence we get

$$E^\gamma \left[(F - E^\gamma[F])^2 \right] = \frac{1}{N^2} \sum_{x \in D} \sum_{y \in D} \delta e^{-\delta} e^{-\lambda^* \delta A_\gamma(x,y)} \lambda^* \delta A_\gamma(x,y) \leq \frac{1}{N} c \xi^d \delta^2$$

The Lemma follows from the Chebishev inequality. □

Proof of Theorem 5.4 We fix \bar{x} as in the hypothesis of the Theorem and we call $\bar{D} = D(\bar{x})$.

Recalling (5.21) we write

$$|N^{-1} \mathcal{M}_D - \delta v_{k,n\delta}(x)| \leq |S_N - \delta v_{k,n\delta}(x)| + \frac{1}{N} R_{\bar{D}} + \frac{1}{N} T_{\bar{D}} \tag{5.24}$$

where

$$S_N = \frac{1}{N} \sum_{x \in \bar{D}} \mathbf{1}_{|\underline{t}(x)|=1}, \tag{5.25}$$

The term $R_{\bar{D}}$ has been treated in Lemma 5.5 and the one with $T_{\bar{D}}$ in Lemma 5.6.

We first compute $\mathbb{E}^\gamma[S_N]$

$$E^\gamma[S_N] = e^{-\delta} \delta \frac{|\bar{D}|}{N} = e^{-\delta} \delta v_{k,n\delta}(\bar{x}) \tag{5.26}$$

Since the clocks are independent we get

$$\mathbb{E}^\gamma \left[(S_N - \mathbb{E}^\gamma[S_N])^2 \right] = \frac{1}{N^2} \sum_{x \in \bar{D}} \{P^\gamma(|\underline{t}(x)| = 1) - P^\gamma(|\underline{t}(x)| = 1)^2\} \leq c \frac{\delta}{N} \tag{5.27}$$

Since $1 - e^{-\delta} \leq c \frac{\delta}{N}$, given any $\theta > 0$ and using (5.27) we get

$$P^\gamma \left[|S_N - \delta v_{k,n\delta}(x)| > N^{-\theta} \right] \leq c \frac{\delta}{N} + c \frac{\delta}{N^{1-2\theta}} \tag{5.28}$$

□

6 Proof of Theorem 3.2

The aim in this section is to study the time-continuum limit of the densities when first $N = (\gamma^{-1} \xi)^d \rightarrow \infty$ and then $\delta \rightarrow 0$. To this end we study the process in the time interval

$$[0, T] = \bigcup_{n < \delta^{-1} T} [n\delta, (n + 1)\delta).$$

Define

$$\begin{aligned} \mathcal{X}'_{N,\delta}(n) = & \bigcap_{i < k} \bigcap_{x,y \in \Omega_\gamma} \left\{ |N^{-1} \mathcal{M}_{C(x),D(y);i}(n) \right. \\ & \left. - \lambda^* \delta N A_\gamma(x,y) v_{k,n\delta}(x) v_{i,n\delta}(y) \right\} \leq c(N^{-\theta} + \delta^{1+\epsilon}) \end{aligned} \tag{6.1}$$

$$\mathcal{X}''_{N,\delta}(n) = \bigcap_{x \in \Omega_\gamma} \left\{ |N^{-1} \mathcal{M}_{D(x)}(n) - \delta v_{k,n\delta}(x)| \leq c(N^{-\theta} + \delta^{1+\epsilon}) \right\}, \tag{6.2}$$

Lemma 6.1 *Let*

$$\mathcal{G}_\gamma = \bigcap_{n\delta \leq T} (\mathcal{X}'_{N,\delta}(n) \cap \mathcal{X}''_{N,\delta}(n)) \tag{6.3}$$

then

$$P^\gamma[\mathcal{G}_\gamma] \geq 1 - \delta^{-1} T c(N^{-1+2\theta} + \delta^{1+\epsilon}) \tag{6.4}$$

Proof Observe that the sets in the curly brackets in (6.1) and (6.2) are constant on the cubes, then the lemma follows from Theorems 5.1 and 5.4. \square

We first take $N \rightarrow \infty$ and then $\delta \rightarrow 0$, thus the probability of \mathcal{G}_γ is as close to 1 as we want if for any δ small enough we take N sufficiently large.

To underline the dependence of v on γ we write below $v^\gamma_{i,n\delta}(y)$ and we rewrite (3.5) as

$$\begin{aligned} v^\gamma_{i,n\delta}(x) &= \frac{1}{N} \sum_{y \in C(x)} \mathbf{1}_{\{U_{n\delta}(y)=i\}}, \\ \underline{v}^\gamma(n\delta) &= \{v^\gamma_{i;n\delta}(x), x \in \Omega_\gamma, i \in \{0, \dots, k\}\}, \quad n\delta \in [0, T]. \end{aligned} \tag{6.5}$$

For $1 \leq i \leq k$ we call

$$\begin{aligned} F_\gamma(x, i; \underline{v}^\gamma(n\delta)) &:= \\ &\{v^\gamma_{i-1;n\delta}(x) - v^\gamma_{i;n\delta}(x)\} \sum_y \lambda^* A_\gamma(y, x) v^\gamma_{k;n\delta}(y) - v^\gamma_{k;n\delta}(x) \mathbf{1}_{i=k} \end{aligned} \tag{6.6}$$

$$F_\gamma(x, 0; \underline{v}^\gamma(n\delta)) := -v^\gamma_{0;n\delta} \sum_y \lambda^* A_\gamma(y, x) v^\gamma_{k;n\delta}(y) + v^\gamma_{k;n\delta}(x) \tag{6.7}$$

Recalling (4.7), (4.8) and (4.9) in \mathcal{G}_γ we have that for any $n \leq \delta^{-1}T$ and $i \in \{0, 1, \dots, k\}$,

$$|v^\gamma_{i;n\delta}(x) - \{v^\gamma_{i;(n-1)\delta}(x) + \delta F_\gamma(x, i; \underline{v}^\gamma((n-1)\delta))\}| \leq c(N^{-\theta} + \delta^{1+\epsilon}) \tag{6.8}$$

having used that $\sum_y A_\gamma(y, x) v^\gamma_{k;n\delta}(y) = \sum_C N A_\gamma(y_C, x) v^\gamma_{k;n\delta}(y_C)$, with y_C any point in C .

Define $u^\gamma(x, i; n\delta), i \in \{0, 1, \dots, k\}, n\delta \leq T$ as the solution of the equations

$$\begin{aligned} u^\gamma(x, i; n\delta) &= u^\gamma(x, i; (n-1)\delta) + \delta F_\gamma(x, i; \underline{u}^\gamma((n-1)\delta)) \\ u^\gamma(x, i; 0) &= v^\gamma_{i;0}(x) \end{aligned} \tag{6.9}$$

Observe that since the initial datum is constant on the cubes C of the partition then also $u^\gamma(x, i; n\delta)$ is constant on the cubes for any n . We thus may call $u_i(C, n) = u^\gamma(x, i; n\delta)$ with any $x \in C$.

Lemma 6.2 *Let $\epsilon < u^\gamma(x, i; 0) < 1 - \epsilon$ (see Definition 3.1), let $T > 0$ and ϵ' such that*

$$(1 - \delta)^{\delta^{-1}T} \epsilon =: \epsilon' \tag{6.10}$$

Then $\epsilon' < u^\gamma(x, i; n\delta) < 1 - \epsilon'$ for all x, i and $n \leq \delta^{-1}T$.

Proof Let C be a cube of the basic partition, and let

$$K_n(C) = \sum_x \lambda^* A_\gamma(x, y) u^\gamma(x, k; n\delta), \quad y \in C \tag{6.11}$$

observing that the right hand side does not depend on which y we take in C . We then have

$$u_i(C, n + 1) - u_i(C, n) = K_n(C) \left(u_{i-1}(C, n) - u_i(C, n) \right) \delta - \delta u_i(C, n) \mathbf{1}_{i=k} \tag{6.12}$$

with $u_{-1}(C, n) \equiv 0$. We call

$$\theta(n) = \min_i u_i(C, n), \quad \Theta(n) = \max_i u_i(C, n) \tag{6.13}$$

The upper bound

Let i such that $\Theta(n + 1) = u_i(C, n + 1)$ and j such that $\Theta(n) = u_j(C, n)$. By (6.12) bounding $u_{i-1}(C, n) \leq u_j(C, n)$ and dropping the last term $-\delta u_i(C, n) \mathbf{1}_{i=k} \leq 0$ we get

$$\begin{aligned} u_i(C, n + 1) - u_j(C, n) &\leq u_i(C, n) - u_j(C, n) + K_n(C) [u_j(C, n) - u_i(C, n)] \delta \\ &\leq -(u_j(C, n) - u_i(C, n)) \left(1 - K_n(C) \delta \right) \end{aligned} \tag{6.14}$$

It follows by induction from (6.14) that $K_n(C) \leq \lambda^*$ for all n . Thus the right hand side of (6.14) is negative for δ small enough and therefore $\Theta(n) \leq \Theta(0) \leq \epsilon$ which proves the upper bound.

The lower bound

Let i such that $\theta(n + 1) = u_i(C, n + 1)$ and j such that $\theta(n) = u_j(C, n)$. We prove below that for δ small enough

$$u_i(C, n + 1) \geq u_j(C, n) - u_j(C, n) \delta \quad \text{namely} \quad \theta(n + 1) \geq \theta(n)(1 - \delta) \tag{6.15}$$

From (6.15) we get for any n such that $n\delta^{-1} \leq T$

$$\theta(n) \geq (1 - \delta)^{\delta^{-1}T} \epsilon =: \epsilon' \tag{6.16}$$

having used Definition 3.1. □

Proof of (6.15) Recalling the definitions of i and j from (6.12) we have

$$\begin{aligned} u_i(C, n + 1) - u_j(C, n) &= u_i(C, n) - u_j(C, n) + K_n(C) [u_{i-1}(C, n) - u_i(C, n)] \delta \\ &\quad - \delta u_i(C, n) \mathbf{1}_{i=k} \end{aligned} \tag{6.17}$$

We bound from below $u_{i-1}(C, n) \geq u_j(C, n)$. We also write the last term as $[(u_i(C, n) - u_j(C, n)) + u_j(C, N)] \delta$ having bounded $\mathbf{1}_{i=k} \leq 1$ and get

$$\theta(n + 1) - \theta(n) \geq \{ [u_i(C, n) - u_j(C, n)] (1 - K_n(C) \delta - \delta) \} - u_j(C, n) \delta \tag{6.18}$$

For δ small enough the curly bracket term is positive and (6.15) is proved. □

Let

$$\| \underline{v}^\gamma(n\delta) - \underline{u}^\gamma(n\delta) \| := \max_{x,i} | v_{i;n\delta}^\gamma(x) - u^\gamma(x, i; n\delta) | \tag{6.19}$$

Lemma 6.3 *In \mathcal{G}_γ we have*

$$\sup_{n:n\delta < T} \| \underline{v}^\gamma(n\delta) - \underline{u}^\gamma(n\delta) \| \leq cT \delta^{-1} (N^{-\theta} + \delta^{1+\epsilon}) \tag{6.20}$$

which vanishes in the limit $N \rightarrow \infty$ and then $\delta \rightarrow 0$.

Proof There is c so that

$$\max_{x,i} |F_\gamma(x, i; \underline{v}^\gamma(n\delta)) - F_\gamma(x, i; \underline{u}^\gamma(n\delta))| \leq c \|\underline{v}^\gamma(n\delta) - \underline{u}^\gamma(n\delta)\| \tag{6.21}$$

and therefore by (6.8)

$$\|\underline{v}^\gamma(n\delta) - \underline{u}^\gamma(n\delta)\| \leq c(1 + \delta) \|\underline{v}^\gamma((n - 1)\delta) - \underline{u}^\gamma((n - 1)\delta)\| + c(N^{-\theta} + \delta^{1+\epsilon}) \tag{6.22}$$

By iteration

$$\|\underline{v}^\delta(n) - \underline{u}^\delta(n)\| \leq \sum_{m=2}^n [c(1 + \delta)]^m (N^{-\theta} + \delta^{1+\epsilon}) \tag{6.23}$$

Since

$$[c(1 + \delta)]^{T\delta^{-1}} \leq c' \equiv c'(T) \tag{6.24}$$

and $n \leq T\delta^{-1}$, we get

$$\|\underline{v}^\gamma(n\delta) - \underline{u}^\gamma(n\delta)\| \leq c'T\delta^{-1}(N^{-2\theta} + \delta^{1+\epsilon}) \tag{6.25}$$

which concludes the prove the Lemma. □

We next define $w^\gamma(x, i, t)$, $t \in [0, T]$, which is the time continuous analogue of u^γ , as the solution of

$$\frac{d}{dt} w^\gamma(x, i, t) = F_\gamma(x, i; \underline{w}^\gamma(t)), \quad \underline{w}^\gamma(t) = \{w^\gamma(x, i, t), x \in \Omega_\gamma, i \in \{0, \dots, k\}\} \tag{6.26}$$

with initial condition $\underline{w}^\gamma(0) = \underline{v}^\gamma(0)$. Then

$$|w^\gamma(x, i, n\delta) - \{w^\gamma(x, i, (n - 1)\delta) + \delta F_\gamma(x, i; \underline{w}^\gamma((n - 1)\delta))\}| \leq c\delta^2 \tag{6.27}$$

and

$$\|\underline{w}^\gamma(n\delta) - \underline{u}^\gamma(n\delta)\| \leq c'T\delta^{-1}\delta^2 \tag{6.28}$$

Therefore

$$\|\underline{w}^\gamma(n\delta) - \underline{v}^\gamma(n\delta)\| \leq cT(\delta + \delta^{-1}N^{-\theta} + \delta^\epsilon), \quad \forall n \leq T\delta^{-1} \tag{6.29}$$

Proof of Theorem 3.2 We call

$$\omega^\gamma(r, i, t) := w^\gamma(\gamma^{-1}r, i, t), \quad r \in \Omega_\gamma^* \tag{6.30}$$

and we observe that ω^γ satisfies

$$\frac{d}{dt} \omega^\gamma(r, i, t) = f_\gamma(r, i; \underline{\omega}^\gamma(t)), \quad \underline{\omega}^\gamma(t) = \{\omega^\gamma(r, i, t), r \in \Omega_\gamma^*, i = 0, \dots, k, t \in [0, T]\} \tag{6.31}$$

with initial condition $\omega^\gamma(r, i, 0) = w^\gamma(r, i, 0)$ and

$$f_\gamma(r, i; \underline{\omega}^\gamma(t)) := \{\omega^\gamma(r, i - 1, t) - \omega^\gamma(r, i, t)\} \sum_{C^\xi} \lambda^* N A_\gamma(\gamma^{-1}r', \gamma^{-1}r) \omega^\gamma(r', k, t) \tag{6.32}$$

where r' is any point in C^ξ . We have used that w^γ is constant on the cubes $C^{\gamma, \xi}$ as well as A_γ . We now observe that (recall (3.6))

$$\lim_{\gamma \rightarrow 0} \gamma^{-d} A_\gamma(\gamma^{-1}r', \gamma^{-1}r) = \xi^d A_\xi(r', r) \tag{6.33}$$

Thus, since $N = (\gamma^{-1}\xi)^d$ we get that

$$\lim_{\gamma \rightarrow 0} \|\underline{\omega}^\gamma(t) - \underline{\varphi}_\xi(t)\| = 0, \quad \underline{\varphi}_\xi(t) = \{\varphi_\xi(r, i, t), r \in \Omega, i \in \{0, \dots, k\}\} \tag{6.34}$$

From Lemma 6.1, (6.29) and (6.34) we then get

$$\lim_{\delta \rightarrow 0} \lim_{\gamma \rightarrow 0} P^\gamma \left[\|\underline{\varphi}_\xi(t) - v^\gamma(n\delta)\| \leq T(\delta + \delta^{-1}N^{-\theta} + \delta^\epsilon) \right] = 1$$

which proves the Theorem. □

7 Stronger Version of Theorems 3.2 and 3.3

In this Section we prove Theorem 3.3 together with a stronger version of Theorem 3.2.

We introduce some new notation and definitions besides those in the previous sections.

We fix a time $t = n_0\delta_0$ for some n_0 and δ_0 . Since we consider the parameter δ of the form 2^{-n_2} with $n_2 \in \mathbb{N}$ we then have that for any $\delta < \delta_0$ there is m so that $t = m\delta$.

We define $v_{C, i; t}$ with $C \in \pi_\xi$, $\xi = 2^{-n_3}$, $n_3 \in \mathbb{N}$, $i \in \{0, 1, \dots, k\}$.

$$v_{C, i; t} = \frac{1}{|C|} \sum_{r \in C} \mathbf{1}_{U_t(r)=i}, \quad |C| = \xi^d \tag{7.1}$$

where by an abuse of notation we have called $U(r)$, $r = \gamma x$, the potential $U(x)$; $U_t(r)$ is the potential at time t .

So far we have studied the one-body correlation functions. In the next theorem we study the many body space-time correlations, namely the law $\mathcal{P}_{\underline{v}}^{\gamma, \delta, \xi}$ of the finite dimensional distribution $\underline{v} = (v_{C_\ell, i_\ell; t_\ell}, \ell = 1, 2, \dots, m)$ in the limit first $\gamma \rightarrow 0$ then $\delta \rightarrow 0$ and finally $\xi \rightarrow 0$.

Theorem 7.1 *With the above notation*

$$\lim_{\xi \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{\gamma \rightarrow 0} P_{\underline{v}}^{\gamma, \delta, \xi} = P_{\underline{v}} \tag{7.2}$$

where $P_{\underline{v}}$ is supported by $\underline{w} = (w_1, \dots, w_m)$ with

$$w_\ell = \frac{1}{|C_\ell|} \int_{C_\ell} dr \rho(r, i_\ell; t_\ell), \quad \ell = 1, 2, \dots, m \tag{7.3}$$

ρ is the solution of (2.2)-(2.4).

Proof Recalling the definition of ω^γ in (6.30) we write

$$v_{C_\ell, i_\ell; t_\ell} = \{v_{C_\ell, i_\ell; t_\ell} - \omega^\gamma(r, i_\ell; t_\ell)\} + \omega^\gamma(r, i_\ell; t_\ell) \quad r \in C_\ell \tag{7.4}$$

In the set \mathcal{G}_γ defined in (6.3) the curly bracket is uniformly bounded by $c(N^{-\theta} + \delta^\epsilon)$ and by Lemma 6.1 $P^\gamma(\mathcal{G}_\gamma) \rightarrow 1$ as $\gamma \rightarrow 0$. Thus in the limits $N \rightarrow \infty$ and then $\delta \rightarrow 0$ the curly bracket goes to 0.

From (6.34) $\omega^\gamma(r, i_\ell; t_\ell)$ converges as $\gamma \rightarrow 0$ to $\varphi(r, i_\ell; t_\ell) \equiv \varphi_\xi(r, i_\ell; t_\ell)$ solution of (3.7), (3.8), (3.9) where the convolution term $A \equiv A_\xi$ is defined in (3.6) and

$$\lim_{\xi \rightarrow 0} A_\xi(r, r') = \lim_{\xi \rightarrow 0} \frac{1}{\xi^{2d}} \int_{C^\xi(r)} dr_1 \int_{D^\xi(r')} dr'_1 J(r_1, r'_1) = J(r, r') \tag{7.5}$$

From (7.5) it is easy to prove the

$$\lim_{\xi \rightarrow 0} \varphi_\xi(r, i_\ell; t_\ell) = w(r, i_\ell; t_\ell)$$

and the Theorem is proved. □

7.1 Positive, Real Valued Times

Even though the set \mathcal{T} is dense in $[0, T]$, yet it sounds non physical to restrict times to \mathcal{T} . The problem can be fixed easily using a variable time mesh.

To explain the idea we refer first to the simpler case of a single time $t \in [0, T]$ as in Theorem 3.2. Suppose $t \notin \mathcal{T}$. We then consider a mesh $\delta \in \{2^{-n}t, n \in \mathbb{N}\}$, and similarly a second mesh $\delta' \in \{2^{-n}(T - t), n \in \mathbb{N}\}$. We can then use the proof of Theorem 3.2 in $[0, t]$ where the mesh is δ and again the proof of Theorem 3.2 in $[t, T]$ with the mesh δ' . The extension to the case of Theorem 3.3 is similar. We have m times $0 < t_1, \dots < t_m < T$ we then consider a mesh $\delta_1 \in \{2^{-n}t_1, n \in \mathbb{N}\}, \dots, \delta_{m+1} \in \{2^{-n}t_m, n \in \mathbb{N}\}$, and use the proof of Theorem 3.2 in each one of the above time intervals.

8 Extensions

In this Section we study the macroscopic limit of other infection/recovery models.

8.1 Additional Recovery Jumps

Here we consider the case where also individual with potential $i < k$ may recover i.e. the individual at site x with $U(x) = i$ recovers at rate λ_i and the potential after recovery becomes $U(x) = 0$.

The macroscopic equations are:

$$\frac{d}{dt} \rho_i(r, t) = -\lambda_i \rho_i(r, t) + \{\rho_{i-1}(r, t) - \rho_i(r, t)\} \int_{\Omega} dr' \lambda^* \rho_k(r', t) J(r', r) \tag{8.1}$$

when $1 \leq i \leq k - 1$; for $i = k$

$$\frac{d}{dt} \rho_k(r, t) = -\rho_k(r, t) + \rho_{k-1}(r, t) \int_{\Omega} dr' \lambda^* \rho_k(r', t) J(r', r) \tag{8.2}$$

and for $i = 0$:

$$\frac{d}{dt} \rho_0(r, t) = \sum \lambda_i \rho_i(r, t) + \rho_k(r, t) - \rho_0(r, t) \int_{\Omega} dr' \lambda^* \rho_k(r', t) J(r', r) \tag{8.3}$$

The proof is similar to the proof of Theorem 7.1 and omitted.

8.2 The Excitatory-Inhibitory Network Model

Referring to a neural network here we consider excitatory and inhibitory neurons; both neurons have a potential in $\{0, \dots, k\}$. When an excitatory neuron with potential k fires the potentials of all the other neurons with potential $< k$ increase by 1. Similarly when an inhibitory neuron with potential k fires the potentials of all the other neurons with potential > 0 and $< k$ decrease by 1. The rates of firing are $\lambda_1^* k J_\gamma^{(j)}(x, y)$, $j = 1$ for excitatory and $j = 2$ for inhibitory. Besides that neurons with potential k decay at rate 1 to a state with potential 0.

For this model we derive the following macroscopic equations. We denote by $\rho_i^1(r, t)$ the limit macroscopic density at position r and time t of the excitatory neurons with potential i and by $\rho_i^2(r, t)$ the limit density of the inhibitory neurons with potential i .

$$\begin{aligned} \frac{d}{dt} \rho_i^{(j)}(r, t) &= \{\rho_{i-1}^{(j)}(r, t) - \rho_i^{(j)}(r, t)\} \int_{\Omega} dr' \lambda_1^* \rho_k^{(1)}(r', t) J^{(1)}(r', r) \\ &\quad + \{\rho_{i+1}^{(j)}(r, t) - \rho_i^{(j)}(r, t)\} \int_{\Omega} dr' \lambda_2^* \rho_k^{(2)}(r', t) J^{(2)}(r', r) \end{aligned} \tag{8.4}$$

when $1 \leq i \leq k - 1$; for $i = k$

$$\frac{d}{dt} \rho_k^{(j)}(r, t) = -\rho_k^{(j)}(r, t) + \rho_{k-1}^{(j)}(r, t) \int_{\Omega} dr' \lambda_1^* \rho_k^{(1)}(r', t) J^{(1)}(r', r) \tag{8.5}$$

for $i = 0$

$$\begin{aligned} \frac{d}{dt} \rho_0^{(j)}(r, t) &= \rho_k^{(j)}(r, t) + \rho_1^{(j)}(r, t) \int_{\Omega} dr' \lambda_2^* \rho_k^{(2)}(r', t) J^{(2)}(r', r) \\ &\quad - \rho_0^{(j)}(r, t) \int_{\Omega} dr' \lambda_1^* \rho_k^{(1)}(r', t) J^{(1)}(r', r) \end{aligned} \tag{8.6}$$

Convergence is in the sense of the finite dimensional distributions as in Theorem 7.1.

It was shown in [1] that in the excitatory-only network, the only nontrivial steady states are spatially uniform and linearly stable. Including inhibitory neurons introduces spatially non-uniform steady states for certain values of the parameters. For such parameters, the spatially uniform steady states become unstable. This can happen as symmetric inhibitory coupling between two regions causes minor differences in activity levels to grow larger: an increase in activity in one region will suppress the other, which in turn releases inhibition on the first and causes its activity level to rise further.

8.3 General Microscopic Model

Place at each site $x \in \mathbb{Z}^d$ a finite-state, continuous-time Markov chain $U(x, t)$ with state space $S = \{0, \dots, k\}$. For any pair of states i and j there is an intrinsic transition rate from i to j , denoted as $g_\gamma^{ij}(x)$, dependent on the scaling parameter γ and location x . Any other site y at any state $U(y, t) = l \in \{0, \dots, k\}$ will have an additive effect $\lambda_{ijl} J_\gamma^{ijl}(y, x)$ on the transition rates at x from i to j . Then together, for any $i \neq j \in S$, the time-dependent transition rate from i to j , denoted $q_{ij}(x, t)$, is given by

$$q_{ij}(x, t) = g_\gamma^{ij}(x) + \sum_l \lambda_{ijl} \sum_{y \in \mathbb{Z}^d} J_\gamma^{ijl}(y, x) \delta(U(y, t) - l),$$

where $\lambda_{ijl} \geq 0$, $g_{\gamma}^{ij}(x) = g^{ij}(\gamma x)$, and $J_{\gamma}^{ijl}(x, y) = \gamma^d J^{ijl}(\gamma x, \gamma y)$. The functions $g^{ij}(x)$ and $J^{ijl}(x, y)$ describe the intrinsic transition rates and the site-to-site interactions in the scaling limit. We take the assumptions on J^{ijl} to be the same as before, and we assume g^{ij} to be continuous and bounded over x .

The Scaling Limit In the scaling limit $\gamma \rightarrow 0$, the state of the system is described by local state distributions $\vec{v}(x, t) = (v_1(x, t), \dots, v_N(x, t))$ which vary continuously in $x \in \mathbb{R}^d$, and evolve in time according to macroscopic equations, for $0 \leq i \leq k$,

$$v_i' = \sum_{j \neq i} \left[g^{ji} + \sum_l \lambda_{jil} R_{jil}(x) \right] v_j - \left[g^{ij} + \sum_l \lambda_{ijl} R_{ijl}(x) \right] v_i \tag{8.7}$$

with

$$R_{ijl}(x) = \int J^{ijl}(y, x) v_k(y) dy. \tag{8.8}$$

Obtaining the Original, Generalized Contact Process The original, generalized contact process can be recovered by choosing particular g^{ij} and J^{ijl} . Let $g^{k0} = 1$ and all other $g^{ij} = 0$ for $i \neq j$. Next, for $0 \leq i < k$, let $\lambda^{i,i+1,k} = \lambda^*$ and all other $\lambda^{ijl} = 0$. Let $J^{i,i+1,k}$ be defined as for the generalized contact process. Then the microscopic and macroscopic equations are the same as for the generalized contact process.

Adding New Features The E-I neuron model from Sect. 8.2 can be recovered by letting $S = \{0, \dots, k\} \times \{0, \dots, k\}$, where the first coordinate corresponds to the E neuron voltage, and second coordinate the I neuron voltage at a site. Excitatory interactions will correspond to setting the terms $\lambda^{(i,j),(i+1,j),(k,l)}$, $\lambda^{(i,j),(i,j+1),(k,l)}$, as well as the associated J terms, and inhibitory interactions will correspond to setting the terms $\lambda^{(i,j),(i-1,j),(l,k)}$, $\lambda^{(i,j),(i,j-1),(l,k)}$, as well as their associated J terms.

An external drive can be added by making nonzero the terms $g^{(i,j),(i+1,j)}$ and $g^{(i,j),(i,j+1)}$. A neuronal leak could be modeled at least approximately by setting $g^{(i,j),(i-1,j)}$ and $g^{(i,j),(i,j-1)}$.

The proofs of all these extensions follow the same steps as in the proof of Theorem 7.1 with new clock processes associated to $g_{\gamma}^{ij}(x)$ and $\lambda_{ijk} J_{\gamma}^{ijk}(y, x)$. As before all these clocks are independent and we can repeat the arguments used in Sects. 3–7.

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

Appendix A Clusters and Branching Processes

In this appendix we will prove Theorem 4.3. For completeness we first recall some definitions and notation.

A configuration is the set of arrows and marked points with their multiplicity which are described respectively by integer valued functions $m(z, z')$ and $m(z'')$, $m(z, z') = 0$ if the

arrow (z, z') is absent and $m(z'') = 0$ if z'' is not a marked point. Then the probability of a configuration is

$$P(\{m(z, z'), m(z'')\}) = \left\{ \prod_{(z, z')} e^{-\lambda^* \delta A(z, z')} \frac{[\lambda^* \delta A(z, z')]^{m(z, z')}}{m(z, z')!} \right\} \left\{ \prod_{z''} e^{-\delta} \frac{\delta^{m(z'')}}{m(z'')!} \right\} \tag{A1}$$

where $P = P^{A_\gamma, \gamma}$ and $A(z, z') = A_\gamma(z, z')$.

A cluster \mathbf{C} is a maximal connected set of arrows with their multiplicity and marked points with their multiplicity. Let $V = V(\mathbf{C})$ be the set of arrows in \mathbf{C} and let $m(z, z')$ the multiplicity of the arrow (z, z') and $m(z'')$ the multiplicity of the marked point z'' . Call T_V the union of starting and endpoints of the arrows in V . Maximality means that there is no arrow starting from T_V^c and ending at T_V . Then

$$P(\mathbf{C}) = \left\{ \prod_{(z, z') \in V} e^{-\lambda^* \delta A(z, z')} \frac{[\lambda^* \delta A(z, z')]^{m(z, z')}}{m(z, z')!} \right\} \left\{ \prod_{z'' \in T_V} e^{-\delta} \frac{\delta^{m(z'')}}{m(z'')!} \right\} \left\{ \prod_{z \in T_V^c, z' \in T_V} e^{-\lambda^* \delta A(z, z')} \right\} \tag{A2}$$

With reference to (4.1) we fix an arrow (x, y) and write

$$\sum_{j \geq 1} \delta^{-aj} \sum_{\mathbf{C}_j \ni (x, y)} P[\mathbf{C}_j] = S \tag{A3}$$

where

$$S := \sum_{\mathbf{C} \ni (x, y)} P[\mathbf{C}] \delta^{-a|\mathbf{C}|}, \quad a \in (0, 1) \tag{A4}$$

where $|\mathbf{C}|$ is the number of elements in \mathbf{C} , namely

$$|\mathbf{C}| = \sum_{(z, z') \in V} m(z, z') + \sum_{z'' \in T_V} m(z'') \tag{A5}$$

The purpose is to prove

$$S \leq c \delta^{1-a-2\epsilon} \frac{1}{N} \tag{A6}$$

which then implies (4.1).

We perform the sum over $\mathbf{C} \ni (x, y)$ by first summing over the multiplicities and get

$$S = \sum_{V \ni (x, y)} \left\{ \prod_{(z, z') \in V} e^{-\lambda^* \delta A(z, z')} \left[e^{\lambda^* \delta^{1-a} A(z, z')} - 1 \right] \right\} \left\{ \prod_{z'' \in T_V} e^{-\delta} \delta^{1-a} \right\} \times \left\{ \prod_{z \in T_V^c, z' \in T_V} e^{-\lambda^* \delta A(z, z')} \right\} \tag{A7}$$

Since $A(z, z') \leq c \frac{\xi^d}{N}$,

$$|e^{\lambda^* \delta^{1-a} A(z, z')} - 1| \leq c \lambda^* \delta^{1-a} \frac{\xi^d}{N}, \quad e^{-\delta} e^{\delta^{1-a}} \leq 1 + c \delta^{1-a}, \quad e^{-\lambda^* \delta A(z, z')} \leq 1 \tag{A8}$$

We also split $\delta^{1-a} = \delta^{1-a-2\epsilon} \delta^{2\epsilon}$, $\epsilon > 0$ such that $1 - a - 2\epsilon > 0$, and get

$$\begin{aligned} S &\leq \delta^{1-a-2\epsilon} \sum_{V \ni (x, y)} \left\{ \prod_{(z, z') \in V} \left(c \lambda^* \delta^{2\epsilon} \frac{\xi^d}{N} \right) \right\} \left\{ \prod_{z' \in T_V} (1 + c \delta^{1-a}) \right\} \\ &\leq \delta^{1-a-2\epsilon} \sum_{V \ni (x, y)} \left\{ \prod_{(z, z') \in V} \left(c' \lambda^* \delta^{2\epsilon} \frac{\xi^d}{N} \right) \right\} \end{aligned} \tag{A9}$$

with $c' > c(1 + c\delta^{1-a})$. The first factor because there is at least one arrow (namely (x, y) which starts from x). We will prove that the term multiplying $\delta^{1-a-2\epsilon}$ is bounded by c/N and thus prove (A6). The proof will exploit the branching structure of V .

We call x the root of the branching, $(x, z_1^1), \dots, (x, z_{n_1}^1)$, $z_1^1 = y$, the arrows which start from the root x ($n_1 \geq 1$ because $V \ni (x, y)$) and $z_1^1, \dots, z_{n_1}^1$ the nodes of the first generation. From each node z_i^1 of the first generation may or may not start new arrows: if no arrow starts from all the nodes z_i^1 then the branching ends, otherwise we call $z_1^2, \dots, z_{n_2}^2$ the nodes which are the endpoints of the new arrows: these are the nodes of the second generation. Notice that there may be arrow which go back to x , in that case that arrow will not produce descendants because they are already included in the arrows of the first generation. Analogously we call $\{z_1^i, \dots, z_{n_i}^i\}$ the endpoints of the arrows starting from nodes of the $(i - 1)$ -th generation. The branching ends when no arrow starts from the nodes of the last generation.

In terms of the branching the configuration are described by the following parameters:

$$\begin{aligned} k; \quad &(n_i, 1 \leq i \leq k); \quad (z_j^i, 1 \leq i \leq k, 1 \leq j \leq n_i); \\ &(R_j^i, 1 \leq i \leq k - 1, 1 \leq j \leq n_i) \end{aligned}$$

which must satisfy

$$\sum_{j=1}^{n_i} R_j^i = n_{i+1} \tag{A10}$$

where

- $k \geq 1$ is the number of generations,
- z_j^i are the position of the nodes,
- R_j^i is the number of arrows which start from z_j^i .

Writing (A9) in terms of the branching we get

$$S \leq \delta^{1-a-2\epsilon} \sum_{k \geq 1} \sum_{(n_i)} \sum_{(z_j^i)} \sum_{(R_j^i)} \left(c' \lambda^* \delta^{2\epsilon} \frac{\xi^d}{N} \right)^{n_1} \left\{ \prod_{i=1}^{k-1} \left(c' \lambda^* \delta^{2\epsilon} \frac{\xi^d}{N} \right)^{R_1^i + \dots + R_{n_i}^i} \right\}$$

By (A10) and since z_j^i is the endpoint of an arrow, so that it may have at most N values, then the sum over (z_j^i) is bounded by $N^{n_1-1+n_2+\dots+n_k}$ (recall that $z_1^1 = y$) we get

$$\sum_{(z_j^i)} \left(\frac{1}{N}\right)^{n_1} \left\{ \prod_{i=1}^{k-1} \left(\frac{1}{N}\right)^{R_1^i+\dots+R_{n_i}^i} \right\} \leq \frac{1}{N}$$

We thus have

$$\begin{aligned} S &\leq \frac{1}{N} \delta^{1-a-2\epsilon} \sum_{k \geq 1} \sum_{(n_i)} \sum_{(R_j^i)} (c' \lambda^* \delta^{2\epsilon} \xi^d)^{n_1} \prod_{i=1}^{k-1} (c' \lambda^* \delta^{2\epsilon} \xi^d)^{R_1^i+\dots+R_{n_i}^i} \\ &\leq \frac{1}{N} \delta^{1-a-2\epsilon} \sum_{k \geq 1} \sum_{(n_i)} [c' \lambda^* \xi^d]^{n_1+\dots+n_k} \sum_{(R_j^i)} \delta^{2\epsilon n_1} \prod_{i=1}^{k-1} (\delta^{2\epsilon})^{R_1^i+\dots+R_{n_i}^i} \end{aligned}$$

We first estimate the sum on (R_j^{k-1}) which satisfy (A10) and get

$$\sum_{(R_j^{k-1})} (\delta^{2\epsilon})^{R_1^{k-1}+\dots+R_{n_{k-1}}^{k-1}} \leq \left[\sum_{m=0}^{\infty} (\delta^{2\epsilon})^m \right]^{n_{k-1}} \leq (1 + c\delta^{2\epsilon})^{n_{k-1}} \tag{A11}$$

writing $\delta^{2\epsilon} = \delta^\epsilon \delta^\epsilon$ and using (A10) we get

$$\sum_{(R_j^{k-2})} (\delta^{2\epsilon})^{R_1^{k-2}+\dots+R_{n_{k-2}}^{k-2}} \leq \delta^{\epsilon n_{k-1}} \left[\sum_{m=0}^{\infty} (\delta^\epsilon)^m \right]^{n_{k-2}} \leq \delta^{\epsilon n_{k-1}} (1 + c\delta^\epsilon)^{n_{k-2}} \tag{A12}$$

The other sums on (R_j^i) with $i < k - 2$ are estimated as in (A12) getting

$$\begin{aligned} S &\leq \frac{1}{N} \delta^{1-a-2\epsilon} \sum_{k \geq 1} \sum_{(n_i)} [c' \lambda^* \xi^d]^{n_1+\dots+n_k} [\delta^\epsilon (1 + c\delta^\epsilon)]^{n_1+\dots+n_{k-1}} \\ &= \frac{1}{N} \delta^{1-a-2\epsilon} \sum_{k \geq 1} \sum_{(n_i)} [c' \lambda^* \xi^d \delta^{\epsilon/2}]^{n_1+\dots+n_k} [\delta^{\epsilon/2} (1 + c\delta^\epsilon)]^{n_1+\dots+n_{k-1}} \\ &\leq \frac{1}{N} \delta^{1-a-2\epsilon} \sum_{k \geq 1} \left[\sum_{n=1}^{\infty} [\delta^{\epsilon/2} (1 + c\delta^\epsilon)]^n \right]^k \\ &\leq \frac{1}{N} \delta^{1-a-2\epsilon} \sum_{k \geq 1} (c\delta^{\epsilon/2})^k \end{aligned} \tag{A13}$$

where for δ small enough we have bounded $c' \lambda^* \xi^d \delta^{\epsilon/2} < 1$. Thus (A6) follows from (A13). □

The proof of (4.2) is reduced to that of (4.1) when we write

$$\text{l.h.s. of (4.2)} = \frac{1}{N} \sum_x \sum_{m(x) \geq 1} e^{-\delta} \frac{\delta^{m(x)}}{m(x)!} + \frac{1}{N} \sum_{x,y} S_{x,y} \tag{A14}$$

where $S_{x,y} \equiv S$ as in (A4), having made explicit its dependence on x, y .

The first term in (A14) is the contribution of clusters with only the marked point x , the other clusters give rise to the second term in (A14). (4.2) then follows from (A14) and (A6). □

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