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Heat Flow in a Periodically Forced, Thermostatted Chain

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Abstract: We investigate the properties of a harmonic chain in contact with a thermal bath at one end and subjected, at its other end, to a periodic force. The particles also undergo a random velocity reversal action, which results in a finite heat conductivity of the system. We prove the approach of the system to a time periodic state and compute the heat current, equal to the time averaged work done on the system, in that state. This work approaches a finite positive value as the length of the chain increases. Rescaling space, the strength and/or the period of the force leads to a macroscopic temperature profile corresponding to the stationary solution of a continuum heat equation with Dirichlet-Neumann boundary conditions.

1. Introduction

The conversion of mechanical energy to heat, accompanied by the production of entropy, is a very common phenomenon in nature. It occurs on all scales of space and time: from ocean tides to the movement of a charged particle in a fluid under the influence of an electric field. It happens each time we rub our hands or scratch our head. A fully microscopic description of the phenomenon is desirable but very complicated. Here we study an example of this phenomenon in a very simple microscopic model system. In particular, we consider a linear chain (system) of $n+1$ particles, labelled by $x = 0, \dots, n$, in contact with a thermal reservoir at one end and acted upon by a periodic force at its other end.

The interaction of the system with the reservoir is stochastic, modeled as usual by an Ornstein-Uhlenbeck process (the Langevin force). The action of the external force on the other hand, is deterministic, described by a time periodic Hamiltonian. There is

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also a stochastic velocity flip acting on the particles of the system to model non linear interactions, inducing a diffusive behaviour of the energy and producing a finite heat conductivity.

We prove the existence of a unique periodic state approached, as $t \to +\infty$, from any initial state. The average work done on the system over a period is equal to the time averaged heat flux, J_n, going into the reservoir. This is computed explicitly for all n. The diffusive behaviour of the energy implies that the heat flux J_n is proportional to the microscopic gradient ot the temperature. To maintain a spatially constant time averaged macroscopic current J in the limit $n \to \infty$ one needs to have a spatially constant time averaged temperature gradient proportional to J. This can be achieved by appropriately scaling the force and/or period θ_n as a function of n. This leads to a macroscopic spatial temperature profile $T(u)$, where $u = x/n \in [0, 1]$ is the scaled spatial coordinate, given by the stationary solution of the heat equation with a fixed temperature $T(0) = T_-$ (the temperature of the heat reservoir that is placed at the left endpoint of the chain) and a fixed energy current J entering the system on the right, $u = 1$ (where the periodic force is applied). The thermal conductivity κ in the heat equation can be computed explicitly by the Kubo formula for this system (see [2] as well as the comments in Section 10). It is independent of the temperature, so, as a result, $T(u)$ is linear in u, with $\kappa T'(u) = -J$. As already mentioned above, it is the presence of the stochastic flip in the bulk that is responsible for the conversion of the work done by the periodic force into heat that is diffusively transported through the system. Detailed study of the thermalization in the bulk for such stochastic dynamics has been studied in [15].

We note that our setting differs from the typical setup, in which the *stationary* macroscopic energy transport has been studied. E.g. in [3,5,12,17] the chain is placed between heat baths at different temperatures. The stationary temperature profile depends then on the boundary temperatures. As far as we know, the present article is the first to derive a rigorous macroscopic limit for a system in a periodic state induced by a periodic external force. Periodic states of a system under external periodic forcing have been previously considered in [14].

1.1. Description of the model. The configuration of particle positions and momenta are described by

$$(\mathbf{q}, \mathbf{p}) = (q_0, \dots, q_n, p_0, \dots, p_n) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}. \tag{1.1}$$

We should think of the positions q_x as relative displacement from a point, say x in a finite lattice $\{0, 1, \dots, n\}$. The total energy of the chain is defined by the Hamiltonian: $\mathcal{H}_n(\mathbf{q}, \mathbf{p}) := \sum_{x=0}^{n} \mathcal{E}_x(\mathbf{q}, \mathbf{p})$, where the microscopic energy is given by

$$\mathcal{E}_x(\mathbf{q}, \mathbf{p}) := \frac{p_x^2}{2} + \frac{1}{2}(q_x - q_{x-1})^2 + \frac{\omega_0^2 q_x^2}{2}, \quad x = 0, \dots, n, \tag{1.2}$$

with the pinning constant $\omega_0 > 0$. We adopt the convention that $q_{-1} := q_0$.

The microscopic dynamics of the process $\{(\mathbf{q}(t), \mathbf{p}(t))\}_{t \geqslant 0}$ describing the total chain is given in the bulk by

$$\begin{aligned} \dot{q}_x(t) &= p_x(t), && x \in \{0, \dots, n\}, \\ \mathrm{d}p_x(t) &= \left(\Delta_N q_x - \omega_0^2 q_x\right) \mathrm{d}t - 2p_x(t-)\mathrm{d}N_x(\gamma t), && x \in \{1, \dots, n-1\}, \end{aligned} \tag{1.3}$$

and at the boundaries by

$$\begin{aligned} \mathrm{d}p_0(t) &= \Big(q_1(t) - q_0(t) - \omega_0^2 q_0\Big)\mathrm{d}t - 2\gamma p_0(t)\mathrm{d}t + \sqrt{4\gamma T_-}\mathrm{d}\widetilde{w}_-(t) \\ \mathrm{d}p_n(t) &= \Big(q_{n-1}(t) - q_n(t) - \omega_0^2 q_n(t)\Big)\mathrm{d}t + \mathcal{F}_n(t)\mathrm{d}t - 2p_n(t-)\mathrm{d}N_n(\gamma t). \end{aligned} \tag{1.4}$$

Here Δ_N is the Neumann discrete laplacian, corresponding to the choice $q_{n+1} := q_n$ and $q_{-1} = q_0$. We assume that the forcing $\mathcal{F}_n(t)$ is θ_n-periodic, with the period $\theta_n = n^b\theta$, and the amplitude n^a, i.e.

$$\mathcal{F}_n(t) = n^a \mathcal{F}\left(\frac{t}{\theta_n}\right). \tag{1.5}$$

Here $\theta > 0$ and the scaling exponents $a \in \mathbb{R}$, $b \geqslant 0$ are to be adjusted later. We assume $\mathcal{F}(t)$ is a smooth 1-periodic function such that

$$\int_0^1 \mathcal{F}(t)\mathrm{d}t = 0, \qquad \int_0^1 \mathcal{F}(t)^2\mathrm{d}t > 0. \tag{1.6}$$

Processes $\{N_x(t)\}$, $x = 1, \ldots, n$ are independent, Poisson of intensity 1, while $\widetilde{w}_-(t)$ is a standard one dimensional Wiener process, independent of the Poisson processes. The parameter $\gamma > 0$ regulates the intensity of the random perturbations and the Langevin thermostat. We have choosen the same parameter in order to simplify notations, it does not affect the results concerning the macroscopic properties of the dynamics.

The generator of the dynamics is given by

$$\mathcal{G}_t = \mathcal{A}_t + \gamma S_{\text{flip}} + 2\gamma S_-, \tag{1.7}$$

where

$$\mathcal{A}_t = \sum_{x=0}^{n} p_x \partial_{q_x} + \sum_{x=0}^{n} (\Delta_N q_x - \omega_0^2 q_x)\partial_{p_x} + \mathcal{F}_n(t)\partial_{p_n}, \tag{1.8}$$

and

$$S_{\text{flip}} F(\mathbf{p}, \mathbf{q}) = \sum_{x=1}^{n} \Big(F(\mathbf{p}^x, \mathbf{q}) - F(\mathbf{p}, \mathbf{q})\Big), \tag{1.9}$$

where $F : \mathbb{R}^{2(n+1)} \to \mathbb{R}$ is a bounded and measurable function, $\mathbf{p}^x$ is the velocity configuration with sign flipped at the x component, i.e. $\mathbf{p}^x = (p_0', \ldots, p_n')$, with $p_y' = p_y$, $y \neq x$ and $p_x' = -p_x$. Furthermore,

$$S_- = T_- \partial_{p_0}^2 - p_0 \partial_{p_0}. \tag{1.10}$$

The energy currents are given by

$$\mathcal{G}_t \mathcal{E}_x = j_{x-1,x} - j_{x,x+1}, \tag{1.11}$$

with

$$j_{x,x+1} := -p_x(q_{x+1} - q_x), \qquad \text{if} \quad x \in \{0, \ldots, n-1\}$$

and at the boundaries

$$j_{-1,0} := 2\gamma\left(T_- - p_0^2\right), \qquad j_{n,n+1} := -\mathcal{F}_n(t)\, p_n. \tag{1.12}$$

1.2. Main results. Our first result concerns the existence and uniqueness of a periodic stationary state for the system. Fix $n \geqslant 1$. Following [8] Section 3.2, we define a *periodic stationary probability measure* $\{\mu_t^P, t \in [0, +\infty)\}$ as a solution of the forward equation $\partial_t \mu_t^P = \mathcal{G}_t^* \mu_t^P$ such that $\mu_{t+\theta_n}^P = \mu_t^P$. This condition is equivalent to

$$\int_0^{\theta_n} ds \int_{\mathbb{R}^{2(1+n)}} \mathcal{G}_s F(\mathbf{q}, \mathbf{p}) \mu_s^P(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}) = 0, \tag{1.13}$$

for any smooth test function $F : \mathbb{R}^{2(n+1)} \to \mathbb{R}$.

Suppose that $\{(\mathbf{q}(t), \mathbf{p}(t))\}_{t\geqslant 0}$ is the solution of (1.3)–(1.4) initially distributed according to μ_0^P. Given a measurable function $F : \mathbb{R}^{2(n+1)} \to \mathbb{R}$ integrable w.r.t. each measure $\{\mu_s^P, s \in [0, +\infty)\}$ we denote

$$\overline{F}(t) := \mathbb{E} F\Big(\mathbf{q}(t), \mathbf{p}(t)\Big) = \int_{\mathbb{R}^{2(n+1)}} F(\mathbf{q}, \mathbf{p}) \mu_t^P(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}), \quad t \geqslant 0, \tag{1.14}$$

where $\mathbb{E}$ is the expectation w.r.t. $\mathbb{P}$—the probability measure corresponding to the noises and with initial data distributed by μ_0^P. The function $\overline{F}(t)$ is θ_n-periodic. We denote its time average by

$$\langle\langle F \rangle\rangle_n := \frac{1}{\theta_n} \int_0^{\theta_n} \overline{F}(t) \mathrm{d}t. \tag{1.15}$$

The subscript n in the notation of the average $\langle\langle\cdot\rangle\rangle_n$ will be sometimes omitted, when it is obvious from the context.

Theorem 1.1. *For a fixed $n \geqslant 1$ there exists a unique periodic stationary state $\{\mu_s^P, s \in [0, +\infty)\}$ for the system (1.3)–(1.4). The measures μ_s^P are absolutely continuous with respect to the Lebesgue measure* $\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}$ *and the density* $\mu_s^P(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}) = f_s^P(\mathbf{q}, \mathbf{p})\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}$ *is strictly positive. Furthermore* $\min_x \langle\langle p_x^2 \rangle\rangle_n \geqslant T_-$.

The proof of Theorem 1.1 is contained in Appendix A.

From (1.11) we conclude that the time averaged energy current $\langle\langle j_{x,x+1}\rangle\rangle$ is constant for $x = -1, \ldots, n$. Denote therefore

$$J_n^{a,b} := \langle\langle j_{x,x+1}\rangle\rangle, \quad x = -1, \ldots, n. \tag{1.16}$$

In particular

$$J_n^{a,b} = -\frac{n^a}{\theta_n} \int_0^{\theta_n} \mathcal{F}\left(\frac{s}{\theta_n}\right) \overline{p}_n(s) \mathrm{d}s = 2\gamma\Big(T_- - \langle\langle p_0^2 \rangle\rangle\Big). \tag{1.17}$$

We prove, see Theorem 3.1 below, that if $b - a = 1/2$, $a \leqslant 0$ and $b \geqslant 0$ (recall that $\theta_n = n^b\theta$), then

$$n J_n^{a,b} = J^{a,b} + o(1), \quad \text{as } n \to +\infty, \tag{1.18}$$

where $J^{a,b} < 0$ is a constant given by an explicit formula, see (3.3).

In our main result, see Theorem 3.4 below, we prove the convergence, in the weak sense, of the time averaged energy distribution $\langle\langle \mathcal{E}_x \rangle\rangle$ to a linear macroscopic profile

$$\langle\langle \mathcal{E}_{[nu]} \rangle\rangle \underset{n\to\infty}{\longrightarrow} T(u) = T_- - \frac{4\gamma J u}{D}, \quad u \in [0, 1], \tag{1.19}$$

where the constant $D > 0$ is the thermal diffusion coefficient given by formula (3.10) (or equivalent formulas (3.11) and (10.3)). We also prove energy equipartition (see Lemma 8.3), so that the difference $\langle\langle \mathcal{E}_x \rangle\rangle - \langle\langle p_x^2 \rangle\rangle$ converge weakly to 0 as $n \to \infty$, and as a result (1.19) is deduced from the same convergence for the averaged kinetic energy (cf Theorem 8.1, where the convergence is in a stronger L^2 sense).

Our second result, see Theorem 9.1 below, deals with the question of the vanishing of the fluctuations of the kinetic energy functional in the case when the period of the force is of a fixed microscopic size. More precisely, supposing that $b = 0$ and $a = -1/2$, we prove that there exists a constant $C > 0$ such that

$$\sum_{x=0}^{n} \int_0^\theta \left(\overline{p_x^2}(t) - \langle\langle p_x^2 \rangle\rangle\right)^2 \mathrm{d}t \leqslant \frac{C}{n^2}, \quad n = 1, 2, \dots \tag{1.20}$$

1.3. About the proof. The macroscopic equation for the energy transport emerges from an exact *fluctuation-dissipation* decomposition of the energy current

$$j_{x,x+1} = \mathcal{G}_t \mathfrak{f}_x - \frac{1}{4\gamma} \nabla \mathfrak{F}_x, \tag{1.21}$$

where $\mathfrak{F}_x$ and $\mathfrak{f}_x$ are local second order polynomials in the variables $\{q_{x+j}, p_{x+j}, j = -1, 0, 1\}$ (see (5.1), (5.2) and (5.3) for the precise definitions, valid also at the boundaries). After taking the time average we obtain

$$\langle\langle j_{x,x+1} \rangle\rangle = -\frac{1}{4\gamma} \nabla \langle\langle \mathfrak{F}_x \rangle\rangle. \tag{1.22}$$

Then we establish first (1.18), that takes care of the left hand side of (1.22), i.e. $n\langle\langle j_{x,x+1} \rangle\rangle \to J^{a,b}$. The explicit calculation involved in computing $J^{a,b}$ rely only on the first moments of the periodic states (see Sect. 4).

Note that the equilibrium average at temperature T of $\mathfrak{F}_x$ equals DT (with $D > 0$ given by (3.10)). Thus, all we need to prove is a kind of a local equilibrium that allows to conclude that $\langle\langle \mathfrak{F}_{[nu]} \rangle\rangle \sim DT(u)$, $u \in [0, 1]$. This is the main part of the work. It involves proving the convergence of the second moments of the positions and momenta. The most difficult part is to establish an a priori upper bound for the time average of the total energy that proves it does not grow faster than the size of the system n (cf. (7.10)).

In Sect. 6 we derive a closed system of equations for the time averages of the covariance matrix (6.17) that involves the discrete Neumann laplacian Δ_N. After performing some manipulations with these equations we obtain that the time averaged position covariances are given by the Green's function $\left(\omega_0^2 - \Delta_N\right)^{-1}(x, x')$ mutiplied by the local temperature (i.e. $\langle\langle p_x^2 \rangle\rangle$) plus an error that is proportional to the averaged current, that is small (of order $O(1/n)$). In the bulk we have that

$$\left(\omega_0^2 - \Delta_N\right)^{-1}(x, x') \underset{n\to\infty}{\longrightarrow} \left(\omega_0^2 - \Delta\right)^{-1}(x - x'). \tag{1.23}$$

Here Δ is the discrete laplacian on $\mathbb{Z}$, that gives the covariance matrix of the positions of the *infinite* system in equilibrium. But in order to obtain the correct bounds on the total energy we need a careful control the behavior of the Green's function at the boundaries, cf Lemma B.2 in Appendix B.

Once the energy bound is established, the next step is to prove local equilibrium, which is contained in Proposition 8.2. After this step the proof of the main result follows directly, see Theorem 8.1.

In Appendix A we prove the existence of the periodic measure. The manipulations done with the equations for the covariance matrix use some ideas from [5]. The harmonic dynamics with self-consistent Langevin reservoirs considered in that article has a similar covariance equations of the corresponding stationary state, see Sect. 10.

Section 9 contains the proof of the result concerning the vanishing size of time variance of the kinetic energy, see Theorem 9.1. Appendix C is devoted to the presentation of the proofs of auxiliary facts formulated in Sect. 9. Some details of the calculations from this section are presented in the internet supplement [9].

2. Some Preliminaries and Notation

2.1. The dynamics of periodic means. Define the averages in the periodic state:

$$\begin{aligned} \overline{p}_x(s) &:= \int_{\mathbb{R}^{2(n+1)}} p_x \mu_s^P(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}), \\ \overline{q}_x(s) &:= \int_{\mathbb{R}^{2(n+1)}} q_x \mu_s^P(\mathrm{d}\mathbf{q}, \mathrm{d}\mathbf{p}), \quad x = 0, \dots, n. \end{aligned} \tag{2.1}$$

They satisfy

$$\begin{aligned} \dot{\overline{q}}_x &= \overline{p}_x, \\ \dot{\overline{p}}_x &= \left(\Delta_{\mathrm{N}} - \omega_0^2\right)\overline{q}_x - 2\gamma \overline{p}_x + \delta_{x,n}\mathcal{F}_n(t), \quad x \in \{0, \dots, n\}. \end{aligned} \tag{2.2}$$

Here Δ_{N} is the Neumann discrete laplacian, subject to the boundary condition $q_{-1} = q_0$, $q_n = q_{n+1}$. We can rewrite the above system using a matrix notation. Let

$$\overline{\mathbf{q}}(t) = \begin{pmatrix} \overline{q}_0(t) \\ \vdots \\ \overline{q}_n(t) \end{pmatrix}, \qquad \overline{\mathbf{p}}(t) = \begin{pmatrix} \overline{p}_0(t) \\ \vdots \\ \overline{p}_n(t) \end{pmatrix}.$$

and $(\mathbf{q}, \mathbf{p})$ be the vector of initial data. We can write that

$$\begin{pmatrix} \overline{\mathbf{q}}(t) \\ \overline{\mathbf{p}}(t) \end{pmatrix} = e^{-At} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} + \int_0^t e^{-A(t-s)} \mathcal{F}_n(s)\, \mathrm{e}_{p,n+1} \mathrm{d}s.$$

Here A is a 2×2 block matrix made of $(n+1) \times (n+1)$ matrices of the form

$$A = \begin{pmatrix} 0 & -\mathrm{Id}_{n+1} \\ -\Delta_{\mathrm{N}} + \omega_0^2 & 2\gamma \mathrm{Id}_{n+1} \end{pmatrix} \tag{2.3}$$

where Id_{n+1} is the $(n+1) \times (n+1)$ identity matrix. We also let $\mathrm{e}_{q,\ell}$ and $\mathrm{e}_{p,\ell}$, $\ell = 1, \dots, n+1$ be the $1 \times 2(n+1)$ column vectors whose components are given by

$$\mathrm{e}_{q,\ell,\ell'} = \delta_{\ell,\ell'} \quad \text{and} \quad \mathrm{e}_{p,\ell,\ell'} = \delta_{n+1+\ell,\ell'}, \quad \ell' = 1, \dots, 2n+2. \tag{2.4}$$

Proposition 2.1. *The spectrum of matrix A is contained in the half plane* $\operatorname{Re}\lambda > 0$. *Thus, there exists $c > 0$ such that*

$$(\lambda + A)^{-1} = \int_0^{+\infty} e^{-(\lambda+A)t} \mathrm{d}t \tag{2.5}$$

is well defined in the half-plane $\operatorname{Re}\lambda > -c$, *with the integral on the right hand side of (2.5) absolutely convergent.*

The proof of the result can be found in Appendix A of [5].

2.2. Time harmonics of the periodic means. Consider the Fourier coefficients of the periodic means

$$\begin{aligned} \widetilde{p}_x(\ell) &= \frac{1}{\theta_n} \int_0^{\theta_n} e^{-2\pi i \ell t/\theta_n} \overline{p}_x(t)\mathrm{d}t, \\ \widetilde{q}_x(\ell) &= \frac{1}{\theta_n} \int_0^{\theta_n} e^{-2\pi i \ell t/\theta_n} \overline{q}_x(t)\mathrm{d}t, \quad \ell \in \mathbb{Z}. \end{aligned} \tag{2.6}$$

They satisfy

$$\begin{aligned} &\frac{2\pi i \ell}{\theta_n} \widetilde{q}_x(\ell) = \widetilde{p}_x(\ell), \\ &\frac{2\pi i \ell}{\theta_n} \widetilde{p}_x(\ell) = \left(\Delta_{\mathrm{N}} - \omega_0^2\right)\widetilde{q}_x(\ell) - 2\gamma\, \widetilde{p}_x(\ell) + n^a \widetilde{\mathcal{F}}(\ell)\delta_{x,n}, \quad x \in \{0, \dots, n\}. \end{aligned} \tag{2.7}$$

Here

$$\widetilde{\mathcal{F}}(\ell) = \int_0^1 e^{-2\pi i \ell t} \mathcal{F}(t)\mathrm{d}t. \tag{2.8}$$

In the particular case when $\ell = 0$ we have

$$\langle\langle q_x \rangle\rangle_n = \widetilde{q}_x(0) \quad \text{and} \quad \langle\langle p_x \rangle\rangle_n = \widetilde{p}_x(0) = 0, \quad x = 0, \dots, n. \tag{2.9}$$

The last equality follows from the first equation of (2.2). Combining the first and the second equations of (2.7) we get

$$0 = -L^n_{\omega_0,\theta_n,\ell} \widetilde{q}_x(\ell) + \; n^a \widetilde{\mathcal{F}}(\ell)\delta_{x,n}, \quad x = 0, \dots, n. \tag{2.10}$$

Here

$$L^n_{\omega_0,\theta,\ell} := \left[\omega_0^2 - \left(\frac{2\pi \ell}{\theta}\right)^2 + i\frac{4\pi \ell \gamma}{\theta} \right] - \Delta \tag{2.11}$$

with the Neumann boundary conditions $\widetilde{q}_{-1}(\ell) = \widetilde{q}_0(\ell)$, $\widetilde{q}_n(\ell) = \widetilde{q}_{n+1}(\ell)$.

2.3. Green's function corresponding to $L^n_{\omega_0,\theta,\ell}$. Denote by $G^n_{\omega_0,\theta,\ell}(x,y)$ the Green's functions corresponding to $L^n_{\omega_0,\theta,\ell}$. It is defined as the solution of

$$\delta_{x,y} = L^n_{\omega_0,\theta,\ell} G^n_{\omega_0,\theta,\ell}(x,y), \quad x,y = 0,\ldots,n. \tag{2.12}$$

This function is given explicitly by

$$G^n_{\omega_0,\theta,\ell}(x,y) = \sum_{j=0}^{n} \frac{\psi_j(x)\psi_j(y)}{\lambda_j + \omega_0^2 - \left(2\pi\ell\theta^{-1}\right)^2 + 4\gamma\pi i\ell\theta^{-1}} \tag{2.13}$$

where λ_j and ψ_j are the respective eigenvalues and eigenfunctions for the discrete Neumann laplacian $-\Delta_{\mathrm{N}}$. They are given by

$$\lambda_j = 4\sin^2\left(\frac{\pi j}{2(n+1)}\right), \quad \psi_j(x) = \left(\frac{2-\delta_{0,j}}{n+1}\right)^{1/2} \cos\left(\frac{\pi j(2x+1)}{2(n+1)}\right), \quad x,j = 0,\ldots,n. \tag{2.14}$$

If $\ell = 0$, then (2.13) defined the Green's function of $\omega_0^2 - \Delta_{\mathrm{N}}$. We will denote it by $G^n_{\omega_0}(x,y)$.

2.4. Green's function of the lattice laplacian. Recall that the lattice gradient and laplacian of any $f : \mathbb{Z} \to \mathbb{R}$ are defined as $\nabla f_x = f_{x+1} - f_x$ and $\Delta f_x = f_{x+1} + f_{x-1} - 2f_x$, $x \in \mathbb{Z}$, respectively.

Suppose that $\omega_0 > 0$. Consider the Green's function of $-\Delta + \omega_0^2$. It is given by, see e.g. [13, (27)],

$$\begin{aligned} G_{\omega_0}(x) = \left(-\Delta + \omega_0^2\right)^{-1}(x) \quad &= \int_0^1 \left\{4\sin^2(\pi u) + \omega_0^2\right\}^{-1} \cos(2\pi u x)du \\ = \frac{1}{\omega_0\sqrt{\omega_0^2+4}} &\left\{1 + \frac{\omega_0^2}{2} + \omega_0\sqrt{1+\frac{\omega_0^2}{4}}\right\}^{-|x|}, \quad x \in \mathbb{Z}. \end{aligned} \tag{2.15}$$

2.5. Some notation. We adopt the following convention. For two sequences (a_n) and (b_n) of real positive numbers we denote $a_n \approx b_n$, $n \geqslant 1$ if there exists $C > 1$ such that $C^{-1}a_n \leqslant b_n \leqslant Ca_n$, for all $n \geqslant 1$.

3. Periodic Stationary Energy Transport

3.1. Asymptotics of the time average of the mean current. Our first result gives an explicit formula for the asymptotics of the time average of the mean current. In what follows we shall also be concerned with the functional

$$I_n^{a,b} := \frac{n^a}{\theta_n} \int_0^{\theta_n} \overline{q}_n(t)\mathcal{F}(t/\theta_n)\mathrm{d}t, \tag{3.1}$$

therefore we give its exact asymptotics, as $n \to +\infty$.

Theorem 3.1. *Suppose that* $\sum_\ell \ell^2|\widetilde{\mathcal{F}}(\ell)|^2 < +\infty$ *and*

$$b - a = \frac{1}{2}, \quad a \leqslant 0 \quad and \quad b \geqslant 0. \tag{3.2}$$

Then,

$$\lim_{n\to+\infty} nJ_n^{a,b} = J^{a,b} := -\left(\frac{2\pi}{\theta}\right)^2 \sum_{\ell\in\mathbb{Z}} \ell^2 \mathcal{Q}^{a,b}(\ell), \tag{3.3}$$

with $\mathcal{Q}^{a,b}(\ell)$ *given by, cf (2.8),*

$$\mathcal{Q}^{-1/2,0}(\ell) = 4\gamma|\widetilde{\mathcal{F}}(\ell)|^2 \int_0^1 \cos^2\left(\frac{\pi z}{2}\right)\left\{\left[4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2 - \left(\frac{2\pi\ell}{\theta}\right)^2\right]^2 + \left(\frac{4\gamma\pi\ell}{\theta}\right)^2\right\}^{-1} \mathrm{d}z \tag{3.4}$$

and

$$\mathcal{Q}^{b-1/2,b}(\ell) = 4\gamma|\widetilde{\mathcal{F}}(\ell)|^2 \int_0^1 \cos^2\left(\frac{\pi z}{2}\right)\left[4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2\right]^{-2} \mathrm{d}z, \quad when\ b > 0. \tag{3.5}$$

Furthermore, we have

$$I_n^{a,a-1/2} = \mathfrak{J}^{a,a-1/2} n^{2a} + o(n^{2a}), \tag{3.6}$$

where

$$\mathfrak{J}^{0,-1/2} := 2\sum_\ell |\widetilde{\mathcal{F}}(\ell)|^2 \int_0^1 \cos^2\left(\frac{\pi z}{2}\right)\left\{4\sin^2(\pi z) + \left[\omega_0^2 - \left(\frac{2\pi\ell}{\theta}\right)^2\right]\right\} \times \left\{\left[4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2 - \left(\frac{2\pi\ell}{\theta}\right)^2\right]^2 + \left(\frac{4\gamma\pi\ell}{\theta}\right)^2\right\}^{-1} dz$$

and for $b > 0$

$$\mathfrak{J}^{b-1/2,b} := 2\sum_\ell |\widetilde{\mathcal{F}}(\ell)|^2 \int_0^1 \cos^2\left(\frac{\pi z}{2}\right)\left\{4\sin^2(\pi z) + \omega_0^2\right\}\left\{4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2\right\}^{-2} dz.$$

The proof of the results is given in Sect. 4.1.

Remark 3.2. It can be shown that, see [9, Appendix D],

$$\mathcal{Q}^{-1/2,0}(\ell) = \frac{\theta|\widetilde{\mathcal{F}}(\ell)|^2}{2\pi\ell}\mathrm{Im}\left(\left\{\frac{2}{\lambda(\omega_0)\sqrt{1+4/\lambda(\omega_0)}} + \frac{1}{2}\right\}\left\{1 + \frac{\lambda(\omega_0)}{2}\left(1 + \sqrt{1 + \frac{4}{\lambda(\omega_0)}}\right)\right\}^{-1}\right),$$

with

$$\lambda(\omega_0) := \omega_0^2 - \left(\frac{2\pi\ell}{\theta}\right)^2 + i\left(\frac{4\gamma\pi\ell}{\theta}\right).$$

Furthermore,

$$\mathcal{Q}^{b-1/2,b}(\ell) = \frac{2\gamma|\widetilde{\mathcal{F}}(\ell)|^2(4+\omega_0^2)}{(\omega_0^4+4\omega_0^2+8)^{3/2}}.$$

From Theorem 3.1 and the definition of $j_{-1,0}$, see (1.12), we immediately conclude the following.

Corollary 3.3. *We have*

$$T_- - \langle\langle p_0^2\rangle\rangle_n = \frac{J^{a,b}}{2\gamma n} + o\left(\frac{1}{n}\right) \tag{3.7}$$

3.2. Asymptotic profile of the periodic averages of the means of the energy function. The following result holds.

Theorem 3.4. *Under the assumptions of Theorem 3.1 we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_x \varphi\left(\frac{x}{n}\right)\langle\langle p_x^2\rangle\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_x \varphi\left(\frac{x}{n}\right)\langle\langle\mathcal{E}_x\rangle\rangle = \int_0^1 \varphi(u)T(u)\mathrm{d}u, \tag{3.8}$$

with

$$T(u) = T_- - \frac{4\gamma J u}{D}, \quad u\in[0,1], \tag{3.9}$$

for any $\varphi \in C[0,1]$. Here J is given by (3.3) and D is defined by, cf (2.15),

$$D = 1 - \omega_0^2\Big(G_{\omega_0}(0) + G_{\omega_0}(1)\Big). \tag{3.10}$$

We present the proof of the theorem in Sect. 8.

A simple calculation, using (2.15), yields an explicit formula for the coefficient D, cf [5, (4.18)],

$$D = \frac{2}{2+\omega_0^2+\omega_0\sqrt{\omega_0^2+4}}. \tag{3.11}$$

Therefore,

$$\lim_{\omega_0\to+\infty} D(\omega_0) = 0, \qquad \lim_{\omega_0\to 0+} D(\omega_0) = 1.$$

In the case $a = -1/2, b = 0$ and θ constant, we can improve the statement of the Theorem 3.4 as we do not neet the time average over the period, more precisely we prove in Sect. 9 that

$$\lim_{n\to\infty}\frac{1}{n}\sum_x\int_0^\theta\left(\overline{p_x^2}(t) - T\left(\frac{x}{n}\right)\right)^2\mathrm{d}t = 0. \tag{3.12}$$

4. The First Moments of Positions and Momenta

4.1. Proof of Theorem 3.1. We only show (3.3). The proof of (3.6) follows from analogous calculations. Using (1.17), the Plancherel identity and then the first equation of (2.7) we get

$$\begin{aligned} J_n^{a,b} &= -n^a \sum_{\ell} \widetilde{\mathcal{F}}(\ell)\widetilde{p}_n(\ell)^* \\ &= \frac{2\pi i n^a}{\theta_n} \sum_{\ell} \widetilde{\mathcal{F}}(\ell)\ell\widetilde{q}_n(\ell)^* = \frac{2\pi i n^a}{\theta_n} \sum_{\ell}\sum_{x} \delta_{x,n}\widetilde{\mathcal{F}}(\ell)\ell\widetilde{q}_x(\ell)^*. \end{aligned} \tag{4.1}$$

Thanks to (2.10) we can further write

$$\begin{aligned} J_n^{a,b} &= \frac{2\pi i}{\theta_n} \sum_{\ell}\sum_{x} \ell\widetilde{q}_x(\ell)^* L^n_{\omega_0,\theta_n,\ell}\widetilde{q}_x(\ell) \\ &= \frac{2\pi i}{\theta_n} \sum_{\ell}\ell \sum_{x} \left\{ |\nabla\widetilde{q}_x(\ell)|^2 + \left[\omega_0^2 - \left(\frac{2\pi\ell}{\theta_n}\right)^2\right] |\widetilde{q}_x(\ell)|^2 \right\} \\ &\quad - 2\gamma \left(\frac{2\pi}{\theta_n}\right)^2 \sum_{\ell}\ell^2 \sum_{x} |\widetilde{q}_x(\ell)|^2. \end{aligned} \tag{4.2}$$

Using the parity of $|\nabla\widetilde{q}_x(\ell)|^2 + \left[\omega_0^2 - \left(\frac{2\pi\ell}{\theta_n}\right)^2\right] |\widetilde{q}_x(\ell)|^2$ we get

$$J_n^{a,b} = -2\gamma \left(\frac{2\pi}{\theta_n}\right)^2 \sum_{\ell}\ell^2 \sum_{x} |\widetilde{q}_x(\ell)|^2. \tag{4.3}$$

From (2.10) we have

$$n^a \widetilde{\mathcal{F}}(\ell)\delta_{x,n} = L^n_{\omega_0,\theta_n,\ell}\widetilde{q}_x(\ell), \quad x = 0, \dots, n. \tag{4.4}$$

Hence, by (2.12),

$$n^a \widetilde{\mathcal{F}}(\ell) G^n_{\omega_0,\theta_n,\ell}(x,n) = \widetilde{q}_x(\ell), \quad x = 0, \dots, n. \tag{4.5}$$

It follows, by (2.13),

$$\begin{aligned} \sum_{x=0}^{n} |\widetilde{q}_x(\ell)|^2 &= n^{2a}|\widetilde{\mathcal{F}}(\ell)|^2 \sum_{x=0}^{n} \left| G^n_{\omega_0,\theta_n,\ell}(x,n)\right|^2 \\ &= \sum_{j=0}^{n} \frac{n^{2a}|\widetilde{\mathcal{F}}(\ell)|^2 \psi_j^2(n)}{[\lambda_j + \omega_0^2 - \left(2\pi\ell\theta_n^{-1}\right)^2]^2 + \left(4\gamma\pi\ell\theta_n^{-1}\right)^2}. \end{aligned} \tag{4.6}$$

A straightforward calculation, using formula (2.13), yields

$$\begin{aligned} G^n_{\omega_0,\theta_n,\ell}(0,n) &= \frac{2}{n+1}\sum_{k=1}^{n}(-1)^k \cos^2\left(\frac{\pi k}{2(n+1)}\right)\left\{4\sin^2\left(\frac{\pi k}{2(n+1)}\right) + \omega_0^2 \right. \\ &\quad \left. - \left(\frac{2\pi\ell}{\theta_n}\right)^2 + \frac{4\gamma\pi i\ell}{\theta_n}\right\}^{-1} + O\left(\frac{1}{n+1}\right) = o(1). \end{aligned}$$

Therefore, we conclude that

$$\sum_x |\widetilde{q}_x(\ell)|^2 = 2|\widetilde{\mathcal{F}}(\ell)|^2 n^{2a} \int_0^1 \cos^2\left(\frac{\pi z}{2}\right) \times \left\{\left[4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2 - \left(\frac{2\pi\ell}{\theta_n}\right)^2\right]^2 + \left(\frac{4\gamma\pi\ell}{\theta_n}\right)^2\right\}^{-1} dz + o\left(n^{2a}\right). \tag{4.7}$$

From (4.3) and (4.7) we get that for a, b satisfying (3.2) we conclude that

$$J_n^{a,b} = -\frac{4\gamma}{n}\left(\frac{2\pi}{\theta}\right)^2 \sum_\ell \ell^2 |\widetilde{\mathcal{F}}(\ell)|^2 \int_0^1 \frac{\cos^2\left(\frac{\pi z}{2}\right)}{\left[4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2 - \left(\frac{2\pi\ell}{\theta_n}\right)^2\right]^2 + \left(\frac{4\gamma\pi\ell}{\theta_n}\right)^2} dz + o\left(\frac{1}{n}\right)$$

and Theorem 3.1 follows. □

4.2. L^2 norms of the position and momentum averages. Denote

$$\langle\langle \overline{q}_x^2 \rangle\rangle := \frac{1}{\theta_n}\int_0^{\theta_n} \overline{q}_x^2(s)ds = \sum_\ell |\widetilde{q}_x(\ell)|^2,$$
$$\langle\langle \overline{p}_x^2 \rangle\rangle := \frac{1}{\theta_n}\int_0^{\theta_n} \overline{p}_x^2(s)ds = \sum_\ell |\widetilde{p}_x(\ell)|^2. \tag{4.8}$$

Using (2.7) we get

$$\sum_x |\widetilde{p}_x(\ell)|^2 = \left(\frac{2\pi\ell}{\theta_n}\right)^2 \sum_x |\widetilde{q}_x(\ell)|^2. \tag{4.9}$$

By virtue of (4.7) we get

$$\sum_x |\widetilde{p}_x(\ell)|^2 = \frac{2}{n}\left(\frac{2\pi\ell|\widetilde{\mathcal{F}}(\ell)|}{\theta}\right)^2 \int_0^1 \cos^2\left(\frac{\pi z}{2}\right) \times \left\{\left[4\sin^2\left(\frac{\pi z}{2}\right) + \omega_0^2 - \left(\frac{2\pi\ell}{\theta_n}\right)^2\right]^2 + \left(\frac{4\gamma\pi\ell}{\theta_n}\right)^2\right\}^{-1} dz + o\left(\frac{1}{n}\right). \tag{4.10}$$

We have proven therefore the following.

Proposition 4.1. *Under the assumptions of Theorem 3.1 we have*

$$\sum_{x=0}^n \langle\langle \overline{q}_x^2 \rangle\rangle \approx n^{2a} \quad \textit{and} \quad \sum_{x=0}^n \langle\langle \overline{p}_x^2 \rangle\rangle \approx \frac{1}{n}, \quad n \geqslant 1. \tag{4.11}$$

5. The Second Moments for the Momentum and Position Variables

5.1. Fluctuation-dissipation relations. Define

$$\begin{aligned} \mathfrak{f}_x &:= \frac{1}{4\gamma}(q_{x+1}-q_x)(p_x+p_{x+1}) + \frac{1}{4}(q_{x+1}-q_x)^2, \qquad x = 0,\dots,n-1, \\ \mathfrak{F}_x &= p_x^2 + (q_{x+1}-q_x)(q_x-q_{x-1}) - \omega_0^2 q_x^2, \qquad x = 0,\dots,n, \end{aligned} \tag{5.1}$$

with the convention that $q_{-1} = q_0$, $q_n = q_{n+1}$. Then

$$\mathcal{G}_t \mathfrak{f}_x = \frac{1}{4\gamma}\nabla\mathfrak{F}_x + j_{x,x+1} + \frac{\delta_{x,n-1}}{4\gamma} n^a \mathcal{F}(t/\theta_n)(q_n - q_{n-1}), \quad x = 0,\dots,n-1. \tag{5.2}$$

After performing the expectation and time averaging we get

$$\begin{aligned} \langle\langle \mathfrak{F}_x \rangle\rangle = \langle\langle \mathfrak{F}_0 \rangle\rangle + \sum_{y=0}^{x-1} \langle\langle \nabla\mathfrak{F}_y \rangle\rangle = \langle\langle \mathfrak{F}_0 \rangle\rangle - 4\gamma J_n x \\ + \frac{\delta_{x,n} n^a}{\theta_n}\int_0^{\theta_n} \mathcal{F}(t/\theta_n)\left(\overline{q}_{n-1}(t) - \overline{q}_n(t)\right) \mathrm{d}t, \qquad x = 1,\dots,n. \end{aligned} \tag{5.3}$$

Remark 5.1. Note that the expectation of $\mathfrak{F}_x$ with respect to the Gibbs gaussian measure on the lattice $\mathbb{Z}$, with the Hamiltonian $\sum_{x\in\mathbb{Z}} \mathcal{E}_x$ and the inverse temperature T^{-1}, is given by

$$T\Big[1 - G_{\omega_0}(1) - G_{\omega_0}(0) + 2G_{\omega_0}(1) - \omega_0^2 G_{\omega_0}(0)\Big]. \tag{5.4}$$

Since G_{ω_0} is the Green's function for $\omega_0^2 - \Delta$, where Δ is the free lattice laplacian, we have

$$1 - G_{\omega_0}(2) - G_{\omega_0}(0) + 2G_{\omega_0}(1) - \omega_0^2 G_{\omega_0}(0) = 1 - \omega_0^2\Big(G_{\omega_0}(0) + G_{\omega_0}(1)\Big) = D \tag{5.5}$$

and the expression in (5.4) equals DT.

6. The Covariance Matrix of the Periodic State

6.1. Dynamics of fluctuations. Denote

$$q'_x(t) := q_x(t) - \overline{q}_x(t) \quad \text{and} \quad p'_x(t) := p_x(t) - \overline{p}_x(t) \tag{6.1}$$

for $x = 0,\dots,n$. From (1.3)–(1.4) and (2.2) we get

$$\begin{aligned} \dot{q}'_x(t) &= p'_x(t), \qquad x \in \{0,\dots,n\}, \\ \mathrm{d}p'_x(t) &= \left(\Delta_{\mathrm{N}} q'_x - \omega_0^2 q'_x\right)\mathrm{d}t - 2\gamma p'_x(t)\mathrm{d}t - 2p_x(t-)\mathrm{d}\widetilde{N}_x(\gamma t), \quad x \in \{1,\dots,n\}, \end{aligned} \tag{6.2}$$

and at the left boundary

$$\mathrm{d}p_0'(t) = \left(\Delta_{\mathrm{N}} q_0' - \omega_0^2 q_0'\right)\mathrm{d}t - 2\gamma p_0'(t)\mathrm{d}t + \sqrt{4\gamma T_-}\mathrm{d}\widetilde{w}_-(t). \tag{6.3}$$

Here $\widetilde{N}_x(t) := N_x(t) - t$. Let

$$\mathbf{X}(t) = \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix}, \quad \overline{\mathbf{X}}(t) = \begin{pmatrix} \overline{\mathbf{q}}(t) \\ \overline{\mathbf{p}}(t) \end{pmatrix},$$

and $\mathbf{X}'(t) = \mathbf{X}(t) - \overline{\mathbf{X}}(t)$. Furthermore, we define

$$\Sigma(\mathbf{p}) = \begin{bmatrix} 0_{n+1} & 0_{n+1} \\ 0_{n+1} & D(\mathbf{p}) \end{bmatrix}, \text{ with } D(\mathbf{p}) = \begin{bmatrix} \sqrt{4\gamma T_-} & 0 & 0 & \dots & 0 \\ 0 & -2p_1 & 0 & \dots & 0 \\ 0 & 0 & -2p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2p_n \end{bmatrix}. \tag{6.4}$$

The symbol 0_n denotes the null $n \times n$ matrix. The solution of (6.2)–(6.3) satisfies

$$\mathbf{X}'(t) = e^{-At}\mathbf{X}'(0) + \int_0^t e^{-A(t-s)}\Sigma\left(\mathbf{p}(s-)\right)\mathrm{d}M(s), \quad t \geqslant 0. \tag{6.5}$$

Here A is defined by (2.3) and $(M(t))_{t \geqslant 0}$ is $2(n+1)$-dimensional vector martingale

$$\mathrm{d}M(s) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{d}\widetilde{w}(s) \\ \mathrm{d}\widetilde{N}_1(\gamma s) \\ \vdots \\ \mathrm{d}\widetilde{N}_n(\gamma s) \end{pmatrix}.$$

Suppose that $\mathbf{X}$ is a random vector that is independent of the noise and distributed according to μ_0^P. Denote by $\overline{\mathbf{X}}$ the vector of its means and by $\mathbf{X}' = \mathbf{X} - \overline{\mathbf{X}}$. For any $\ell \geqslant 0$ define $\mathbf{X}_\ell(t)$, $t \geqslant -\ell\theta$ - the solution of (6.2)–(6.3) that satisfies $\mathbf{X}_\ell(-\ell\theta) = \mathbf{X}$. We call such solutions θ-*periodic*. Note that

$$\mathbf{X}_\ell'(t) = e^{-A(t+\ell\theta)}\mathbf{X}' + \int_{-\ell\theta}^t e^{-A(t-s)}\Sigma\left(\mathbf{p}_\ell(s-)\right)\mathrm{d}M(s), \quad t \geqslant -\ell\theta. \tag{6.6}$$

Obviously, $\left(\mathbf{X}_\ell(t+\theta)\right)_{t \geqslant -\ell\theta}$ has the same law as $\left(\mathbf{X}_\ell(t)\right)_{t \geqslant -\ell\theta}$.

6.2. The covariance matrix. Suppose that $\mathbf{X}(t)$ is a θ-periodic solution of (1.3)–(1.4). Define the vector valued function

$$\overline{\mathfrak{p}^2}(t) = \begin{pmatrix} \mathbb{E}p_0^2(t) \\ \vdots \\ \mathbb{E}p_n^2(t) \end{pmatrix}$$

and the covariance matrix

$$S(t) = \begin{bmatrix} S^{(q)}(t) & S^{(q,p)}(t) \\ S^{(p,q)}(t) & S^{(p)}(t) \end{bmatrix}, \tag{6.7}$$

where

$$S^{(q)}(t) = \Big[\mathbb{E}[q'_x(t)q'_y(t)]\Big]_{x,y=0,\dots,n}, \quad S^{(q,p)}(t) = \Big[\mathbb{E}[q'_x(t)p'_y(t)]\Big]_{x,y=0,\dots,n},$$
$$S^{(p)}(t) = \Big[\mathbb{E}[p'_x(t)p'_y(t)]\Big]_{x,y=0,\dots,n} \quad \text{and} \quad S^{(p,q)}(t) = \Big[S^{(q,p)}(t)\Big]^T. \tag{6.8}$$

Obviously both $\overline{\mathfrak{p}^2}(t)$, $S^{(q)}(t)$, $S^{(p)}(t)$ are θ-periodic and the matrices are symmetric.

We shall also consider $\langle\langle\overline{\mathfrak{p}^2}\rangle\rangle$ and $\langle\langle S^{(\alpha)}\rangle\rangle$, $\alpha \in \{q, p, (p,q), (q,p)\}$ the respecive vector and matrices of time averages.

Given a vector $\mathfrak{y} = (y_0, y_1, \dots, y_n)$, define also the matrix valued function

$$D_2(\mathfrak{y}) = 4\gamma \begin{bmatrix} T_- & 0 & 0 & \dots & 0 \\ 0 & y_1 & 0 & \dots & 0 \\ 0 & 0 & y_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & y_n \end{bmatrix}. \tag{6.9}$$

Let $\Sigma_2(\mathfrak{y})$ be the 2×2 block matrix

$$\Sigma_2(\mathfrak{y}) = \begin{bmatrix} 0_{n+1} & 0_{n+1} \\ 0_{n+1} & D_2(\mathfrak{y}) \end{bmatrix}. \tag{6.10}$$

Proposition 6.1. *The following identities hold*

$$S(t) = \int_0^{+\infty} e^{-As}\Sigma_2\big(\overline{\mathfrak{p}^2}(t-s)\big)e^{-A^Ts}\mathrm{d}s, \quad t \geqslant 0 \tag{6.11}$$

and

$$\langle\langle S\rangle\rangle = \int_0^{+\infty} e^{-As}\Sigma_2\big(\langle\langle\mathfrak{p}^2\rangle\rangle\big)e^{-A^Ts}\mathrm{d}s. \tag{6.12}$$

Proof. Formula (6.12) is an obvious consequence of (6.11), so we only prove the latter. From (6.6) we conclude that

$$S(t) = e^{-A(t+\ell\theta)}\mathbb{E}\Big\{\mathbf{X}'(0) \otimes \Big(\mathbf{X}'(0)\Big)^T\Big\}e^{-A^T(t+\ell\theta)} + \int_{-\ell\theta}^{t} e^{-A(t-s)}\Sigma_2\Big(\overline{\mathbf{p}^2}(s)\Big)e^{-A^T(t-s)}\mathrm{d}s, \quad t \geqslant -\ell\theta. \tag{6.13}$$

Letting $\ell \to +\infty$ and using Proposition 2.1 we obtain

$$S(t) = \int_{-\infty}^{t} e^{-A(t-s)}\Sigma_2\Big(\overline{\mathbf{p}^2}(s)\Big)e^{-A^T(t-s)}\mathrm{d}s, \quad t \in \mathbb{R}. \tag{6.14}$$

Changing variables $s' := t - s$ we conclude (6.11). □

6.3. Structure of the covariance matrix of the periodic averages. By (6.12) and partial integration in time, we have

$$\begin{aligned} AS(t) &= -\int_0^\infty \left(\frac{d}{ds}e^{-As}\right)\Sigma_2\big(\overline{\mathfrak{p}^2}(t-s)\big)e^{-A^T s}\mathrm{d}s \\ &= -\int_0^\infty e^{-As}\Sigma_2\big(\overline{\mathfrak{p}^2}(t-s)\big)A^T e^{-A^T s}\mathrm{d}s + \Sigma_2\big(\overline{\mathfrak{p}^2}(t)\big) \\ &\quad -\int_0^\infty e^{-As}\Sigma_2\big(\overline{\mathfrak{p}^2}'(t-s)\big)e^{-A^T s}\mathrm{d}s \\ &= -S(t)A^T + \Sigma_2\big(\overline{\mathfrak{p}^2}(t)\big) - S'(t). \end{aligned} \tag{6.15}$$

Integrating the above relation over the period we conclude that the matrix $\langle\langle S\rangle\rangle$ satisfies the equation

$$A\langle\langle S\rangle\rangle + \langle\langle S\rangle\rangle A^T = \Sigma_2(\langle\langle \mathfrak{p}^2\rangle\rangle). \tag{6.16}$$

It leads to the following equations on the blocks defined in (6.7) (see (2.3) and (6.9)):

$$\begin{aligned} &\langle\langle S^{(q,p)}\rangle\rangle = \Big[\langle\langle S^{(p,q)}\rangle\rangle\Big]^T = -\langle\langle S^{(p,q)}\rangle\rangle, \\ &\langle\langle S^{(q)}\rangle\rangle(\omega_0^2 - \Delta_{\mathrm{N}}) + 2\gamma\langle\langle S^{(q,p)}\rangle\rangle - \langle\langle S^{(p)}\rangle\rangle = 0, \\ &(\omega_0^2 - \Delta_{\mathrm{N}})\langle\langle S^{(q)}\rangle\rangle + 2\gamma\langle\langle S^{(p,q)}\rangle\rangle - \langle\langle S^{(p)}\rangle\rangle = 0 \\ &(\omega_0^2 - \Delta_{\mathrm{N}})\langle\langle S^{(q,p)}\rangle\rangle - \langle\langle S^{(q,p)}\rangle\rangle(\omega_0^2 - \Delta_{\mathrm{N}}) = D_2(\langle\langle \mathfrak{p}^2\rangle\rangle) - 4\gamma\langle\langle S^{(p)}\rangle\rangle. \end{aligned}$$

From here we conclude

$$\begin{aligned} &\langle\langle S^{(q,p)}\rangle\rangle = -\langle\langle S^{(p,q)}\rangle\rangle, \\ &\langle\langle S^{(p)}\rangle\rangle = \frac{1}{2}\Big\{\langle\langle S^{(q)}\rangle\rangle(\omega_0^2 - \Delta_{\mathrm{N}}) + (\omega_0^2 - \Delta_{\mathrm{N}})\langle\langle S^{(q)}\rangle\rangle\Big\}, \\ &4\gamma\langle\langle S^{(q,p)}\rangle\rangle = (\omega_0^2 - \Delta_{\mathrm{N}})\langle\langle S^{(q)}\rangle\rangle - \langle\langle S^{(q)}\rangle\rangle(\omega_0^2 - \Delta_{\mathrm{N}}), \\ &(\omega_0^2 - \Delta_{\mathrm{N}})\langle\langle S^{(q,p)}\rangle\rangle - \langle\langle S^{(q,p)}\rangle\rangle(\omega_0^2 - \Delta_{\mathrm{N}}) = D_2(\langle\langle \mathfrak{p}^2\rangle\rangle) - 4\gamma\langle\langle S^{(p)}\rangle\rangle. \end{aligned} \tag{6.17}$$

Denote

$$\widetilde{S}^{(q,p)}_{j,j'} = \sum_{x,x'=0}^{n}\langle\langle S^{(q,p)}_{x,x'}\rangle\rangle\psi_j(x)\psi_{j'}(x')$$

and analogously define $\widetilde{S}^{(p)}_{j,j'}$ and $\widetilde{S}^{(q)}_{j,j'}$. The eigenvalues of $\omega_0^2 - \Delta_{\mathrm{N}}$ are given by

$$\mu_j = \omega_0^2 + \lambda_j = \omega_0^2 + 4\sin^2\left(\frac{\pi j}{2(n+1)}\right), \quad j = 0, \ldots, n. \tag{6.18}$$

Then we have the inverse relations

$$\langle\langle S^{(\alpha)}_{x,x'}\rangle\rangle = \sum_{j,j'=0}^{n} \widetilde{S}^{(\alpha)}_{j,j'}\psi_j(x)\psi_{j'}(x'). \tag{6.19}$$

With this notation we can rewrite (6.17) as follows

$$\begin{aligned}
\widetilde{S}^{(q,p)}_{j,j'} &= -\widetilde{S}^{(p,q)}_{j,j'},\\
\widetilde{S}^{(p)}_{j,j'} &= \frac{1}{2}(\mu_j+\mu_{j'})\widetilde{S}^{(q)}_{j,j'}\\
4\gamma\widetilde{S}^{(q,p)}_{j,j'} &= \widetilde{S}^{(q)}_{j,j'}(\mu_j-\mu_{j'}),\\
(\mu_j-\mu_{j'})\widetilde{S}^{(q,p)}_{j,j'} &= 4\gamma\widetilde{F}_{j,j'} - 4\gamma\widetilde{S}^{(p)}_{j,j'}.
\end{aligned} \tag{6.20}$$

where

$$\widetilde{F}_{j,j'} := \sum_{y=0}^{n}\psi_j(y)\psi_{j'}(y)\langle\langle p_y^2\rangle\rangle + \Big(T_- - \langle\langle p_0^2\rangle\rangle\Big)\psi_j(0)\psi_{j'}(0). \tag{6.21}$$

By eliminating $\widetilde{S}^{(q,p)}_{j,j'}$ from the above equations we get

$$\widetilde{S}^{(p)}_{j,j'} = \widetilde{F}_{j,j'} - \frac{(\mu_j-\mu_{j'})^2}{(4\gamma)^2}\widetilde{S}^{(q)}_{j,j'}. \tag{6.22}$$

Thus,

$$\widetilde{S}^{(q)}_{j,j'} = \frac{2}{\mu_j+\mu_{j'}}\widetilde{F}_{j,j'} - \frac{(\mu_j-\mu_{j'})^2}{8\gamma^2(\mu_j+\mu_{j'})}\widetilde{S}^{(q)}_{j,j'}. \tag{6.23}$$

It follows that

$$\widetilde{S}^{(p)}_{j,j'} = \Theta(\mu_j,\mu_{j'})\widetilde{F}_{j,j'},\qquad \Theta(\mu_j,\mu_{j'}) = \left[1+\frac{(\mu_j-\mu_{j'})^2}{8\gamma^2(\mu_j+\mu_{j'})}\right]^{-1}, \tag{6.24}$$

and

$$\widetilde{S}^{(q)}_{j,j'} = \frac{2\Theta(\mu_j,\mu_{j'})}{\mu_j+\mu_{j'}}\widetilde{F}_{j,j'}. \tag{6.25}$$

7. Energy Bounds

Throughout the remainder of the paper we shall always assume that the assumptions of Theorem 3.1 are in force. From (6.24) and (6.21) we have

$$\langle\langle S^{(p)}_{x,x}\rangle\rangle = \sum_{j,j'}\Theta(\mu_j,\mu_{j'})\widetilde{F}_{j,j'}\psi_j(x)\psi_{j'}(x) = \sum_y M_{x,y}\langle\langle p_y^2\rangle\rangle + \big(T_- - \langle\langle p_0^2\rangle\rangle\big)M_{x,0} \tag{7.1}$$

where

$$M_{x,y} := \sum_{j,j'=0}^{n} \Theta(\mu_j, \mu_{j'})\psi_j(x)\psi_{j'}(x)\psi_j(y)\psi_{j'}(y). \tag{7.2}$$

From (6.1) we have

$$\langle\langle p_x^2\rangle\rangle = \langle\langle (p_x')^2\rangle\rangle + \langle\langle \overline{p}_x^2\rangle\rangle.$$

We can further write

$$\langle\langle p_x^2\rangle\rangle - \sum_{y=0}^{n} M_{x,y}\langle\langle p_y^2\rangle\rangle = G_x^{(n)}, \quad \text{where}$$

$$G_x^{(n)} := \big(T_- - \langle\langle p_0^2\rangle\rangle\big)M_{0,x} + \langle\langle \overline{p}_x^2\rangle\rangle. \tag{7.3}$$

7.1. A lower bound on matrix $[M_{x,y}]$. The main result of the present section is the following.

Proposition 7.1. *There exists* $c_* > 0$ *such that*

$$\sum_{x,y=0}^{n} (\delta_{x,y} - M_{x,y}) f_y f_x \geqslant c_* \sum_{x=0}^{n-1} (\nabla f_x)^2, \quad \textit{for any } (f_x) \in \mathbb{R}^{n+1},\ n = 1, 2, \ldots. \tag{7.4}$$

Proof. For any sequence $(f_x) \in \mathbb{R}^{n+1}$ we can write

$$\sum_{x,y=0}^{n} (\delta_{x,y} - M_{x,y}) f_y f_x = \sum_{x,y=0}^{n} \sum_{j,j'=0}^{n} \big(1 - \Theta(\mu_j, \mu_{j'})\big)\, \psi_j(x)\psi_{j'}(x)\psi_j(y)\psi_{j'}(y) f_y f_x$$

$$= \sum_{j,j'=0}^{n} \big(1 - \Theta(\mu_j, \mu_{j'})\big) \left(\sum_{x=0}^{n} \psi_j(x) f_x \psi_{j'}(x)\right)^2. \tag{7.5}$$

An elementary argument (cf (6.24)) shows that there exists $C_* > 0$, such that

$$1 - \Theta(\mu_j, \mu_{j'}) \geqslant C_*(\mu_j - \mu_{j'})^2 = C_* \left(\lambda_j - \lambda_{j'}\right)^2.$$

Therefore, see (2.14),

$$\sum_{x,y=0}^{n} (\delta_{x,y} - M_{x,y}) f_y f_x \geqslant C_* \sum_{j,j'=0}^{n} \left(\lambda_j - \lambda_{j'}\right)^2 \left(\sum_{x=0}^{n} \psi_j(x) f_x \psi_{j'}(x)\right)^2$$

$$= C_* \sum_{x,y=0}^{n} \sum_{j,j'=0}^{n} \left(\lambda_j - \lambda_{j'}\right)^2 \psi_j(x)\psi_{j'}(x)\psi_j(y)\psi_{j'}(y) f_x f_y$$

$$= 2C_* \sum_{x,y=0}^{n} \left\{ \sum_{j=0}^{n} \lambda_j^2 \psi_j^2(x)\delta_{x,y} - \sum_{j,j'=0}^{n} \lambda_j\lambda_{j'}\psi_j(x)\psi_{j'}(x)\psi_j(y)\psi_{j'}(y) \right\} f_y f_x$$

$$= 2C_* \sum_{x,y=0}^{n} f_y f_x \Big\{ \delta_{x,y}\langle \Delta_{\mathrm{N}}^2 \delta_x, \delta_y\rangle - \langle \Delta_{\mathrm{N}}\delta_x, \delta_y\rangle^2 \Big\}.$$

As before, here $\Delta_{\rm N}$ is the Neumann laplacian. A careful calculation shows that

$$\delta_{x,y}\langle\Delta_{\rm N}^2\delta_x,\delta_y\rangle-\langle\Delta_{\rm N}\delta_x,\delta_y\rangle^2=\langle\Delta_{\rm N}\delta_x,\delta_y\rangle.$$

Therefore

$$\sum_{x,y=0}^{n}(\delta_{x,y}-M_{x,y})f_yf_x\geqslant 2C_*\sum_{x=0}^{n-1}(\nabla f_x)^2 \tag{7.6}$$

and the conclusion of the proposition follows. □

Proposition 7.2. *There exists $C>0$ such that*

$$\sup_x|G_x^{(n)}|\leqslant\frac{C}{n},\quad n=1,2,\ldots. \tag{7.7}$$

Proof. The result is a straightforward consequence of Corollary 3.3 and Proposition 4.1. □

From Propositions 7.1 and 7.2 we immediately conclude the following.

Corollary 7.3. *There exists $C>0$ such that*

$$\sum_{x=0}^{n-1}[\langle\langle p_x^2\rangle\rangle-\langle\langle p_{x+1}^2\rangle\rangle]^2\leqslant\frac{C}{n+1}\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle,\quad n=1,2,\ldots. \tag{7.8}$$

Proof. Using (7.4) and then (7.3) we get

$$\begin{aligned}\sum_{x=0}^{n-1}[\langle\langle p_x^2\rangle\rangle-\langle\langle p_{x+1}^2\rangle\rangle]^2&\leqslant c_*^{-1}\sum_{x,y=0}^{n}(\delta_{x,y}-M_{x,y})\langle\langle p_y^2\rangle\rangle\langle\langle p_x^2\rangle\rangle\\&=\frac{1}{c_*}\sum_{x=0}^{n}G_x^{(n)}\langle\langle p_x^2\rangle\rangle\leqslant\frac{C}{c_*(n+1)}\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle\end{aligned} \tag{7.9}$$

and the conclusion of the corollary follows. □

7.2. Upper bound on energy. The main objective of the present section is the following.

Theorem 7.4. *Under assumptions of Theorem 3.1, there exists $C>0$ such that*

$$\frac{1}{n+1}\sum_{x=0}^{n}\langle\langle\mathcal{E}_x\rangle\rangle\leqslant C,\quad n=1,2,\ldots. \tag{7.10}$$

Before presenting the proof of Theorem 7.4 in Sect. 7.3 we show some auxiliary results.

Proposition 7.5. *As $n\to+\infty$ we have*

$$\sum_{x=0}^{n}\langle\langle q_x^2\rangle\rangle=\Big(G_{\omega_0}(0)+o(1)\Big)\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle+O(1)\left(\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle\right)^{1/2}+O(1). \tag{7.11}$$

Proof. Since $\langle\langle \overline{q}_y^2\rangle\rangle$ is of order n^{2a}, see (4.11), and $a \leqslant 0$ (cf (3.2)) we only need to prove (5.1) for $\sum_x \langle\langle S^{(q)}_{x,x}\rangle\rangle$. Using (6.25) and then (6.21) we obtain

$$\begin{aligned}\sum_x \langle\langle S^{(q)}_{x,x}\rangle\rangle &= \sum_j \widetilde{S}^{(q)}_{j,j} = \sum_j \frac{1}{\mu_j}\widetilde{F}_{j,j}\\ &= \sum_y\sum_j \frac{\psi_j(y)^2}{\mu_j}\langle\langle p_y^2\rangle\rangle + (T_- - \langle\langle p_0^2\rangle\rangle)\sum_j \frac{\psi_j(0)^2}{\mu_j} \\ &= \sum_y G^n_{\omega_0}(y,y)\langle\langle p_y^2\rangle\rangle + (T_- - \langle\langle p_0^2\rangle\rangle)G_{n,\omega_0}(0,0),\end{aligned} \tag{7.12}$$

where $G^n_{\omega_0}$ is the Green's function of $\omega_0^2 - \Delta_{\rm N}$, see Sect. 2.3. The second term on the utmost right hand side of (7.12) is of order $\frac{1}{n}$. Concerning the first term, thanks to Lemma B.2, it equals

$$\big(G_{\omega_0}(0) + o(1)\big)\sum_{y=0}^n \langle\langle p_y^2\rangle\rangle + \sum_{y=0}^n \widetilde{H}^{(n)}(y)\langle\langle p_y^2\rangle\rangle, \tag{7.13}$$

where $|\widetilde{H}^{(n)}(y)|$ satisfies estimate (B.3). Hence,

$$\begin{aligned}\left|\sum_{y=0}^n \widetilde{H}^{(n)}(y)\langle\langle p_y^2\rangle\rangle\right| &\leqslant \sum_{y=0}^n |\widetilde{H}^{(n)}_y| \left(\sum_{x=0}^{y-1} |\langle\langle p_{x+1}^2\rangle\rangle - \langle\langle p_x^2\rangle\rangle|\right)\\ &\quad + \langle\langle p_0^2\rangle\rangle \sum_{y=0}^n |\widetilde{H}^{(n)}(y)|.\end{aligned} \tag{7.14}$$

Thanks to (B.3) we have

$$H_* := \sup_{n\geqslant 1}\sum_{y=0}^n |\widetilde{H}^{(n)}(y)| < +\infty.$$

Thus, using (3.7) we conclude that the second term on the right hand side of (7.14) stays bounded, as $N \to +\infty$. The first term can be rewritten by changing order of summation and then estimated using the Cauchy-Schwarz inequality and bound (7.8)

$$\begin{aligned}&\sum_{x=0}^n |\langle\langle p_{x+1}^2\rangle\rangle - \langle\langle p_x^2\rangle\rangle| \left(\sum_{y=x+1}^n |\widetilde{H}^{(n)}_y|\right)\\ &\leqslant \left\{\sum_{x=0}^n \left(\sum_{y=x+1}^n |\widetilde{H}^{(n)}_y|\right)^2\right\}^{1/2} \left\{\sum_{x=0}^n [\langle\langle p_{x+1}^2\rangle\rangle - \langle\langle p_x^2\rangle\rangle]^2\right\}^{1/2}\\ &\leqslant H_*\sqrt{n}\left\{\sum_{x=0}^n [\langle\langle p_{x+1}^2\rangle\rangle - \langle\langle p_x^2\rangle\rangle]^2\right\}^{1/2} \leqslant C\left(\sum_{y=0}^n \langle\langle p_y^2\rangle\rangle\right)^{1/2}\end{aligned} \tag{7.15}$$

and the conclusion of the proposition follows. □

Similarly we obtain

Proposition 7.6. *As $n \to +\infty$ we have*

$$\begin{aligned}
\sum_{x=0}^{n}\langle\langle q_x q_{x-1}\rangle\rangle &= \big(G_{\omega_0}(1)+o(1)\big)\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle + O(1)\left(\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle\right)^{1/2} + O(1),\\
\sum_{x=0}^{n}\langle\langle q_x q_{x+1}\rangle\rangle &= \big(G_{\omega_0}(1)+o(1)\big)\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle + O(1)\left(\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle\right)^{1/2} + O(1),\\
\sum_{x=0}^{n}\langle\langle q_{x+1} q_{x-1}\rangle\rangle &= \big(G_{\omega_0}(2)+o(1)\big)\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle + O(1)\left(\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle\right)^{1/2} + O(1).
\end{aligned} \tag{7.16}$$

Proof. We only prove the first formula of (7.16). The second and third ones follow analogously. Using (6.21) and (6.25) we obtain

$$\begin{aligned}
\sum_{x=0}^{n}\langle\langle S^{(q)}_{x,x-1}\rangle\rangle &= \sum_{x=0}^{n}\sum_{j,j'}\widetilde{S}^{(q)}_{j,j'}\psi_j(x)\psi_{j'}(x-1)\\
&= \sum_{j,j'}\frac{2\Theta(\mu_j,\mu_{j'})\Upsilon_{j,j'}(1)}{\mu_j+\mu_{j'}}\sum_{y}\psi_j(y)\psi_{j'}(y)\langle\langle p_y^2\rangle\rangle\\
&\quad + \left(T_- - \langle\langle p_0^2\rangle\rangle\right)\sum_{j,j'}\frac{2\Theta(\mu_j,\mu_{j'})\Upsilon_{j,j'}(1)\psi_j(0)\psi_{j'}(0)}{\mu_j+\mu_{j'}}.
\end{aligned} \tag{7.17}$$

Here, we have used the convention $\langle\langle S^{(q)}_{0,-1}\rangle\rangle = \langle\langle S^{(q)}_{0,0}\rangle\rangle$ and the notation

$$\begin{aligned}
\Upsilon_{j,j'}(1) &:= \frac{1}{2}\sum_{x=0}^{n}\big[\psi_j(x)\psi_{j'}(x-1)+\psi_{j'}(x)\psi_j(x-1)\big],\\
\Upsilon_{j,j'}(2) &:= \frac{1}{2}\sum_{x=0}^{n}\big[\psi_j(x-1)\psi_{j'}(x+1)+\psi_{j'}(x+1)\psi_j(x-1)\big].
\end{aligned} \tag{7.18}$$

A simple application of trigonometric identities leads to the following formulas.

Lemma 7.7. *For $\ell = 1, 2$*

$$\begin{aligned}
&\Upsilon_{0,j'}(\ell) = 0, \quad j' = 1,\dots,n,\\
&\Upsilon_{j,j'}(\ell) = \frac{1-(-1)^{j+j'}}{n+1}\cos\left(\frac{\pi j'\ell}{2(n+1)}\right)\cos\left(\frac{\pi j\ell}{2(n+1)}\right), \quad j,j' = 1,\dots,n,\ j\neq j'\\
&\Upsilon_{j,j}(\ell) = \cos\left(\frac{\pi\ell j}{n+1}\right), \quad j = 0,\dots,n.
\end{aligned} \tag{7.19}$$

This result implies that the second term on the right hand side of (7.17) is of order $O(1/n)$. The first term equals

$$\sum_j \frac{\Upsilon_{j,j}(1)}{\mu_j} \sum_y \psi_j^2(y)\langle\langle p_y^2\rangle\rangle + R_n, \quad \text{where}$$
$$R_n := \sum_y \rho_n(y)\langle\langle p_y^2\rangle\rangle \quad \text{and}$$
$$\rho_n(y) := \sum_{j\neq j'} \frac{2\Theta(\mu_j, \mu_{j'})\Upsilon_{j,j'}(1)}{\mu_j + \mu_{j'}} \psi_j(y)\psi_{j'}(y).$$

Using formula for $\psi_j(y)$, see (2.14), and for $\Upsilon_{j,j}$ we can rewrite the above expression in the form

$$\widetilde{g}_n(1)\sum_y \langle\langle p_y^2\rangle\rangle + R_n + \widetilde{R}_n + o(1)\sum_y \langle\langle p_y^2\rangle\rangle, \quad \text{where}$$
$$\widetilde{g}_n(1) := \frac{1}{n+1}\sum_j \frac{1}{\mu_j}\cos\left(\frac{\pi j}{n+1}\right),$$
$$\widetilde{R}_n := \sum_y \widetilde{\rho}_n(y)\langle\langle p_y^2\rangle\rangle \quad \text{and}$$
$$\widetilde{\rho}_n(y) := \frac{2}{n+1}\sum_j \frac{1}{\mu_j}\cos\left(\frac{\pi j}{n+1}\right)\cos\left(\frac{\pi j(2y+1)}{n+1}\right).$$

A simple calculation shows that $\widetilde{g}_n(1)$ is the Riemann sum approximation of order $1/n$ of the integral defining $G_{\omega_0}(1)$, see (2.15). An application of Lemma B.1 yields that both $|\rho_n(y)|$ and $|\widetilde{\rho}_n(y)|$ satisfy the bound (B.3). Repeating the estimates done in (7.14) and (7.15) we conclude that

$$|R_n| + |\widetilde{R}_n| \leqslant C\left(\sum_{y=0}^n \langle\langle p_y^2\rangle\rangle\right)^{1/2},$$

where the constant C is independent of n. The conclusion of the lemma follows. □

7.3. Proof of Theorem 7.4. Using (5.1), (7.16) and (5.5) we obtain

$$\begin{aligned}\sum_{x=0}^n \langle\langle \mathfrak{F}_x\rangle\rangle &= \sum_{x=0}^n \Big[\langle\langle p_x^2\rangle\rangle + \langle\langle q_x q_{x+1}\rangle\rangle + \langle\langle q_x q_{x-1}\rangle\rangle - \langle\langle q_{x-1} q_{x+1}\rangle\rangle - (1+\omega_0^2)\langle\langle q_x^2\rangle\rangle\Big] \\ &= (D + o(1))\sum_{x=0}^n \langle\langle p_x^2\rangle\rangle + O(1)\left(\sum_{x=0}^n \langle\langle p_x^2\rangle\rangle\right)^{1/2} + O(1).\end{aligned} \tag{7.20}$$

On the other hand, from (5.3) and (3.6), we conclude

$$
\begin{aligned}
\sum_{x=0}^{n} \langle\langle \mathfrak{F}_x \rangle\rangle &= (n+1)\langle\langle \mathfrak{F}_0 \rangle\rangle - 2\gamma J_n n(n+1), \\
&+ \frac{n^a}{\theta_n} \int_0^{\theta_n} \mathcal{F}(t/\theta_n) \left(\overline{q}_{n-1}(t) - \overline{q}_n(t) \right) \mathrm{d}t \\
&\leqslant (n+1)\langle\langle p_0^2 \rangle\rangle - 2\gamma n^2 J_n + O(n^{2a}) \leqslant C'(n+1)
\end{aligned}
\tag{7.21}
$$

that gives the bound

$$
\frac{1}{n+1} \sum_{x=0}^{n} \langle\langle p_x^2 \rangle\rangle \leqslant C, \quad n = 1, 2, \dots . \tag{7.22}
$$

The conclusion of the theorem then follows from the above estimate and Proposition 7.5. ◻

7.4. H^1 bound on the energy density. As a direct consequence of (7.22) and Corollary 7.3 we have

Corollary 7.8. *There exists $C > 0$ such that*

$$
\sum_{x=0}^{n-1} [\langle\langle p_x^2 \rangle\rangle - \langle\langle p_{x+1}^2 \rangle\rangle]^2 \leqslant C \tag{7.23}
$$

and

$$
\sup_{x=0,\dots,n} \langle\langle p_x^2 \rangle\rangle_n \leqslant C n^{1/2}, \quad n = 1, 2, \dots \tag{7.24}
$$

Proof. Estimate (7.23) is obvious in light of (7.8). To prove (7.24) note that

$$
\begin{aligned}
\langle\langle p_x^2 \rangle\rangle &\leqslant \sum_{y=1}^{n} |\langle\langle p_y^2 \rangle\rangle - \langle\langle p_{y-1}^2 \rangle\rangle| + \langle\langle p_0^2 \rangle\rangle \\
&\leqslant \sqrt{n} \left\{ \sum_{y=1}^{n} [\langle\langle p_y^2 \rangle\rangle - \langle\langle p_{y-1}^2 \rangle\rangle]^2 \right\}^{1/2} + \langle\langle p_0^2 \rangle\rangle \leqslant C\sqrt{n} + \langle\langle p_0^2 \rangle\rangle.
\end{aligned}
$$

◻

The following result stregthens the above estimates.

Proposition 7.9. *There exists $C > 0$ such that*

$$
\begin{aligned}
&\sum_{x=0}^{n-1} [\langle\langle p_x^2 \rangle\rangle - \langle\langle p_{x+1}^2 \rangle\rangle]^2 \leqslant \frac{C}{n+1}, \quad n = 1, 2, \dots, \\
&\sup_{x=0,\dots,n} \langle\langle p_x^2 \rangle\rangle \leqslant C.
\end{aligned}
\tag{7.25}
$$

Proof. From (7.3) and Propositions 7.1 and 4.1 there exists $C > 0$ such that

$$\sum_{x=0}^{n-1}\Big(\langle\langle p_x^2\rangle\rangle - \langle\langle p_{x+1}^2\rangle\rangle\Big)^2 \leqslant C\left|\sum_{x=0}^{n} G_x^{(n)}\langle\langle p_x^2\rangle\rangle\right|$$

$$\leqslant \frac{C}{n+1} + C\sum_{x=0}^{n}\langle\langle \overline{p}_x\rangle\rangle^2\langle\langle p_x^2\rangle\rangle \tag{7.26}$$

$$\leqslant \frac{C}{n+1} + C\sup_x\langle\langle p_x^2\rangle\rangle\sum_{x=0}^{n}\langle\langle \overline{p}_x\rangle\rangle^2 \leqslant \frac{C}{n+1} + \frac{C}{n+1}\sup_x\langle\langle p_x^2\rangle\rangle, \tag{7.27}$$

Using the Cauchy-Schwarz inequality we conclude

$$\sup_x\langle\langle p_x^2\rangle\rangle \leqslant \langle\langle p_0^2\rangle\rangle + \sum_{x=0}^{n-1}\Big|\langle\langle p_x^2\rangle\rangle - \langle\langle p_{x+1}^2\rangle\rangle\Big|$$

$$\leqslant \langle\langle p_0^2\rangle\rangle + \sqrt{n}\left\{\sum_{x=0}^{n-1}\Big(\langle\langle p_x^2\rangle\rangle - \langle\langle p_{x+1}^2\rangle\rangle\Big)^2\right\}^{1/2} \tag{7.28}$$

Denote $D_n := \sum_{x=0}^{n-1}\Big(\langle\langle p_x^2\rangle\rangle - \langle\langle p_{x+1}^2\rangle\rangle\Big)^2$. We can summarize the inequalities obtained as follows: there exists $C > 0$ such that

$$D_n \leqslant \frac{C}{n+1} + \frac{C}{n+1}\sup_x\langle\langle p_x^2\rangle\rangle,$$

$$\sup_x\langle\langle p_x^2\rangle\rangle \leqslant \langle\langle p_0^2\rangle\rangle + \sqrt{n+1}D_n^{1/2} \leqslant \langle\langle p_0^2\rangle\rangle + C + C\sup_x\langle\langle p_x^2\rangle\rangle^{1/2}, \tag{7.29}$$

for all $n = 1, 2, \ldots$. Thus the second estimate of (7.25) follows, which in turn implies the first estimate of (7.25) as well. □

8. Convergence of the Energy Density

8.1. Identification of the macroscopic energy density limit. Proof of Theorem 3.4. Let

$$e_n(u) := \langle\langle p_x^2\rangle\rangle, \quad u \in \Big[\frac{x}{n+1}, \frac{x+1}{n+1}\Big), \quad x = 0, \ldots, n.$$

By Proposition 7.9 and Corollary 7.4 we have

$$\int_0^1 e_n^2(u)\mathrm{d}u \leqslant \Big(\sup_x\langle\langle p_x^2\rangle\rangle\Big)\Big(\frac{1}{n+1}\sum_{x=0}^{n}\langle\langle p_x^2\rangle\rangle\Big) \leqslant C.$$

Let $\widetilde{e}_n(u)$ be piecewise linear function obtained by the linear interpolation between the nodal points $P_x := \Big(\frac{x}{n+1}, \langle\langle p_x^2\rangle\rangle\Big)$, $x = 0, \ldots, n+1$ (here $p_{n+1} = p_n$). By Proposition 7.9

$$\int_0^1 [\widetilde{e}_n'(u)]^2\mathrm{d}u = (n+1)\sum_{x=0}^{n}\Big(\langle\langle p_{x+1}^2\rangle\rangle - \langle\langle p_x^2\rangle\rangle\Big)^2 \leqslant C, \quad n = 1, 2, \ldots.$$

Therefore $(\widetilde{e}_n(u))$ is relatively compact in both $L^2(0,1)$ and $C[0,1]$. Since

$$\int_0^1 [\widetilde{e}_n(u) - e_n(u)]^2 \mathrm{d}u \leqslant \frac{1}{n+1}\sum_{x=0}^{n}\Big(\langle\langle p_{x+1}^2\rangle\rangle - \langle\langle p_x^2\rangle\rangle\Big)^2 \leqslant \frac{C}{(n+1)^2}$$

also $(e_n(u))$ is relatively compact in $L^2(0,1)$.

For some subsequence $n' \to +\infty$ (which we still denote by n) we have therefore

$$\lim_{n\to+\infty} e_n(u) = e_{\text{thm}}(u), \qquad \text{in the } L^2 \text{ sense}$$

and $e_{\text{thm}} \in C[0,1]$. In order to show the convergence of $\frac{1}{n}\sum_x \varphi\left(\frac{x}{n}\right)\langle\langle p_x^2\rangle\rangle$ in (3.8) it suffices therefore to prove the following.

Theorem 8.1. *We have*

$$e_{\text{thm}}(u) = T(u), \quad u \in [0,1], \tag{8.1}$$

where T is given by (3.9).

Proof. Since $\langle\langle p_0^2\rangle\rangle \to T_-$, compactness of $(\widetilde{e}_n(u))$ in $C[0,1]$ implies that $\lim_{u\to 0} e_{\text{thm}}(u) = T_-$. It follows that we only have to verify that

$$J\varphi(1) = \frac{D}{4\gamma}\int_0^1 \varphi''(u)e_{\text{thm}}(u)du, \tag{8.2}$$

for any $\varphi \in C^2[0,1]$ such that $\operatorname{supp}\varphi \subset (0,1]$ and $\varphi'(1) = 0$. Here and below, for abbreviation sake, we let $J := J^{b-1/2,b}$, $J_n := J_n^{b-1/2,b}$ (see (1.17) and (3.3)).

The left hand side of (8.2) is given by

$$\lim_{n\to\infty} nJ_n\varphi\left(1 - \frac{1}{n}\right) = J\varphi(1). \tag{8.3}$$

On the other hand, from (5.2), for n sufficiently large so $\varphi(1/n) = 0$

$$\begin{aligned} nJ_n\varphi\left(1-\frac{1}{n}\right) &= \sum_{x=1}^{n-2} n\left[\varphi\left(\frac{x+1}{n}\right) - \varphi\left(\frac{x}{n}\right)\right]\langle\langle j_{x,x+1}\rangle\rangle \\ &= -\frac{1}{4\gamma}\sum_{x=1}^{n-2} n\left[\varphi\left(\frac{x+1}{n}\right) - \varphi\left(\frac{x}{n}\right)\right]\langle\langle\nabla\mathfrak{F}_x\rangle\rangle \\ &= \frac{1}{4\gamma}\frac{1}{n}\sum_{x=1}^{n-2} n^2\left[\varphi\left(\frac{x+1}{n}\right) + \varphi\left(\frac{x-1}{n}\right) - 2\varphi\left(\frac{x}{n}\right)\right]\langle\langle\mathfrak{F}_x\rangle\rangle \\ &= \frac{1}{4\gamma}\frac{1}{n}\sum_{x=0}^{n-2}\varphi''\left(\frac{x}{n}\right)\langle\langle\mathfrak{F}_x\rangle\rangle + R_n \end{aligned} \tag{8.4}$$

with

$$|R_n| \leqslant \frac{C}{n^2}\sum_x |\langle\langle\mathfrak{F}_x\rangle\rangle| \underset{n\to\infty}{\longrightarrow} 0. \tag{8.5}$$

following from (7.10).

Then we are left to prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{x=0}^{n-2}\varphi''\left(\frac{x}{n}\right)\left(\langle\langle\mathfrak{F}_x\rangle\rangle - D\langle\langle p_x^2\rangle\rangle\right) = 0. \tag{8.6}$$

This will follows directly if we prove the following □

Proposition 8.2. *For any test function* $\varphi\in C^2([0,1])$ *such that* $\operatorname{supp}\varphi\subset(0,1]$ *and* $\varphi'(1)=0$ *we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{x=0}^{n-2}\varphi''\left(\frac{x}{n}\right)\left(\langle\langle q_xq_{x+\ell}\rangle\rangle - G_{\omega_0}(\ell)\langle\langle p_x^2\rangle\rangle\right) = 0,\qquad \ell=0,1,2. \tag{8.7}$$

Proof. By virtue of (4.11) we have

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{x=0}^{n}\varphi\left(\frac{x}{n}\right)\langle\langle\overline{q}_x\overline{q}_{x+\ell}\rangle\rangle = 0. \tag{8.8}$$

It suffices therefore to prove that

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{x=0}^{n}\varphi\left(\frac{x}{n}\right)\int_0^t\left\{\langle\langle q_x'q_{x+\ell}'\rangle\rangle - G_{\omega_0}(\ell)\langle\langle p_x^2\rangle\rangle\right\}\mathrm{d}s = 0. \tag{8.9}$$

We first prove (8.9) for $\ell=0$. By (6.25) we have

$$\langle\langle(q_x')^2\rangle\rangle = \sum_{j,j'}\widetilde{S}^{(q)}_{j,j'}\psi_j(x)\psi_{j'}(x) = H_n(x) + O\Big(\frac{1}{n}\Big), \tag{8.10}$$

with

$$\begin{aligned}H_n(x) &= \sum_{y=0}^{n}\sum_{j,j'}\Phi\left(\frac{j}{n+1},\frac{j'}{n+1}\right)\psi_j(y)\psi_{j'}(y)\psi_j(x)\psi_{j'}(x)\langle\langle p_y^2\rangle\rangle\\ &= H_n^1(x) + \frac{1}{n+1}H_n^2(x) + \frac{\Phi\,(0,0)}{(n+1)^2}\sum_{y=0}^{n}\langle\langle p_y^2\rangle\rangle.\end{aligned} \tag{8.11}$$

Here $\Phi\left(\frac{j}{n+1},\frac{j'}{n+1}\right) = \frac{2\Theta(\mu_j,\mu_{j'})}{\mu_j+\mu_{j'}}$ and $H_n^j(x) := \sum_{y=0}^n K_j^{(n)}(x,y)\langle\langle p_y^2\rangle\rangle$, with

$$\begin{aligned}K_1^{(n)}(x,y) &:= \frac{1}{(n+1)^2}\sum_{j,j'=0}^{n}\Phi\left(\frac{j}{n+1},\frac{j'}{n+1}\right)\left[\cos\left(\frac{\pi j(y-x)}{n+1}\right)+\cos\left(\frac{\pi j(y+x+1)}{n+1}\right)\right]\\ &\times\left[\cos\left(\frac{\pi j'(y-x)}{n+1}\right)+\cos\left(\frac{\pi j'(y+x+1)}{n+1}\right)\right],\\ K_2^{(n)}(x,y) &:= -\frac{1}{n+1}\sum_{j=0}^{n}\Phi\left(\frac{j}{n+1},0\right)\left[\cos\left(\frac{\pi j(y-x)}{n+1}\right)+\cos\left(\frac{\pi j(y+x+1)}{n+1}\right)\right].\end{aligned} \tag{8.12}$$

Using Lemma B.1 we obtain that

$$|K_j^{(n)}(x,y)| \leqslant C\left(\frac{1}{\chi_n^2((x-y)/2)}+\frac{1}{\chi_n^2((x+y)/2)}\right), \quad x,y=0,\dots,n,\ n=1,2,\dots \tag{8.13}$$

for $j=1,2$. In particular the above estimate implies that

$$K_{j,*} := \sup_{x,n}\sum_{y=0}^{n}|K_j^{(n)}(x,y)| < +\infty, \quad j=1,2. \tag{8.14}$$

In consequence, by virtue of (7.25),

$$|H_n^j(x)| \leqslant \sum_{y=0}^{n}|K_j^{(n)}(x,y)|\langle\langle p_y^2\rangle\rangle \leqslant K_{j,*}\sup_{y,n}\langle\langle p_y^2\rangle\rangle =: H_{j,*} < +\infty \tag{8.15}$$

and the term corresponding to $H_n^2(x)$ is negligible, as $n\to+\infty$.

Choose $\delta\in(0,1)$ sufficiently small, so that $\varphi(u)=0$, when $u\in(0,\delta)$. Then, there exists $C>0$ and $n_0\geqslant 1$ such that

$$\begin{aligned}\frac{1}{n}\left|\sum_{x\in(0,\delta n)\cup((1-\delta)n,n)}\varphi''\left(\frac{x}{n}\right)\langle\langle (q_x')^2\rangle\rangle\right| &\leqslant \frac{1}{n}\sum_{x\in((1-\delta)n,n)}\left|\varphi''\left(\frac{x}{n}\right)\right|\left(|H_n(x)|+O\left(\frac{1}{n}\right)\right)\\ &\leqslant C\|\varphi''\|_\infty\delta, \quad n=n_0+1,n_0+2,\dots.\end{aligned}$$

For $\delta n\leqslant x\leqslant(1-\delta)n$ inequality (8.13) implies that there exists $C>0$ such that $K_1^{(n)}(x,y)=\overline{K}_1^{(n)}(x,y)+O(1/n^2)$, where

$$\overline{K}_1^{(n)}(x,y) := \frac{1}{4(n+1)^2}\sum_{j,j'=-n-1}^{n}\Phi\left(\frac{j}{n+1},\frac{j'}{n+1}\right)\cos\left(\frac{\pi j(y-x)}{n+1}\right)\cos\left(\frac{\pi j'(y-x)}{n+1}\right), \tag{8.16}$$

and, by Lemma B.1, there exists $C>0$ such that

$$|\overline{K}_1^{(n)}(x,y)| \leqslant \frac{C}{1+(x-y)^2}, \quad y=0,\dots,n,\ x\in(\delta n,(1-\delta)n) \tag{8.17}$$

for $n=1,2,\dots$. To prove (8.7) it suffices therefore to show that for any $\delta\in(0,1/2)$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n}\right)\left(\overline{H}_n^1(x)-G_{\omega_0}(0)\langle\langle p_x^2\rangle\rangle\right)=0, \tag{8.18}$$

where $\overline{H}_n^1(x) := \sum_{y=0}^{n}\overline{K}_1^{(n)}(x,y)\langle\langle p_y^2\rangle\rangle$. Using Cauchy-Schwarz inequality, estimates (8.17) and (7.25) we conclude, that

$$\begin{aligned}\sum_{y=0}^{n}\left|\langle\langle p_y^2\rangle\rangle-\langle\langle p_x^2\rangle\rangle\right||\overline{K}_1^{(n)}(x,y)| &\leqslant \sum_{y=0}^{n}|y-x|^{1/2}\left|\sum_{z=0}^{n-1}\left(\langle\langle p_{z+1}^2\rangle\rangle-\langle\langle p_z^2\rangle\rangle\right)^2\right|^{1/2}|\overline{K}_1^{(n)}(x,y)|\\ &\leqslant \frac{C}{n^{1/2}}\sum_{y=0}^{n}|y-x|^{1/2}|\overline{K}_1^{(n)}(x,y)| \leqslant \frac{C'}{n^{1/2}}\end{aligned} \tag{8.19}$$

Therefore

$$\frac{1}{n+1}\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n+1}\right)\sum_{y=0}^{n}\langle\langle p_y^2\rangle\rangle\overline{K}_1^{(n)}(x,y)$$
$$=\frac{1}{n+1}\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n+1}\right)\langle\langle p_x^2\rangle\rangle\sum_{y=0}^{n}\overline{K}_1^{(n)}(y,x)+O\left(\frac{1}{\sqrt{n}}\right).$$

Another application of Lemma B.1 allows us to conclude that

$$|\sum_{y=0}^{n}\overline{K}_1^{(n)}(y,x)-\sum_{y=x-n-1}^{x+n}\overline{K}_1^{(n)}(y,x)|\leqslant\frac{C}{n^2},\quad \delta n\leqslant x\leqslant(1-\delta)n, \tag{8.20}$$

Using elementary trigonometric identities we conclude that

$$2\sum_{z=-n-1}^{n}\cos\left(\frac{\pi jz}{n+1}\right)\cos\left(\frac{\pi j'z}{n+1}\right)=(n+1)\Big(\delta_{j,-j'}+\delta_{j,j'}\Big).$$

Hence, by virtue of Lemma B.2, we get

$$\begin{aligned}\sum_{y=x-n-1}^{x+n}\overline{K}_1^{(n)}(y,x)&=\frac{1}{n+1}\sum_{j=0}^{n}\Phi\left(\frac{j}{n+1},\frac{j}{n+1}\right)\\&=\frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{\mu_j}=G_{\omega_0}(0)+o(1),\end{aligned}\tag{8.21}$$

for $\delta n\leqslant x\leqslant(1-\delta)n$. We have proven therefore that

$$\frac{1}{n}\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n}\right)\overline{H}_n^1(x)=\frac{1}{n}G_{\omega_0}(0)\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n}\right)\langle\langle p_x^2\rangle\rangle+o(1)$$

and (8.18) follows. This ends the proof of (8.7) for $\ell=0$. □

Proof for $\ell\neq 0$.

The proof is similar to the previously considered case, so we only sketch it. As before, by (6.25) we have

$$\langle\langle q_x'q_{x+\ell}'\rangle\rangle=\sum_{j,j'}\widetilde{S}_{j,j'}^{(q)}\psi_j(x)\psi_{j'}(x+\ell)+O(1/n)=H_{n,\ell}(x)+O(1/n), \tag{8.22}$$

with

$$\begin{aligned}H_{n,\ell}(x)&=\sum_{y=0}^{n}\sum_{j,j'}\frac{2\Theta(\mu_j,\mu_{j'})}{\mu_j+\mu_{j'}}\psi_j(y)\psi_{j'}(y)\psi_j(x)\psi_{j'}(x+\ell)\langle\langle p_y^2\rangle\rangle\\&=\sum_{y=0}^{n}\overline{K}^{(n,\ell)}(x,y)\langle\langle p_y^2\rangle\rangle+O\left(\frac{1}{n}\right).\end{aligned}\tag{8.23}$$

Here

$$\overline{K}^{(n,\ell)}(x,y) := \frac{1}{4(n+1)^2}\sum_{j,j'=-n-1}^{n}\Phi\left(\frac{j}{n+1},\frac{j'}{n+1}\right)$$
$$\cos\left(\frac{\pi j(y-x)}{n+1}\right)\cos\left(\frac{\pi j'(y-x-\ell)}{n+1}\right). \quad (8.24)$$

Using Lemma B.1, in a manner similar to what we have done in the argument leading up to (8.18), we conclude that it suffices to show that for any $\delta \in (0, 1/2)$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n}\right)\left(\overline{H}_{n,\ell}(x) - G_{\omega_0}(\ell)\langle\langle p_x^2\rangle\rangle\right) = 0, \quad (8.25)$$

with $\overline{H}_{n,\ell}(x) := \sum_{y=0}^{n}\overline{K}^{(n,\ell)}(x,y)\langle\langle p_y^2\rangle\rangle$. Furthermore, thanks to estimates analogous to those leading up to (8.26), we get

$$\Big|\sum_{y=0}^{n}\overline{K}^{(n,\ell)}(y,x) - \sum_{y=x-n-1}^{x+n}\overline{K}^{(n,\ell)}(y,x)\Big| \leqslant \frac{C}{n^2}, \quad \delta n \leqslant x \leqslant (1-\delta)n. \quad (8.26)$$

Using elementary trigonometric identities we conclude that

$$2\sum_{z=-n-1}^{n}\cos\left(\frac{\pi jz}{n+1}\right)\cos\left(\frac{\pi j'(z-\ell)}{n+1}\right) = (n+1)\cos\left(\frac{j\ell}{n+1}\right)\left(\delta_{j,-j'}+\delta_{j,j'}\right).$$

Hence,

$$\sum_{y=x-n-1}^{x+n}\overline{K}_1^{(n)}(y,x) = \frac{1}{n+1}\sum_{j=0}^{n}\Phi\left(\frac{j}{n+1},\frac{j}{n+1}\right)\cos\left(\frac{j\ell}{n+1}\right)$$
$$= G_{\omega_0}(\ell) + o(1), \quad (8.27)$$

for $\delta n \leqslant x \leqslant (1-\delta)n$. We have proven therefore that

$$\frac{1}{n}\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n}\right)\overline{H}_{n,\ell}(x) = \frac{1}{n}G_{\omega_0}(\ell)\sum_{\delta n\leqslant x\leqslant(1-\delta)n}\varphi''\left(\frac{x}{n}\right)\langle\langle p_x^2\rangle\rangle + o(1)$$

and (8.25) follows. This ends the proof of (8.7) for $\ell \neq 0$.

Finally, to finish the proof of Theorem 3.4 we show the following result, that is a form of the equipartition property of the energy.

Lemma 8.3. *Suppose that* $\varphi \in C^1[0,1]$. *Then,*

$$\lim_{n\to+\infty}\frac{1}{n+1}\sum_{x=0}^{n}\varphi\left(\frac{x}{n+1}\right)\left(\langle\langle p_x^2\rangle\rangle - \langle\langle r_x^2\rangle\rangle - \omega_0^2\langle\langle q_x^2\rangle\rangle\right) = 0. \quad (8.28)$$

Here $r_x := q_x - q_{x-1}$, $x = 1,\dots,n$ *and* $r_0 := 0$.

Proof. After a simple calculation we obtain

$$\langle\langle p_x^2\rangle\rangle = \langle\langle \omega_0^2 q_x^2 - (\Delta_{\rm N} q_x) q_x\rangle\rangle - \frac{\delta_{x,n} n^a}{\theta_n}\int_0^{\theta_n} \overline{q}_x(t)\mathcal{F}(t/\theta)\mathrm{d}t \tag{8.29}$$

for $x = 0, \ldots, n$. Therefore, by (3.6) (see also (3.1))

$$\begin{aligned} &\frac{1}{n+1}\sum_{x=0}^{n}\varphi\left(\frac{x}{n+1}\right)\left(\langle\langle p_x^2\rangle\rangle - \langle\langle r_x^2\rangle\rangle - \omega_0^2\langle\langle q_x^2\rangle\rangle\right)\\ &\quad = \frac{1}{n+1}\sum_{x=0}^{n}\varphi\left(\frac{x}{n+1}\right)\langle\langle r_x q_x\rangle\rangle - \frac{1}{n+1}\sum_{x=1}^{n}\varphi\left(\frac{x-1}{n+1}\right)\langle\langle r_x q_{x-1}\rangle\rangle \\ &\quad - \frac{1}{n+1}\sum_{x=0}^{n}\varphi\left(\frac{x}{n+1}\right)\langle\langle r_x^2\rangle\rangle + O\left(\frac{1}{n^{1-a}}\right). \end{aligned} \tag{8.30}$$

Since $r_0 = 0$ the last expression equals

$$\frac{1}{n+1}\sum_{x=1}^{n}\left[\varphi\left(\frac{x}{n+1}\right) - \varphi\left(\frac{x-1}{n+1}\right)\right]\langle\langle r_x q_{x-1}\rangle\rangle + O\left(\frac{1}{n^{1-a}}\right) = O\left(\frac{1}{n}\right),$$

by virtue of (7.10). This ends the proof of (8.28). □

9. Vanishing Time Variance of the Kinetic Energy

A natural question is about the time averaging of the energy functional. We prove that the time variance of the average kinetic energy vanishes as $n \to \infty$. We consider only the case when $b = 0$ and $a^{-1/2}$ in the dynamics described by (1.3)–(1.5).

Theorem 9.1. *Under the assumption stated above there exists a constant $C > 0$ such that*

$$\sum_{x=0}^{n}\frac{1}{\theta}\int_0^{\theta}\left(\overline{p_x^2}(t) - \langle\langle p_x^2\rangle\rangle\right)^2 \mathrm{d}t \leqslant \frac{C}{n^2}, \quad n = 1, 2, \ldots. \tag{9.1}$$

Proof. From (A.14) we get

$$\overline{p_x^2}(t) = T_- M_{x,0} + \sum_{x'=1}^{n}\int_0^{\theta}\mathfrak{g}_{x,x'}(s)\overline{p_{x'}^2}(t-s)\mathrm{d}s + \overline{p_x}^2(t), \tag{9.2}$$

where $\mathfrak{g}_{x,x'}(s)$ is defined in (A.13). Averaging over the t variable we get

$$\langle\langle p_x^2\rangle\rangle = T_- M_{x,0} + \sum_{x'=1}^{n}\int_0^{\theta}\mathfrak{g}_{x,x'}(s)\mathrm{d}s\langle\langle p_{x'}^2\rangle\rangle + \langle\langle\overline{p_x}^2\rangle\rangle. \tag{9.3}$$

Then, denoting

$$\begin{aligned} V_x(t) &:= \overline{p_x^2}(t) - \langle\langle p_x^2\rangle\rangle,\\ v_x(t) &:= \overline{p_x}^2(t) - \langle\langle\overline{p_x}^2\rangle\rangle, \end{aligned}$$

we can write

$$V_x(t) = \sum_{x'=1}^{n} \int_0^{\theta} \mathfrak{g}_{x,x'}(s) V_{x'}(t-s)\mathrm{d}s + v_x(t). \tag{9.4}$$

For $m \in \mathbb{Z}$ define

$$\begin{aligned} M_{x,y}(m) &:= \int_0^{\theta} \mathfrak{g}_{x,y}(s) e^{-2\pi i m s/\theta} \mathrm{d}s \\ &= 4\gamma \int_0^{+\infty} e^{-2\pi i m s/\theta} \Big([e^{-As}]_{x+n+1,y+n+1} \Big)^2 \mathrm{d}s \end{aligned} \tag{9.5}$$

(cf (A.13)) and

$$\widetilde{V}_x(m) := \frac{1}{\theta} \int_0^{\theta} e^{-2\pi \mathrm{i} m t/\theta} V_x(t)\mathrm{d}t \qquad \text{and} \qquad \widetilde{v}_x(m) := \frac{1}{\theta} \int_0^{\theta} e^{-2\pi \mathrm{i} m t/\theta} v_x(t)\mathrm{d}t.$$

Note that obviously $\widetilde{V}_x(0) = 0$ and $M_{x,y}(0) = M_{x,y}$, see (A.16). From (9.4) we get

$$\widetilde{V}_x(m) = \sum_{x'=1}^{n} M_{x,x'}(m) \widetilde{V}_{x'}(m) + \widetilde{v}_x(m). \tag{9.6}$$

Multiplying both sides by $\widetilde{V}_x^{\star}(m)$ and summing over x we get

$$\sum_{x=0}^{n} |\widetilde{V}_x(m)|^2 = \sum_{x=0}^{n} \sum_{x'=1}^{n} M_{x,x'}(m) \widetilde{V}_{x'}(m) \widetilde{V}_x^{\star}(m) + \sum_{x=0}^{n} \widetilde{v}_x(m) \widetilde{V}_x^{\star}(m). \tag{9.7}$$

Hence,

$$\begin{aligned} &\sum_{x,x'=0}^{n} \Big(\delta_{x,x'} - M_{x,x'}(m) \Big) \widetilde{V}_{x'}(m) \widetilde{V}_x^{\star}(m) \\ &= -\widetilde{V}_0(m) \sum_{x=0}^{n} M_{x,0}(m) \widetilde{V}_x^{\star}(m) + \sum_{x=0}^{n} \widetilde{v}_x(m) \widetilde{V}_x^{\star}(m). \end{aligned} \tag{9.8}$$

We have the following.

Lemma 9.2. *There exists a constant $\mathfrak{C} > 0$, such that*

$$\Big| \sum_{x,y=0}^{n} (\delta_{x,y} - M_{x,y}(m)) f_y^{\star} f_x \Big| \geqslant \mathfrak{C} \sum_{x=0}^{n} |f_x|^2, \quad (f_0, \ldots, f_n) \in \mathbb{C}^{n+1} \tag{9.9}$$

for $m \neq 0$ and $n = 1, 2, \ldots$.

The lemma is proven in Appendix C, where we also prove the following

Lemma 9.3. *We have*

$$\mathfrak{M} := \sup_m \left\{ \sum_{x=0}^{n} |M_{x,0}(m)|^2 \right\}^{1/2} < +\infty. \tag{9.10}$$

Lemma 9.4. *There exists a constant* $\mathfrak{v}_* > 0$ *such that*

$$\left\{\sum_{x=0}^{n} \langle\langle v_x^2 \rangle\rangle\right\}^{1/2} \leqslant \frac{\mathfrak{v}_*}{n}, \quad n = 1, 2, \ldots \tag{9.11}$$

Lemma 9.5. *There exists a constant* $\mathfrak{V}_* > 0$ *such that*

$$\langle\langle V_0^2 \rangle\rangle^{1/2} \leqslant \frac{\mathfrak{V}_*}{n^{1/2}} \left\{\sum_{x=0}^{n} \langle\langle V_x^2 \rangle\rangle\right\}^{1/4} \tag{9.12}$$

The lemmas are proven in Appendix C.

We show how to apply these to finish the proof of the theorem.

From (9.8), (9.9) and the Cauchy-Schwarz inequality we conclude that

$$\mathfrak{C} \sum_{x=0}^{n} |\widetilde{V}_x(m)|^2 \leqslant \left[|\widetilde{V}_0(m)| \left\{\sum_{x=0}^{n} |M_{x,0}(m)|^2\right\}^{1/2} + \left\{\sum_{x=0}^{n} |\widetilde{v}_x(m)|^2\right\}^{1/2} \right] \left\{\sum_{x=0}^{n} |\widetilde{V}_x(m)|^2\right\}^{1/2}, \tag{9.13}$$

i.e.

$$\mathfrak{C}^2 \sum_{x=0}^{n} |\widetilde{V}_x(m)|^2 \leqslant \left[\mathfrak{M} |\widetilde{V}_0(m)| + \left\{\sum_{x=0}^{n} |\widetilde{v}_x(m)|^2\right\}^{1/2} \right]^2 \leqslant 2\mathfrak{M}^2 |\widetilde{V}_0(m)|^2 + 2 \sum_{x=0}^{n} |\widetilde{v}_x(m)|^2. \tag{9.14}$$

Summing over m and using (9.12) together with (9.11) we obtain

$$\begin{aligned} \mathfrak{C}^2 \sum_{x=0}^{n} \langle\langle V_x^2 \rangle\rangle \leqslant 2\mathfrak{M}^2 \langle\langle V_0^2 \rangle\rangle + 2 \sum_{x=0}^{n} \langle\langle v_x^2 \rangle\rangle &\leqslant \frac{2(\mathfrak{M}\mathfrak{V}_*)^2}{n} \left\{\sum_{x=0}^{n} \langle\langle V_x^2 \rangle\rangle\right\}^{1/2} + \frac{\mathfrak{v}_*^2}{n^2} \\ &\leqslant \frac{(\mathfrak{M}\mathfrak{V}_*)^4}{\mathfrak{C}^2 n^2} + \frac{\mathfrak{C}^2}{2} \sum_{x=0}^{n} \langle\langle V_x^2 \rangle\rangle + \frac{\mathfrak{v}_*^2}{n^2}, \end{aligned} \tag{9.15}$$

and (9.1) follows. □

10. Concluding Remarks

In this article we studied the energy transport in the periodic state of a pinned harmonic chain with bulk dynamics perturbed by a random flip of the signs of the velocities. Work was done on the system by a periodic forcing acting on the right hand side of the chain and the heat was absorbed by a heat bath coupled to the system via a Langevin stochastic thermostat at temperature T_- on the left. The asymptotic temperature profile (3.9) should be seen as the stationary solution of the heat equation:

$$\begin{aligned} \partial_t T(t,u) &= \frac{D}{4\gamma} \partial_u^2 T(t,u) \qquad u \in (0,1) \\ T(t,0) &= T_-, \quad \partial_u T(t,1) = -\frac{4\gamma J}{D}, \quad T(0,u) = T_0(u), \end{aligned} \tag{10.1}$$

where J is given by (3.3). In a companion work [10] we prove that, under proper conditions on the initial distribution, for any compactly supported test function $\varphi \in C([0, +\infty) \times [0, 1])$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=0}^{n} \int_0^{+\infty} \varphi\left(t, \frac{x}{n}\right) \mathcal{E}_x(n^2 t) \mathrm{d}t = \int_0^{+\infty} \int_0^1 \varphi(t, u) T(t, u) \mathrm{d}t \mathrm{d}u, \quad \text{in probability,} \tag{10.2}$$

with $T(t, u)$ the solution of (10.1). Notice the diffusive rescaling of space and time.

The diffusion coefficient appearing in (10.1) (defined in (1.10)) has been computed in a different way in [1, Theorem formula (74)] and [7, Theorem 3.2, formula (3.21)]. It turns out that it equals to the diffusivity of a phonon performing a random walk on the integer lattice with random scattering generated by the noise. As a result

$$D = 2 \int_{\mathbb{T}} \left(\frac{\omega'(k)}{2\pi} \right)^2 \mathrm{d}k \tag{10.3}$$

where $\omega'(k)/(2\pi)$ is a group velocity of a phonon of frequency k belonging to the one dimensional unit torus $\mathbb{T}$, that is the interval $[-1/2, 1/2]$ with identified endpoints. Here $\omega(k) = \sqrt{\omega_0^2 + 4\sin^2(\pi k)}$ is the dispersion relation of the harmonic lattice considered in the present paper. In fact, if we consider a more general type of noise that allows to scatter the phonons of given frequency k, with the total scattering kernel $R(k)$ (in the case of the flip noise $R(k) \equiv 1$) we would have

$$D = 2 \int_{\mathbb{T}} \left(\frac{\omega'(k)}{2\pi} \right)^2 \frac{\mathrm{d}k}{R(k)}.$$

As noted before the velocity reversals introduced in the dynamics serve the purpose of making the heat conductivity finite. In their absence the harmonic crystal has an infinite conductivity [17]. The velocity reversals are thus an idealized substitute for the anharmonicities, impurities and other defects which scatter phonons and produce a finite conductivity and establishes the validity of Fourier's law in real solids.

An alternative way of modeling anharmonicity for achieving a finite conductivity is the introduction of "self consistent" reservoirs. This was introduced in [4,16] and fully analyzed in [5] for describing the heat flow in a harmonic crystal in contact with two heat reservoirs at different temperatures and no external force. In that model one introduces, in addition to the external reservoirs, also "internal" Langevin reservoirs for each particle. Letting T_x be the temperature of the reservoir at position $x = 0, \ldots, n$, with the same coupling γ as used here, Eq. (2.2) remains unchanged while Eq. (1.11) will have an extra term, $2\gamma(T_x - p_x^2)$, on its right hand side. Solving for the periodic first and second moments of the system the internal T_x, $x = 1, \ldots, n-1$, are then determined by the requirement that the time average of the internal heat flux, given by the term in the square bracket above, vanishes in the stationary state. As a result, there is no contribution to the average current from these internal reservoirs and the limiting macroscopic behavior is the same as in a corresponding velocity flip model. This approach can be modified by considering a periodic, instead of a constant, self-consistent temperature profile. This will make the dynamics of the first and second moments of the position and momenta variables identical with that of the flip model. An important property of the self consistent reservoir model is that the periodic measure is Gaussian, consequently it is determined by the first and second moments already computed here.

Various possible extension of the present model are presented in the review article [11]: forcing acting on a particle in the bulk, unpinned dynamics, higher dimensional lattice, anharmonic interactions.

Appendix A. The proof of Theorem 1.1

Since the result does not depend on the scaling factor n^a, standing by the force $\mathcal{F}(t)$, and the period size θ_n we assume that $a = 0$ and $\theta_n = \theta$. Given a Borel probability measure μ on $\mathbb{R}^{2(n+1)}$ (see (1.1)) we denote by $\big(\mathbf{q}_\mu(t), \mathbf{p}_\mu(t)\big)$ the solution of (1.3)–(1.4)) such that $\big(\mathbf{q}_\mu(0), \mathbf{p}_\mu(0)\big)$ is distributed according to μ. Denote then by $\big(\overline{\mathbf{q}}_\mu(t), \overline{\mathbf{p}}_\mu(t)\big)$ and $C_\mu(t)$ the vector of averages and matrix of the mixed second moments of the solution, correspondingly. They are defined by formulas (2.1) and a 2×2 block matrix

$$C_\mu(t) = \begin{bmatrix} C_\mu^{(q)}(t) & C_\mu^{(q,p)}(t) \\ C_\mu^{(p,q)}(t) & C_\mu^{(p)}(t) \end{bmatrix}$$

Each block is an $(n + 1) \times (n + 1)$ matrix

$$\begin{aligned} &C_\mu^{(q)}(t) = [q_x(t)q_{x'}(t)]_{x,x'=0,\dots,n} \quad C_\mu^{(p)}(t) = \mathbb{E}[p_x(t)p_{x'}(t)]_{x,x'=0,\dots,n}, \\ &C_\mu^{(q,p)}(t) = \mathbb{E}[q_x(t)p_{x'}(t)]_{x,x'=0,\dots,n} \end{aligned}$$

and $C_\mu^{(p,q)}(t) = [C_\mu^{(q,p)}(t)]^T$, where the initial state is taken to be μ.
By similar calculation as done in (6.15), their evolution is described by the system of linear differential equations with periodic forcing

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \overline{\mathbf{q}}_\mu(t) \\ \overline{\mathbf{p}}_\mu(t) \end{pmatrix} = -A\begin{pmatrix} \overline{\mathbf{q}}_\mu(t) \\ \overline{\mathbf{p}}_\mu(t) \end{pmatrix} + \mathcal{F}(t/\theta)\mathrm{e}_{p,n+1}, \\ &\frac{\mathrm{d}}{\mathrm{d}t}C_\mu(t) = -AC_\mu(t) - C_\mu(t)A^T + \Sigma_2\big(\mathbf{c}_{2,\mu}(t)\big) + \mathcal{F}(t/\theta)F(t), \end{aligned} \tag{A.1}$$

where

$$\mathbf{c}_{2,\mu}(t) = \begin{pmatrix} C_{0,0,\mu}^{(p)}(t) \\ \vdots \\ C_{n,n,\mu}^{(p)}(t) \end{pmatrix} \tag{A.2}$$

and

$$F(t) := \begin{bmatrix} 0 & \overline{\mathbf{q}}_\mu(t) \otimes \mathrm{e}_{p,n+1} \\ \mathrm{e}_{p,n+1} \otimes \overline{\mathbf{q}}_\mu(t) & \mathrm{e}_{p,n+1} \otimes \overline{\mathbf{p}}_\mu(t) + \overline{\mathbf{p}}_\mu(t) \otimes \mathrm{e}_{p,n+1} \end{bmatrix} \tag{A.3}$$

Here $\mathrm{e}_{p,n+1}$ and Σ_2 are defined in (2.4) and (6.10) respectively. Suppose that we are given a vector $\overline{X} \in \mathbb{R}^{2(n+1)}$ and a symmetric non-negative definite $2(n+1) \times 2(n+1)$ matrix $S \geqslant \overline{X} \otimes \overline{X}$. Then, equations (A.1) describe the evolution of the first two moments of the solution of (1.3)–(1.4)) whose initial distribution is a random vector with the first two moments given by $\overline{X}$ and S, respectively.

A.1. The existence and uniqueness of the periodic mean and second moment. In the first step we show the existence of a periodic solution of (A.1) that corresponds to the mean and covariance of a certain probability evolution.

Proposition A.1. *There exists a unique vector* $\overline{\mathbf{X}}_{\mathrm{per}} = (\overline{\mathbf{q}}_{\mathrm{per}}, \overline{\mathbf{p}}_{\mathrm{per}}) \in \mathbb{R}^{2(n+1)}$ *and a non-negative symmetric matrix* $C_{\mathrm{per}} \geqslant \overline{\mathbf{X}}_{\mathrm{per}} \otimes \overline{\mathbf{X}}_{\mathrm{per}}$ *such that the solution of (A.1) with*

$$\Big((\overline{\mathbf{q}}(0), \overline{\mathbf{p}}(0)), C(0)\Big) = \Big((\overline{\mathbf{q}}_{\mathrm{per}}, \overline{\mathbf{p}}_{\mathrm{per}}), C_{\mathrm{per}}\Big)$$

satisfies

$$\Big((\overline{\mathbf{q}}(0), \overline{\mathbf{p}}(0)), C(0)\Big) = \Big((\overline{\mathbf{q}}(\theta), \overline{\mathbf{p}}(\theta)), C(\theta)\Big). \tag{A.4}$$

In addition, we have

$$C^{(p)}_{x,x}(t) \geqslant T_-, \quad x = 0, \ldots, n. \tag{A.5}$$

The remaining part of this section is devoted to the proof of this results.

A.1.1. The existence of the periodic first moment Let

$$\begin{pmatrix} \overline{\mathbf{q}} \\ \overline{\mathbf{p}} \end{pmatrix} := \int_{-\infty}^{0} \mathcal{F}(s/\theta) e^{As} \mathrm{e}_{p,n+1} \mathrm{d}s. \tag{A.6}$$

Thanks to Proposition 2.1 the vector $(\overline{\mathbf{q}}, \overline{\mathbf{p}})$ is well defined. One can easily check that the solution of the first equation of (A.1) starting from the vector is given by

$$\overline{\mathbf{X}}(t) = \begin{pmatrix} \overline{\mathbf{q}}(t) \\ \overline{\mathbf{p}}(t) \end{pmatrix} := \int_{-\infty}^{t} \mathcal{F}(s/\theta) e^{-A(t-s)} \mathrm{e}_{p,n+1} \mathrm{d}s. \tag{A.7}$$

and is therefore θ-periodic. In fact, thanks to Proposition 2.1 the periodic solution has to be unique. Since the coordinates of $\overline{\mathbf{X}}(t)$ satisfy the first equation of (A.1) we conclude that the matrix $\overline{\mathbf{X}}_2(t) := \overline{\mathbf{X}}(t) \otimes \overline{\mathbf{X}}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{\mathbf{X}}_2(t) = -A\overline{\mathbf{X}}_2(t) - \overline{\mathbf{X}}_2(t) A^T + \mathcal{F}(t/\theta) F(t),$$

and it is given by the formula

$$\overline{\mathbf{X}}_2(t) = \int_{-\infty}^{t} \mathcal{F}(s/\theta) e^{-A(t-s)} F(s) e^{-A^T(t-s)} \mathrm{d}s, \quad t \in \mathbb{R}. \tag{A.8}$$

A.1.2. The existence of the periodic second moment Now we are going to establish the existence of a periodic second moment. Suppose that $C(t)$ is a periodic solution of the second equation of (A.1). Using the argument made in the proof of Proposition 6.1 we can conclude that it satisfies the equation

$$\begin{aligned} C(t) &= \int_{-\infty}^{t} e^{-A(t-s)} \left(\Sigma_2\big(\mathbf{c}_2(s)\big) + \mathcal{F}(s/\theta)F(s)\right) e^{-A^T(t-s)} \mathrm{d}s \\ &= \int_{-\infty}^{t} e^{-A(t-s)} \Sigma_2\big(\mathbf{c}_2(s)\big) e^{-A^T(t-s)} \mathrm{d}s + \overline{\mathbf{X}}_2(t) \\ &= \int_0^{\infty} e^{-As} \Sigma_2\big(\mathbf{c}_2(t-s)\big) e^{-A^T s} \mathrm{d}s + \overline{\mathbf{X}}_2(t) \\ &= \sum_{\ell=0}^{\infty} \int_0^{\theta} e^{-A(s+\ell\theta)} \Sigma_2\big(\mathbf{c}_2(t-s)\big) e^{-A^T(s+\ell\theta)} \mathrm{d}s + \overline{\mathbf{X}}_2(t) \\ &\qquad, \quad t \in \mathbb{R}, \end{aligned} \tag{A.9}$$

where the matrix Σ_2 is defined by (6.10), $\mathbf{c}_2(s)$ relates to $C(s)$ via (A.2) and $F(s)$ is defined by (A.3), using $(\overline{\mathbf{q}}(t), \overline{\mathbf{p}}(t))$ instead of $(\overline{\mathbf{q}}_\mu(t), \overline{\mathbf{p}}_\mu(t))$. Conversely, any periodic symmetric matrix valued function $C(t)$ satisfying (A.9) is a periodic solution to the second equation of (A.1).
For $x, x', y = 0, \ldots, n$ define

$$g_{x,x',y}(s) := \sum_{\ell=0}^{+\infty} \left[e^{-A(s+\ell\theta)}\right]_{x+n+1,y+n+1} \left[e^{-A^T(s+\ell\theta)}\right]_{y+n+1,x'+n+1}. \tag{A.10}$$

Consider the following linear mapping: $\mathcal{L} : [C(\mathbb{T}_\theta)]^{n+1} \to [C(\mathbb{T}_\theta)]^{n+1}$, where $\mathbb{T}_\theta := \theta\mathbb{T}$ is the torus of size θ, that assigns to a given vector of θ-periodic functions $\mathbf{T}(s) = [T_0(s), \ldots, T_n(s)]$ a vector valued function

$$\mathcal{L}\mathbf{T} := (\mathfrak{G}_0\mathbf{T}, \ldots, \mathfrak{G}_n\mathbf{T}), \tag{A.11}$$

where

$$\mathfrak{G}_x\mathbf{T}(t) = \sum_{y=0}^{n} \int_0^{\theta} \mathfrak{G}_{x,y}(s) T_y(t-s)\, \mathrm{d}s. \tag{A.12}$$

Here

$$\mathfrak{G}_{x,y}(s) = 4\gamma g_{x,x,y}(s), \quad y = 0, \ldots, n. \tag{A.13}$$

Obviously, from (A.10), we have $\mathfrak{G}_{x,y}(s) \geqslant 0$. Note also that although $g_{x,x',y}(\cdot)$ need not be θ-periodic the functions $\mathfrak{G}_x\mathbf{T}(t)$, $x = 0, \ldots, n$ are θ-periodic. In addition, if $C(t)$ satisfies (A.9), then

$$\mathbf{c}_2(t) = \mathcal{L}\mathcal{T}(\mathbf{c}_2)(t) + \overline{\mathbf{p}}^2(t), \tag{A.14}$$

where for a given $\mathbf{T}^T = (T_0, \ldots, T_n) \in \mathbb{R}^{n+1}$

$$\mathcal{T}(\mathbf{T}) = \begin{pmatrix} T_- \\ T_1 \\ \vdots \\ T_n \end{pmatrix}, \quad \overline{\mathbf{p}}^2(t) = \begin{pmatrix} \overline{p}_0^2(t) \\ \vdots \\ \overline{p}_n^2(t) \end{pmatrix}.$$

Conversely, by finding a solution $\mathbf{c}_2$ of (A.14) one can define then a θ-periodic function $C(t)$ by the right hand side of (A.9). The entries of the function corresponding to $C_{x,x}$, $x = n+1, \ldots, 2n+1$ coincide with the coordinates of the vector $\mathbf{c}_2$, by virtue of (A.14). Thus, the function $C(t)$ solves equation (A.9). We have reduced therefore the problem of finding a periodic solution to the second equation of (A.1) to solving equation (A.14).

A.1.3. Solution of (A.14) Let

$$\mathcal{C}_+ := [\mathbf{T} = (T_0, T_1, \ldots, T_n) : \ T_x \in C(\mathbb{T}_\theta) \text{ and } T_x \geqslant T_-, \ x = 0, \ldots, n].$$

It is a closed subset of $\Big(C(\mathbb{T}_\theta)\Big)^{n+1}$, equipped with the norm

$$|||\mathbf{T}||| := \max\{\|T_x\|_\infty, \ x = 0, \ldots, n\}.$$

Consider the mapping

$$\mathfrak{T} = (\mathfrak{T}_0, \ldots, \mathfrak{T}_n) : \mathcal{C}_+ \to \Big(C(\mathbb{T}_\theta)\Big)^{n+1}, \quad \text{where} \quad \mathfrak{T}\mathbf{T} := \mathcal{L}\mathcal{T}(\mathbf{T}) + \overline{\mathbf{p}}^2. \tag{A.15}$$

Using the notation of (A.12) and (A.14) we have

$$\mathfrak{T}_x(\mathbf{T})(t) := T_- \int_0^\theta \mathfrak{G}_{x,0}(s)\mathrm{d}s + \sum_{x'=1}^n \int_0^\theta \mathfrak{G}_{x,x'}(s) T_{x'}(t-s)\,\mathrm{d}s + \overline{p}_x^2(t), \quad x = 0, \ldots, n.$$

Comparing (6.12) with (A.9), after time averaging over a period, it is easy to identify

$$\int_0^\theta \mathfrak{G}_{x,y}(s)\mathrm{d}s = M_{x,y} \tag{A.16}$$

defined by (7.2). The matrix $[M_{x,y}]_{x,y=0}^n$ is symmetric, bi-stochastic (as can be easily seen from (7.2)). It also follows immediately that

$$1 = \sum_{y=0}^n \int_0^\theta \mathfrak{G}_{x,y}(s)\mathrm{d}s, \quad x = 0, 1, \ldots, n \tag{A.17}$$

and, as a consequence, $\mathfrak{T}\big(\mathcal{C}_+\big) \subset \mathcal{C}_+$. Furthermore, we claim that $M_{x,y} > 0$ for all $x, y = 0, \ldots, n$. Indeed, a simple calculation, using (2.3) and (2.5), yields

$$\begin{aligned}
\big[(\lambda + A)^{-1}\big]_{x+n+1,y+n+1} &= \sum_{j=0}^n \frac{\lambda \psi_j(x)\psi_j(y)}{\lambda^2 + 2\gamma\lambda + \mu_j},\\
\big[(\lambda + A)^{-1}\big]_{x,y+n+1} &= \sum_{j=0}^n \frac{\psi_j(x)\psi_j(y)}{\lambda^2 + 2\gamma\lambda + \mu_j},\\
\big[(\lambda + A)^{-1}\big]_{x+n+1,y} &= -\sum_{j=0}^n \frac{\mu_j\psi_j(x)\psi_j(y)}{\lambda^2 + 2\gamma\lambda + \mu_j},\\
\big[(\lambda + A)^{-1}\big]_{x,y+n+1} &= \sum_{j=0}^n \frac{\psi_j(x)\psi_j(y)}{\lambda^2 + 2\gamma\lambda + \mu_j}.
\end{aligned} \tag{A.18}$$

The poles of the meromorphic functions appearing in (A.18) are given by

$$\lambda_{j,\pm} = -\Big(\gamma \pm \sqrt{\gamma^2 - \mu_j}\Big). \tag{A.19}$$

Suppose that $M_{x,y} = 0$ for some x, y. From (A.16) we conclude then that

$$0 = M_{x,y} = \int_0^\theta \mathfrak{G}_{x,y}(s)\mathrm{d}s = 4\gamma \int_0^{+\infty} \Big[e^{-As}\Big]^2_{x+n+1,\,y+n+1} \mathrm{d}s, \tag{A.20}$$

which in turn would implies that $\Big[e^{-As}\Big]_{x+n+1,\,y+n+1} \equiv 0$ for all $s \geqslant 0$, thus also

$$0 \equiv \big[(\lambda + A)^{-1}\big]_{x+n+1,\,y+n+1} = \sum_{j=0}^{n} \frac{\lambda\psi_j(x)\psi_j(y)}{\lambda^2 + 2\gamma\lambda + \mu_j}.$$

As a result, we conclude that $\psi_j(x)\psi_j(y) = 0$, for all $j = 0, \ldots, n$, which is impossible. We shall show that the mapping $\mathfrak{T}$ has a unique fixed point in $\mathcal{C}_+$ by proving that the mapping is a contraction in the norm $\|\!| \cdot |\!\|$. Indeed, for $\mathbf{T}_j^T = [T_{j,0}, T_{j,1}, \ldots, T_{j,n}]$, $j = 1, 2$, we have

$$\begin{aligned}
|\mathfrak{T}_x(\mathbf{T}_1)(t) - \mathfrak{T}_x(\mathbf{T}_2)(t)| &= \Big|\sum_{x'=1}^{n} \int_0^\theta \mathfrak{G}_{x,x'}(s)\Big[T_{1,x'}(t-s) - T_{2,x'}(t-s)\Big]\mathrm{d}s\Big| \\
&\leqslant \sum_{x'=1}^{n} \int_0^\theta \mathfrak{G}_{x,x'}(s)\Big|T_{1,x'}(t-s) - T_{2,x'}(t-s)\Big|\mathrm{d}s \\
&\leqslant \sum_{x'=1}^{n} \left(\int_0^\theta \mathfrak{G}_{x,x'}(s)\mathrm{d}s\right) \|T_{1,x'} - T_{2,x'}\|_\infty \\
&\leqslant \left(1 - \int_0^\theta \mathfrak{G}_{x,0}(s)\mathrm{d}s\right) \|\!|\mathbf{T}_1 - \mathbf{T}_2|\!\|, \quad x = 0, \ldots, n.
\end{aligned}$$

Therefore

$$\|\!|\mathfrak{T}(\mathbf{T}_1) - \mathfrak{T}(\mathbf{T}_2)|\!\| \leqslant \rho \|\!|\mathbf{T}_1 - \mathbf{T}_2|\!\|,$$

where

$$\rho := \max\left[1 - \int_0^\theta \mathfrak{G}_{x,0}(s)\mathrm{d}s,\ x = 0, \ldots, n\right] < 1.$$

We have proven that $\|\mathfrak{T}(\mathbf{T}_1) - \mathfrak{T}(\mathbf{T}_2)\|_\infty \leqslant \rho\|\mathbf{T}_1 - \mathbf{T}_2\|_\infty$ and the existence of a unique fixed point follows. This ends the proof of Proposition A.1. □

A.2. The end of the proof of Theorem 1.1. Suppose now that ν is a probability law whose first and second moments are θ-periodic, e.g. it could be a Gaussian distribution with the mean and the second moment given by $\mathbf{P}_{\rm per}$ and $C_{\rm per}$, respectively. Denote by

$$\mathcal{P}_{s,t}F(\mathbf{q},\mathbf{p}) = \int_{\mathbb{R}^{2(n+1)}} F(\mathbf{q}',\mathbf{p}')\mathcal{P}_{s,t}(\mathbf{q},\mathbf{p};\mathrm{d}\mathbf{q}',\mathrm{d}\mathbf{p}')$$

the evolution family of transition probability operators corresponding to the dynamics described by (1.3)–(1.4). Consider the event $E := [N_x(\theta) = 0,\ x = 1,\dots,n]$. We have $\mathbb{P}[E] > 0$. Suppose that the dynamics starts at $(\mathbf{q},\mathbf{p})$. Then, for any $F \geqslant 0$ we can write

$$\mathcal{P}_{0,\theta}F(\mathbf{q},\mathbf{p}) \geqslant \mathbb{E}\Big[F\big(\mathbf{q}(\theta),\mathbf{p}(\theta)\big),\ E\Big] = \mathbb{P}[E]\mathcal{Q}_{0,\theta}F(\mathbf{q},\mathbf{p}), \tag{A.21}$$

where $\mathcal{Q}_{s,t}$ is the transition probability operator for the non-homogeneous Ornstein-Uhlenbeck dynamics that corresponds to the generator $\mathcal{G}_t^{(g)} = \mathcal{A}_t + 2\gamma S_-$, see (1.7) and (1.8). Using the hypoellipticity of $\mathcal{G}_t^{(g)}$, see [9, Section A.3], one can prove that there exist strictly positive transition probability density kernels $\rho_{s,t}$ corresponding to $\mathcal{Q}_{s,t}$, see [9, Section A.2] for more details. Thanks to (A.21) we conclude that

$$\mathcal{P}_{0,\theta}(\mathbf{q},\mathbf{p};\mathrm{d}\mathbf{q}',\mathrm{d}\mathbf{p}') \geqslant c_*\rho_{0,\theta}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}', \tag{A.22}$$

where $c_* := \mathbb{P}[E]$. Then, $\nu_{0,t} := \nu\mathcal{P}_{0,t}$ describes the law of $(\mathbf{q}(t),\mathbf{p}(t))$ with the prescribed initial data. Thanks to Proposition A.1 we can see that the total energy $\mathcal{H}(t) := \sum_{x=0}^n \mathcal{E}_x(t)$ (see (1.2)) is a Lyapunov function for the above system, since $\mathbb{E}\mathcal{H}(t)$ is θ-periodic. The above implies that the family of laws $\{\nu_{0,t},\ t \geqslant 0\}$ is tight in $\mathbb{R}^{2(n+1)}$. Thus, also the family $\mu_N := N^{-1}\int_0^{N\theta}\nu_{0,s}\mathrm{d}s$ is tight. Suppose that μ_∞ is its limiting measure, i.e. there exists a sequence $N' \to +\infty$ such that $\mu_{N'} \to \mu_\infty$, in the topology of weak convergence. Since $\mathcal{P}_{s,t}$ has the Feller property one can easily conclude that $\mu_\infty\mathcal{P}_{0,\theta} = \mu_\infty$. Hence $\mu_s^P := \mu_\infty\mathcal{P}_{0,s}$, $s \in [0,+\infty)$ is a periodic stationary state.
Suppose that $\mu(\mathrm{d}\mathbf{q},\mathrm{d}\mathbf{p}) = f(\mathbf{q},\mathbf{p})\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}$, where f is a C^∞ smooth probability density. One can show, using the regularity theory of stochastic differential equations, that $\mu\mathcal{P}_{0,\theta}$ is absolutely continuous w.r.t. the Lebesgue measure and its density is also C^∞ smooth, see e.g. [6, Corollary III.3.4, p. 303]. This allows us to conclude further that $\mu\mathcal{P}_{0,\theta}$ is absolutely continuous, provided that μ is absolutely continuous. We shall denote by $\mathcal{P}_{0,\theta}$ the corresponding operator induced on $L^1(\mathbb{R}^{2(n+1)})$. The operator $\mathcal{Q}_{0,\theta}$ corresponding to the Gaussian dynamics transforms μ_∞ into an absolutely continuous measure. Thanks to (A.22) we conclude that

$$\mu_\infty(\mathrm{d}\mathbf{q}',\mathrm{d}\mathbf{p}') = \mu_\infty\mathcal{P}_{0,\theta}(\mathrm{d}\mathbf{q}',\mathrm{d}\mathbf{p}') \geqslant c_*\mu_\infty\mathcal{Q}_{0,\theta}(\mathbf{q}',\mathbf{p}')\mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}'. \tag{A.23}$$

Therefore the singular part of μ_∞ is of at most mass $1 - c_*$. Since $\mathcal{P}_{0,\theta}$ transforms the space of abolutely continuous measures into itself, both the singular and absolutely continuous parts of μ_∞, after normalization, become invariant under $\mathcal{P}_{0,\theta}$. Iterating this procedure we conclude, after m steps, that the singular part can be of at most mass $(1 - c_*)^m$, which eventually leads to the conclusion that the measure μ_∞ is absolutely continuous. The respective density is positive, due to (A.22). This ends the proof of Theorem 1.1. □

Appendix B. Green Functions Convergence

Recall that G_{ω_0} and $G^n_{\omega_0}$ are the Green's functions corresponding to $\omega_0^2-\Delta$ and $\omega_0^2-\Delta_{\rm N}$, where Δ is the free lattice laplacian on $\mathbb{Z}$ and $\Delta_{\rm N}$ is the Neumann discrete laplacian on $\{0, 1, \dots, n\}$, see Sects. 2.3 and 2.4, respectively.

B.1. Estimantes on oscillating sums. Define $\chi_n(x)$ as the $n+1$-periodic extension of $\chi_n(x) := (1+x)\wedge(n+2-x)$, $x \in [0, n+1]$. Suppose that $\Phi : \mathbb{R}^2 \to \mathbb{C}$ is a θ, θ'-periodic function in each variable respectively. Denote

$$H^{(n)}_{x,x'} = \frac{1}{(n+1)^2}\sum_{j,j'=0}^{n}\Phi\left(\frac{\theta j}{n+1}, \frac{\theta' j'}{n+1}\right)\exp\left\{\frac{2i\pi jx}{n+1}\right\}\exp\left\{\frac{2i\pi j'x'}{n+1}\right\}$$

for $x, x' \in \mathbb{Z}$.

Lemma B.1. *Suppose that Φ is of C^k-class for some $k \geqslant 1$. Then, there exists C such that*

$$\left|H^{(n)}_{x,x'}\right| \leqslant \frac{C}{\chi_n^{\ell}(x)\chi_n^{\ell'}(x')}, \quad x, x' \in \mathbb{Z},\ n \geqslant 1,\ \ell, \ell' \geqslant 0,\ \ell+\ell' \leqslant k. \tag{B.1}$$

Proof. To simplify the notation we suppose that $\theta = \theta' = 1$. Summation by parts yields

$$\begin{aligned} H^{(n)}_{x,x'} &= \frac{1}{(n+1)^2}\sum_{j,j'=0}^{n}\Phi\left(\frac{j}{n+1}, \frac{j'}{n+1}\right)\frac{\exp\left\{\frac{2i\pi x}{n+1}\right\}-1}{\exp\left\{\frac{2i\pi x}{n+1}\right\}-1}\exp\left\{\frac{2i\pi jx}{n+1}\right\}\exp\left\{\frac{2i\pi j'x'}{n+1}\right\} \\ &= \frac{1}{(n+1)^2[\exp\left\{\frac{i\pi x}{n+1}\right\}-1]}\sum_{j,j'=0}^{n}\left[\Phi\left(\frac{j-1}{n+1}, \frac{j'}{n+1}\right)-\Phi\left(\frac{j}{n+1}, \frac{j'}{n+1}\right)\right] \\ &\quad\times \exp\left\{\frac{2i\pi jx}{n+1}\right\}\exp\left\{\frac{2i\pi j'x}{n+1}\right\} \end{aligned}$$

Since Φ is of C^1 class

$$\left|\Phi\left(\frac{j-1}{n+1}, \frac{j'}{n+1}\right)-\Phi\left(\frac{j}{n+1}, \frac{j'}{n+1}\right)\right| \leqslant \frac{C}{n+1}$$

for some constant $C > 0$. In addition, there exists $c > 0$ such that

$$\left|\exp\left\{\frac{2i\pi x}{n+1}\right\}-1\right| \geqslant \frac{c\chi_n(x)}{n+1}$$

for $x, x' \in \mathbb{Z}$, $n \geqslant 1$. Thus, there exists $C > 0$ such that

$$|H^{(n)}_{x,x'}| \leqslant \frac{C}{\chi_n(x)}, \quad x, x' \in \mathbb{Z}.$$

Iterating this argument in the regularity degree k of Φ we conclude (B.1). □

B.2. Application. An application concerns the approximation of the Green's function G_{ω_0} by $G^n_{\omega_0}$ along the diagonal.

Lemma B.2. *We have*

$$G^n_{\omega_0}(y,y) = G_{\omega_0}(0) + \widetilde{H}^{(n)}(y) + O\Big(\frac{1}{n}\Big), \quad y=0,\dots,n,\ n \geqslant 1. \tag{B.2}$$

Here for some constant $C>0$ we have

$$|\widetilde{H}^{(n)}(y)| \leqslant \frac{C}{\chi^2(y)}, \quad y=0,\dots,n,\ n\geqslant 1. \tag{B.3}$$

Proof. Using the definition of the Green's function (2.13) (with $\ell = 0$) and formulas (2.14) we obtain

$$G^n_{\omega_0}(y,y) = \frac{1}{n+1}\sum_{j=0}^{n} \Xi\left(\frac{j}{n+1}\right)\left[1+\cos\left(\frac{\pi j(2y+1)}{n+1}\right)\right] + O\Big(\frac{1}{n}\Big),$$

where

$$\Xi(u) = \left\{4\sin^2\left(\frac{\pi u}{2}\right) + \omega_0^2\right\}^{-1}.$$

As a result we write $G^n_{\omega_0}(y,y)$ in the form (B.2), with

$$\widetilde{H}^{(n)}_y = \frac{1}{2(n+1)}\sum_{j=-n-1}^{n} \cos\left(\frac{\pi j(2y+1)}{n+1}\right)\Xi\left(\frac{j}{n+1}\right).$$

Estimate (B.3) is then a consequence of Lemma B.1. □

Appendix C. Proofs of Lemmas 9.2, 9.3 and 9.5

C.1. Proof of Lemma 9.2. For $m \in \mathbb{Z}$ and $g = (g_0,\dots,g_n) \in \mathbb{R}^{n+1}$ define

$$S(m) := \int_0^{+\infty} e^{-2\pi i m s/\theta} e^{-As}\Sigma(g) e^{-A^T s} \mathrm{d}s = \begin{bmatrix} S^q(m) & S^{qp}(m) \\ S^{pq}(m) & S^p(m)\end{bmatrix},$$

where Σ_2 is defined in (6.10). Note that

$$\sum_{y=0}^{n} M_{x,y}(m) g_y = S^p_{x,x}(m). \tag{C.1}$$

Arguing similarly as in the proof of (6.16) we get

$$AS(m) + S(m)A^T + 2i\alpha_m S(m) = \Sigma_2(g), \tag{C.2}$$

where $\alpha_m := \pi m/\theta$. Denote

$$\widetilde{S}^{q,p}_{j,j'}(m) = \sum_{x,x'=0}^{n} S^{q,p}_{x,x'}(m)\psi_j(x)\psi_{j'}(x')$$

Following the same manipulations as those leading to (6.20) we obtain

$$\begin{aligned}
&\widetilde{S}^{p}_{j,j'}(m) = \frac{1}{2}\widetilde{S}^{q}_{j,j'}(m)\Big[\mu_{j'} + \mu_j + 2(\gamma + i\alpha_m)i\alpha_m\Big],\\
&\widetilde{S}^{q}_{j,j'}(m)\Big[\mu_{j'} - \mu_j - 4i\alpha_m(\gamma + i\alpha_m)\Big] + 4(\gamma + i\alpha_m)\widetilde{S}^{qp}_{j,j'}(m) = 0,\\
&\widetilde{S}^{qp}_{j,j'}(m)(\mu_j - \mu_{j'}) + 2i\alpha_m\widetilde{S}^{q}_{j,j'}(m)\mu_{j'} + 2(2\gamma + i\alpha_m)\widetilde{S}^{p}_{j,j'}(m) = \widetilde{F}_{j,j'}
\end{aligned} \tag{C.3}$$

and $\widetilde{F}_{j,j'} = \sum_{x=0}^{n}\psi_j(x)\psi_{j'}(x)g_x$. Solving the above system using the procedure used to deal with (6.20) we obtain

$$\widetilde{S}^{p}_{j,j'}(m) = \sum_y \Theta_m(\mu_j, \mu_{j'})\psi_j(y)g_y\psi_{j'}(y),$$

with

$$\begin{aligned}
\Theta_m(c, c') := \frac{2\gamma}{2\gamma + i\alpha_m}\Bigg\{&\Big[(2\gamma + i\alpha_m)\Big(c + c' + 2(\gamma + i\alpha_m)i\alpha_m\Big)\Big]^{-1}\\
&\times\Big[\frac{1}{4}(\gamma + i\alpha_m)^{-1}(c - c')^2 + i\alpha_m(c + c')\Big] + 1\Bigg\}^{-1}.
\end{aligned} \tag{C.4}$$

Note that $\Theta_0(c, c') = \Theta(c, c')$ defined in (6.24). From (C.1) we conclude that

$$M_{x,y}(m) = \sum_{j,j'=0}^{n}\Theta_m(\mu_j, \mu_{j'})\psi_j(x)\psi_{j'}(x)\psi_j(y)\psi_{j'}(y) \tag{C.5}$$

As in (7.5), for any sequence $(f_x) \in \mathbb{C}^{n+1}$ we can write

$$\sum_{x,y=0}^{n}(\delta_{x,y} - M_{x,y}(m))f_y^{\star}f_x = \sum_{j,j'=0}^{n}\Big(1 - \Theta_m(\mu_j, \mu_{j'})\Big)\left|\sum_{x=0}^{n}\psi_j(x)f_x\psi_{j'}(x)\right|^2.$$

We have

$$\begin{aligned}
&1 - \Theta_m(c, c')\\
&\quad = \Bigg\{i\alpha_m + \Big[c + c' + 2(\gamma + i\alpha_m)i\alpha_m\Big]^{-1}\Big[\frac{1}{4}(\gamma + i\alpha_m)^{-1}(c - c')^2 + i\alpha_m(c + c')\Big]\Bigg\}\\
&\quad\times\Bigg\{\Big[c + c' + 2(\gamma + i\alpha_m)i\alpha_m\Big]^{-1}\Big[\frac{1}{4}\Big(\gamma + i\alpha_m\Big)^{-1}(c - c')^2 + i\alpha_m(c + c')\Big] + (2\gamma + i\alpha_m)\Bigg\}^{-1}.
\end{aligned}$$

Thus $\lim_{m\to+\infty}\Big(1 - \Theta_m(c, c')\Big) = 1$. On the other hand, if $m \neq 0$, an easy calculation shows that $1 - \Theta_m(c, c') = 0$ implies that $(c - c')^2 = 8\alpha_m^2(\alpha_m^2 + \gamma^2)$ and $c + c' = 2\alpha_m^2$, where $\alpha_m = \pi m/\theta$. But this would clearly lead to a contradiction, as then we would have $|c - c'| > \sqrt{2}(c + c')$, which is clearly impossible (remember that $c, c' > 0$). Hence, there exists $\mathfrak{C}_* > 0$ such that

$$|1 - \Theta_m(c, c')| \geqslant \mathfrak{C}_*, \quad |m| \geqslant 1,\ c, c' \in [0, \omega_0^2 + 4].$$

This ends the proof of (9.9). □

C.2. Proof of Lemma 9.3. Using (C.5) we obtain

$$\sum_{x=0}^{n} |M_{x,0}(m)|^2 = \sum_{j_1,\dots,j_4=0}^{n} \Theta_m(\mu_{j_1},\mu_{j_2})\Theta_m(\mu_{j_3},\mu_{j_4}) \prod_{k=1}^{4} \psi_{j_k}(0) \sum_{x=0}^{n}\prod_{k=1}^{4} \psi_{j_k}(x). \tag{C.6}$$

Applying elementary trigonometric identities we conclude that

$$\sum_{x=0}^{n}\prod_{k=1}^{4} \psi_{j_k}(x) = \frac{1}{(n+1)^2}\left\{\prod_{k=1}^{4}(2-\delta_{0,j_k})\right\}^{1/2} \sum_{x=0}^{n}\prod_{k=1}^{4}\cos\left(\frac{\pi j_k(2x+1)}{2(n+1)}\right)$$
$$= \frac{1}{2^5(n+1)}\left\{\prod_{k=1}^{4}(2-\delta_{0,j_k})\right\}^{1/2} \sum_{\iota_1,\dots,\iota_4\in\{-1,1\}} \cos\left(\frac{\pi}{2(n+1)}\sum_{k=1}^{4}\iota_k j_k\right) 1_{2(n+1)\mathbb{Z}}\Big(\sum_{k=1}^{4}\iota_k j_k\Big)$$

Therefore we can write

$$\sum_{x=0}^{n} |M_{x,0}(m)|^2 = \sum_{\iota,\iota_1\in\{-1,1\}}\sum_{\iota',\iota_1'\in\{-1,1\}} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}u' \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_1' \mathfrak{V}_m(u,u')\mathfrak{V}_m^{\star}(u_1,u_1') \tag{C.7}$$

$$\times \exp\left\{\mathrm{i}\pi\Big(\iota u + \iota' u' + \iota_1 u_1 + \iota_1' u_1'\Big)\right\} \sum_{q\in\mathbb{Z}} \delta_q\Big(\iota u + \iota' u' + \iota_1 u_1 + \iota_1' u_1'\Big) + O_m\left(\frac{1}{n}\right), \tag{C.8}$$

where $O_m\left(\frac{1}{n}\right) \leqslant \frac{C}{n}$ for some constant $C > 0$, independent of n and m, and

$$\mathfrak{V}_m(u,u') := \Theta_m\left(\omega_0^2 + 4\sin^2\left(\frac{\pi u}{2}\right), \omega_0^2 + 4\sin^2\left(\frac{\pi u'}{2}\right)\right)\cos\left(\frac{\pi u}{2}\right)\cos\left(\frac{\pi u'}{2}\right).$$

We claim that there exists $C > 0$, independent of n and m, such that $\mathfrak{V}_m(u,u') \leqslant C$ for all $u, u' \in [0, \omega_0^2 + 4]$. Indeed, as can be seen directly from (C.4), we have $\lim_{m\to+\infty}\Theta_m(c,c') = 0$ uniformly in $c, c' \in [0, \omega_0^2+4]$. On the other hand the function $\mathbb{R}\times[0,\omega_0^2+4]^2 \ni (m,c,c') \to \Theta_m(c,c')$ is bounded on compact set. If otherwise, this would imply that there exist $(m,c,c') \in \mathbb{R}\times[0,\omega_0^2+4]^2$ such that

$$0 = \left[(2\gamma + i\alpha_m)\Big(c + c' + 2(\gamma + i\alpha_m)i\alpha_m\Big)\right]^{-1}$$
$$\times\left[\frac{1}{4}(\gamma + i\alpha_m)^{-1}(c-c')^2 + i\alpha_m(c+c')\right] + 1.$$

An easy calculation gives $c + c' = 2\alpha_m^2$ and $(c-c')^2 = 8(\alpha_m^2+\gamma^2)\alpha_m^2$, where $\alpha_m = \pi m/\theta$. This leads to a contradiction, as then $|c-c'| > \sqrt{2}(c+c')$ (but both $c, c' > 0$). Thus the conclusion of the lemma follows. □

C.3. Proof of Lemma 9.4. From (A.18) we obtain

$$[e^{-At}]_{x+n+1,x'+n+1} = \sum_{j=0}^{n} E_j(t)\psi_j(x)\psi_j(x'),$$

where (cf (6.18))

$$E_j(t) := \frac{1}{2\sqrt{\gamma^2-\mu_j}}\Big[-\lambda_{j,+}\exp\big\{\lambda_{j,+}t\big\}+\lambda_{j,-}\exp\big\{\lambda_{j,-}t\big\}\Big],\quad \text{if } \mu_j\neq\gamma^2.$$

In the case $\mu_j=\gamma^2$ (then $\lambda_{j,\pm}=\gamma$, cf (A.19)) we have $E_j(t):=(1-\gamma t)e^{-\gamma t}$. Using (A.6) we obtain therefore

$$\begin{aligned}\overline{p}_x(t) &= \frac{1}{n^{1/2}}\sum_{j=0}^{n}\int_0^{+\infty}\mathcal{F}((t-s)/\theta)[e^{-As}]_{x+n+1,2n+1}\mathrm{d}s\\ &= \frac{1}{n^{1/2}}\sum_{j=0}^{n}\psi_j(x)\psi_j(n)\int_0^{+\infty}\mathcal{F}((t-s)/\theta)E_j(s)\mathrm{d}s. \end{aligned}\tag{C.9}$$

From (C.9) we conclude that there exists $\mathfrak{p}_*>0$ such that

$$\sup_{t\in\mathbb{R},x=0,\dots,n}|\overline{p}_x(t)|\leqslant\frac{\mathfrak{p}_*}{n^{1/2}},\quad n=1,2,\dots. \tag{C.10}$$

Estimate (9.11) is then a straightforward consequence of (C.10) and (4.11). □

C.4. Proof of Lemma 9.5. Multiplying both sides of (9.4) by $V_x(t)$ and averaging over time we get

$$\begin{aligned}\langle\langle V_x^2\rangle\rangle &= \textstyle\sum_{x'=1}^{n}\int_0^{\theta}\mathfrak{g}_{x,x'}(s)\langle\langle V_x(\cdot)V_{x'}(\cdot-s)\rangle\rangle\mathrm{d}s+\langle\langle V_x v_x\rangle\rangle\\ &\leqslant \textstyle\sum_{x'=1}^{n}M_{x,x'}\langle\langle V_x^2\rangle\rangle^{1/2}\langle\langle V_{x'}^2\rangle\rangle^{1/2}+\langle\langle V_x v_x\rangle\rangle.\end{aligned}\tag{C.11}$$

Summing up over x we obtain

$$\begin{aligned}&\sum_{x,x'=0}^{n}\Big(\delta_{x,x'}-M_{x,x'}\Big)\langle\langle V_x^2\rangle\rangle^{1/2}\langle\langle V_{x'}^2\rangle\rangle^{1/2}+\langle\langle V_0^2\rangle\rangle^{1/2}\sum_{x=0}^{n}M_{x,0}\langle\langle V_x^2\rangle\rangle^{1/2}\\ &\leqslant\sum_{x=0}^{n}\langle\langle V_x v_x\rangle\rangle.\end{aligned}$$

Using (7.4) and the Cauchy-Schwarz inequality we obtain in particular that

$$M_{0,0}\langle\langle V_0^2\rangle\rangle\leqslant\left\{\sum_{x=0}^{n}\langle\langle V_x^2\rangle\rangle\right\}^{1/2}\left\{\sum_{x=0}^{n}\langle\langle v_x^2\rangle\rangle\right\}^{1/2}\leqslant\frac{C}{n}\left\{\sum_{x=0}^{n}\langle\langle V_x\rangle\rangle^2\right\}^{1/2}$$

for some $C > 0$ independent of n. The last estimate follows from (9.11). To finish the proof note that from (7.2) we have

$$M_{0,0} = \frac{4}{(n+1)^2} \sum_{j,j'=0}^{n} (1-\delta_{0,j})(1-\delta_{0,j'})\Theta(\mu_j,\mu_{j'}) \cos^2\left(\frac{\pi j}{2(n+1)}\right) \cos^2\left(\frac{\pi j'}{2(n+1)}\right)$$

$$\approx 4\gamma^2 \int_0^1 \int_0^1 \frac{\left[\omega_0^2 + 2\sin^2(\pi u/2) + 2\sin^2(\pi u'/2)\right] \cos^2(\pi u/2)\cos^2(\pi u'/2) \mathrm{d}u \mathrm{d}u'}{\gamma^2\left(\omega_0^2 + 2\sin^2(\pi u/2) + 2\sin^2(\pi u'/2)\right) + \left(\sin^2(\pi u/2) - \sin^2(\pi u'/2))\right)^2},$$

where the equality holds up to a term of order $O(1/n)$. □

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