Nonlinearity

PAPER

Diffusive limit for a Boltzmann-like equation with non-conserved momentum

To cite this article: R Esposito *et al* 2019 *Nonlinearity* **32** 4834

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **32** (2019) 4834–4852 https://doi.org/10.1088/1361-6544/ab395a

Diffusive limit for a Boltzmann-like equation with non-conserved momentum

R Esposito[1,5], P L Garrido[2], J L Lebowitz[3] and R Marra[4]

[1] International Research Center M&MOCS, Università dell'Aquila, L'Aquila, Italy
[2] Instituto Carlos I de Fisica Teorica y Computacional, Universidad de Granada, E-18071 Granada, Spain
[3] Departments of Mathematics and Physics, Rutgers University, New Brunswick, NJ, United States of America
[4] Dipartimento di Fisica and Unità INFN, Università di Roma Tor Vergata, 00133 Roma, Italy

E-mail: raff.esposito@gmail.com

Received 8 May 2019, revised 24 July 2019
Accepted for publication 7 August 2019
Published 30 October 2019

CrossMark

Recommended by Professor Charles R Doering

Abstract

We consider a kinetic model whose evolution is described by a Boltzmann-like equation for the one-particle phase space distribution $f(x, v, t)$. There are hard-sphere collisions between the particles as well as collisions with randomly fixed scatterers. As a result, this evolution does not conserve momentum but only mass and energy. We prove that the diffusively rescaled $f^\varepsilon(x, v, t) = f(\varepsilon^{-1}x, v, \varepsilon^{-2}t)$ tends, as $\varepsilon \to 0$, to a Maxwellian $M_{\rho,0,T} = \frac{\rho}{(2\pi T)^{3/2}} \exp[-\frac{|v|^2}{2T}]$, where ρ and T are solutions of coupled diffusion equations and estimate the error in $L^2_{x,v}$.

Keywords: kinetic theory, hydrodynamic limit, diffusion equations
Mathematics Subject Classification numbers: 82C40, 76S05

1. Introduction and results

We study a kinetic model investigated by Garrido and Lebowitz in [11] in which only the mass and the energy are conserved by the evolution but not the momentum. This models the flow of a gas (or fluid) in a porous medium. It can also be seen as the Grad–Boltzmann limit of a hard sphere system elastically scattered by randomly distributed obstacles. It thus serves as a simplified example for the derivation of macroscopic equations from mesoscopic kinetic ones: the number of conserved quantities is reduced from five to two. There are at present no

[5] Author to whom any correspondence should be addressed.

rigorous derivations of hydrodynamic equations in the diffusive limit when there are five conserved quantities and density and temperature are space-time dependent to the lowest order. Here we extend the heuristic analysis of this system in two dimensions described in [11] and give a fully rigorous derivation of the appropriate coupled diffusion equations.

The model is defined in the following way:

Let Ω be the three dimensional unit torus. $x \in \Omega$ denotes the position of the particle and $v \in \mathbb{R}^3_v$ its velocity. The kinetic equation on $\Omega \times \mathbb{R}^3_v$ is

$$\partial_t F + v \cdot \nabla F = Q(F), \tag{1.1}$$

with $Q(F) = Q_B(F, F) + \alpha Q_d(F)$, $\alpha \geqslant 0$, where $Q_B(F, H)$ is the symmetrized Boltzmann collision operator for hard spheres [4] defined as

$$Q_B(F,H)(v)$$
$$:= \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}w \int_{\mathbb{S}^2} \mathrm{d}\omega B(v-w,\omega)[F(v')H(w') + F(w')H(v') - F(v)H(w) - F(w)H(v)], \tag{1.2}$$

with $\omega \in \mathbb{S}^2$ and $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. $v' = v - [(v-w)\cdot\omega]\omega$, $w' = v + [(v-w)\cdot\omega]\omega$, and $B(V,\omega) = |V \cdot \omega|$ is *the hard spheres cross section* and $Q_d(F)$ models elastic collisions with randomly distributed infinite mass scatterers at rest whose effect is to remove part of the momentum during collisions. Therefore $Q_d(F)$ is a linear operator conserving only mass and energy, not momentum, so that

$$\int_{\mathbb{R}^3_v} \mathrm{d}v Q_d(F) = 0, \quad \int_{\mathbb{R}^3_v} \mathrm{d}v |v|^2 Q_d(F) = 0, \tag{1.3}$$

for any F, but

$$\int_{\mathbb{R}^3_v} \mathrm{d}v v Q_d(F) \neq 0, \tag{1.4}$$

for some F. We also require that the corresponding entropy dissipation is negative

$$D_d(F) = \int \mathrm{d}x \int \mathrm{d}v Q_d(F) \log F \leqslant 0. \tag{1.5}$$

Let $M_{\rho,u,T}$ be the local Maxwellian with ρ, u and T possibly depending on space and time

$$M_{\rho,u,T} = \frac{\rho}{(2\pi T)^{3/2}} \mathrm{e}^{-\frac{(v-u)^2}{2T}}. \tag{1.6}$$

We set

$$\eta(\rho,u,T) := -D_d(M_{\rho,u,T}) \geqslant 0, \quad \text{with} \quad \eta(\rho,u,T) = 0 \quad \text{iff } u = 0. \tag{1.7}$$

We can model the non momentum conserving collisions with the background by various choices of $Q_d(F)$ [11].

We prefer here, for simplicity of presentation, to consider the operator

$$Q_d(F) = \int_{\mathbb{S}^2} \mathrm{d}\omega [F(v - 2(v\cdot\omega)\omega) - F(v)]|v\cdot\omega|. \tag{1.8}$$

The results in this paper apply to all choices in [11]. Note that, since $|v - 2\omega(\omega\cdot v)| = |v|$, $Q_d(F) = 0$ if F depends on v only through $|v|$. By the Boltzmann H

theorem, $D_B(F) = \int_{\emptyset} \mathrm{d}x \int_{\mathbb{R}^3_v} \mathrm{d}v \log F Q_B(F,F) \leqslant 0$ and vanishes if and only if $F = M_{\rho,u,T}$. Moreover, if $\alpha > 0$, we have also

$$Q_B(F,F) + \alpha Q_d(F) = 0 \quad \text{iff } F = M_{\rho,0,T}. \tag{1.9}$$

In fact, if $Q_B(F,F) + \alpha Q_d(F) = 0$, multiplying by $\log F$ and integrating, we obtain

$$D_B(F) + \alpha D_d(F) = 0.$$

But both are non positive, so we must have $D_B(F) = 0$ and $D_d(F) = 0$. The first implies $F = M_{\rho,u,T}$. By the second of (1.7) then $u = 0$ and we get the conclusion.

From now on we assume $\alpha > 0$.

To look at the behavior of the solution on the diffusive space-time scale [5] we consider the equation for $F^\varepsilon(x,v,t) = F(\varepsilon^{-1}x, v, \varepsilon^{-2}t)$. F^ε so defined satisfies the equation

$$\partial_t F^\varepsilon + \varepsilon^{-1} v \cdot \nabla F^\varepsilon = \varepsilon^{-2} Q(F^\varepsilon), \tag{1.10}$$

and we seek for its solution in the form

$$F^\varepsilon = \mu + \sum_{i=1}^{3} \varepsilon^i F_i + \varepsilon^{5/2}\sqrt{\mu} f, \tag{1.11}$$

where $\mu = M_{\rho,0,T}$.

Note that, by total mass and total energy conservation, there is no loss of generality in assuming

$$\int_{\Omega\times\mathbb{R}^3_v} \mathrm{d}x\mathrm{d}v F^\varepsilon = \int_{\Omega\times\mathbb{R}^3_v} \mathrm{d}x\mathrm{d}v \mu, \quad \int_{\Omega\times\mathbb{R}^3_v} \mathrm{d}x\mathrm{d}v |v|^2 F^\varepsilon = \int_{\Omega\times\mathbb{R}^3_v} \mathrm{d}x\mathrm{d}v |v|^2 \mu. \tag{1.12}$$

We prove that, as $\varepsilon \to 0$, $F^\varepsilon(x,v,t)$ tends to the Maxwellian $M_{\rho,0,T}$ where ρ and T are solutions of the following set of two coupled diffusion equations for the density ρ and the temperature:

$$\begin{cases} \partial_t \rho = \nabla \cdot \left[H \frac{\nabla \rho}{\rho} \right] + \nabla \cdot \left[H' \frac{\nabla T}{T} \right], \\ \frac{3}{2}\rho \partial_t T = \nabla \cdot \left[T H' \frac{\nabla \rho}{\rho} \right] + \nabla \cdot \left[T H_1' \frac{\nabla T}{T} \right], \end{cases} \tag{1.13}$$

where H, H', H_1 are transport coefficients whose expressions are (independent of the index i)

$$H = \int \mathrm{d}v v_i \mathcal{L}^{-1}(\mu v_i); \quad H' = \int \mathrm{d}v v_i \Big(\frac{|v|^2}{2T} - \frac{3}{2}\Big) \mathcal{L}^{-1}(\mu v_i); \tag{1.14}$$

$$H_1' = \int \mathrm{d}v v_i \Big(\frac{|v|^2}{2T} - \frac{3}{2}\Big) \mathcal{L}^{-1}\Big(v_i \Big(\frac{|v|^2}{2T} - \frac{3}{2}\Big)\mu\Big). \tag{1.15}$$

Here

$$\mathcal{L}F = \mathcal{L}_B F - \alpha Q_d(F), \tag{1.16}$$

where

$$\mathcal{L}_B F = -2 Q_B(\mu, F).$$

We note that $\mathcal{L}^{-1}(v_j \mu(\frac{|v|^2}{2T} - \frac{3}{2}))$ and $\mathcal{L}^{-1}(v_j \mu)$ are well defined since the functions $\mu v_j (\frac{|v|^2}{2T} - \frac{3}{2})$ and μv_j are in the space orthogonal to the null space of $\mathcal{L}$. Thus equations (1.14) and (1.15) make sense.

We remark that the transport coefficients H and H' diverge as $\alpha \to 0$ because μv_i are in the null space of $\mathcal{L}_B$, so $\mathcal{L}_B^{-1}$ is not well defined on the function μv_i.

Moreover, we determine also F_1 as

$$F_1 = -\mathcal{L}^{-1}[v \cdot \nabla \mu] + \mu \left[\frac{\rho_1}{\rho} + \frac{|v|^2 - 3T}{2\rho T^2} T_1 \right] \tag{1.17}$$

where ρ_1, T_1 are solutions of linear diffusion equations such that

$$\int_\Omega \mathrm{d}x \rho_1 = 0, \quad \int_\Omega \mathrm{d}x T_1 = 0. \tag{1.18}$$

F_i, for $i > 1$, will be specified later.

The aim of this paper is to show that at the lowest order the behavior of the system is correctly described by the diffusive system (1.13), or, in other words, to prove the validity of the expansion (1.11). The main result of this paper is the following

Theorem 1.1. *Let $t > 0$ be fixed and assume that the solution (ρ, T) to (3.20) and (3.21) have positive lower bounds and there is $C(\rho, T) \ll 1$ such that, for $0 \leqslant k, \ell \leqslant 4$ with $1 \leqslant k + \ell \leqslant 4$,*

$$\sup_{0 \leqslant s \leqslant t} \sum_{k,\ell} (\|\nabla^k \partial_t^\ell \rho(s)\|_2 + \|\nabla^k \partial_t^\ell T(s)\|_2) < C(\rho, T). \tag{1.19}$$

Assume also that the initial value of F is positive and satisfies (4.27) below. Then, if $\varepsilon \ll 1$, (1.10) has a positive solution F, represented by (1.11), such that

$$\|\mu^{-1/2}(F - \mu)\|_2 \leqslant C\varepsilon.$$

Here, for $p \geqslant 1$, the $\|\cdot\|_p$-norm is defined as

$$\|f\|_p = \left[\int \mathrm{d}x\mathrm{d}v |f(x, v, t)|^p \right]^{1/p}; \quad \|f\|_\infty = \operatorname{ess\,sup}_{x,v} |f(x, v, t)|. \tag{1.20}$$

Remark 1.2. Theorem 1.1 holds under the strong assumptions on ρ, T and the bounds (1.19). Using the parabolic structure of equations (3.20) and (3.21) it can be proved (see e.g. [9]) that one can choose sufficiently smooth initial data $\rho_0 > \bar{\rho}_0$ and $T_0 > \bar{T}_0$, for positive constants $\bar{\rho}_0$ and $\bar{T}_0$ such that the assumptions of theorem 1.1 are satisfied in some finite time interval (see proposition 4.1).

2. Strategy of the proof

The proof of theorem (1.1) will be given in section 4 and here we only present the main ideas of the proof.

Once the Maxwellian μ and the terms of the expansion F_i, $i = 1, 2, 3$ in (1.11) are computed, the main technical problem is to obtain bounds uniform in ε for the remainder f, which solves a non linear problem. To deal with the non linearity we use an iterative procedure based on two steps. The first step is to study the linear problem obtained by pretending that the non linear term is computed using the solution of the previous step of the iteration. The aim is to bound the L^2-norm of the solution to the linear problem (see proposition 4.4). The novelty with respect to previous work using this ideas, e.g. [7], is the fact that the Maxwellian μ depends on x, t through ρ and T. This produces a term singular in ε in the inequality for $\|f\|_2^2$ which has to be dealt with. The most dangerous part of this term, depending on $(\mathbf{P}f)^2$ (here $\mathbf{P}$

is the projector on the space spanned by the conserved quantities) vanishes thanks to the fact that the Maxwellian μ has mean velocity $u = 0$.

The other terms which have to be dealt with contain a polynomial of degree three in v which gives troubles for large velocities. To this end, following [10], we introduce a global Maxwellian μ_T with temperature given by the min $T(x,t)$ assumed strictly positive and bound the high velocity tail of f in terms of the L^∞ norm of $h \sim f\sqrt{\mu}/\sqrt{\mu_T}$. Then h is bounded in L^∞ by proposition 4.3. The presence of $\|h\|_\infty$ in the energy inequality is a serious obstacle to obtaining a global in time statement. Theorem 1.1 is in fact established for arbitrary $t > 0$, but with constants depending on t.

Once the linear problem is solved, we need to get bounds on the non linear term. Here we have another novelty with respect to [7]. In that paper the non linear term is bounded in terms of $L^p([0,t],\Omega,\mathbb{R}^3_v)$ norms of f and its time derivative f_t, with $p = 3$ and 6. Here we cannot use this method because the equation for f_t involves a term which is too singular in ε. Therefore we can only use L^2 and L^∞ norms. But the singularity $\varepsilon^{-3/2}$ of the L^∞ norm of h (see proposition 4.3) has to be controlled by a sufficiently high power of ε in front of the non linear term and hence we need to look for a remainder in (1.11) of order $\varepsilon^{5/2}$ while in [7] ε was sufficient. We remark that, as a consequence, in the present case we need to assume some regularity properties of the limiting solution, while in [7] the convergence is proved without such assumption.

Remark 2.1. We conclude this section by observing that if the random collisions operator Q_d is absent ($\alpha = 0$), the problem of deriving in a rigorous way hydrodynamic equations in the diffusive limit, with non homogeneous density and temperature at time zero, is completely open. A formal expansion shows (see e.g. [2, 5, 13]) that the limiting equations are different from the Navier–Stokes equations. In stationary non homogeneous situations there are few results, see for example [1].

3. The expansion

We start presenting the expansion. Following a strategy similar to [7, 8] (where only the first two terms of the expansion are considered) we look for a solution of the form (1.11),(1.12). By substituting (1.11) into equation (1.10) we get

$$
\begin{aligned}
&-\varepsilon^{-2}[Q_B(\mu,\mu)+\alpha Q_d(\mu)]\\
&+\varepsilon^{-1}[v\cdot\nabla\mu+\mathcal{L}F_1]\\
&+\varepsilon^{0}[\partial_t\mu+v\cdot\nabla F_1+\mathcal{L}F_2-Q_B(F_1,F_1)]\\
&+\varepsilon^{1}[\partial_t F_1+v\cdot\nabla F_2+\mathcal{L}F_3-2Q_B(F_1,F_2)]\\
&+\varepsilon^{2}[\partial_t F_2+v\cdot\nabla F_3-2Q_B(F_1,F_3)]-Q_B(F_2,F_2)]\\
&+\varepsilon^{3}[\partial_t F_3-2Q_B(F_2,F_3)]+\varepsilon^4 Q(F_3,F_3)]\\
&+\varepsilon^{5/2}\Big[\partial_t(\sqrt{\mu}f)+\varepsilon^{-1}v\cdot\nabla(\sqrt{\mu}f)+\varepsilon^{-2}\mathcal{L}(\sqrt{\mu}f)\\
&-\varepsilon^{-1}\Big(2Q_B(F_1,\sqrt{\mu}f)+2\varepsilon Q_B(F_2,\sqrt{\mu}f)+2\varepsilon^2 Q_B(F_3,\sqrt{\mu}f)\Big)-\varepsilon^{1/2}Q_B(\sqrt{\mu}f,\sqrt{\mu}f)\Big]=0.
\end{aligned}
$$

We now examine all the terms in the equation above. The most diverging term $\varepsilon^{-2}[Q_B(\mu,\mu)+Q_d(\mu)]$ vanishes because μ is a local Maxwellian with vanishing mean velocity.

To cancel the diverging term of order ε^{-1} we impose

$$v\cdot\nabla\mu+\mathcal{L}F_1=0. \tag{3.1}$$

Since $\int \mathrm{d}v\mathcal{L}F = 0$ and $\int \mathrm{d}v|v|^2\mathcal{L}F = 0$, the previous equation has a solution under the solvability conditons

$$\int \mathrm{d}v v\cdot\nabla\mu = 0, \quad \int \mathrm{d}v|v|^2 v\cdot\nabla\mu = 0. \tag{3.2}$$

But $v\cdot\nabla\mu$ is odd in v, so conditions (3.2) are satisfied and one can write the most general solution to (3.1) in the form

$$F_1 = -\mathcal{L}^{-1}[v\cdot\nabla\mu] + \mu\left[\frac{\rho_1}{\rho} + \frac{|v|^2 - 3T}{2\rho T^2}T_1\right], \tag{3.3}$$

where the second term is the component of F_1 in the null space of $\mathcal{L}$. One can choose ρ_1, T_1 such that

$$\int_\Omega \mathrm{d}x\rho_1 = 0, \quad \int_\Omega \mathrm{d}xT_1 = 0. \tag{3.4}$$

The term of order ε^0 satisfies the equation

$$\partial_t\mu + v\cdot\nabla F_1 - Q_B(F_1, F_1) + \mathcal{L}F_2 = 0. \tag{3.5}$$

This equation has a solution if the following solvability conditions are satisfied

$$\int \mathrm{d}v[\partial_t\mu + v\cdot\nabla F_1] = 0, \quad \int \mathrm{d}v|v|^2[\partial_t\mu + v\cdot\nabla F_1] = 0. \tag{3.6}$$

Hence

$$F_2 = -\mathcal{L}^{-1}\Big[\partial_t\mu + v\cdot\nabla F_1 - Q_B(F_1, F_1)\Big] + \mu\left[\frac{\rho_2}{\rho} + \frac{|v|^2 - 3T}{2\rho T^2}T_2\right], \tag{3.7}$$

where ρ_2 and T_2 are functions of x, t yet undetermined.

By using the expression of F_1 in (3.3) the solvability conditions (3.6) provide two diffusion equations for the density and temperature. In fact, writing (3.6) explicitly we get

$$\int \mathrm{d}v\left\{\partial_t\mu - v\cdot\nabla\left[\mathcal{L}^{-1}[v\cdot\nabla\mu] - \mu\left[\frac{\rho_1}{\rho} + \frac{|v|^2 - 3T}{2\rho T^2}T_1\right]\right]\right\} = 0, \tag{3.8}$$

$$\int \mathrm{d}v|v|^2\left\{\partial_t\mu - v\cdot\nabla\left[\mathcal{L}^{-1}[v\cdot\nabla\mu] - \mu\left[\frac{\rho_1}{\rho} + \frac{|v|^2 - 3T}{2\rho T^2}T_1\right]\right]\right\} = 0 \tag{3.9}$$

The last term in both equations vanishes because it is odd in v. Thus, the equations reduce to

$$\int \mathrm{d}v\left\{\partial_t\mu - v\cdot\nabla\mathcal{L}^{-1}[v\cdot\nabla\mu]\right\} = 0, \tag{3.10}$$

$$\int \mathrm{d}v\frac{|v|^2}{2}\left\{\partial_t\mu - v\cdot\nabla\mathcal{L}^{-1}[v\cdot\nabla\mu]\right\} = 0. \tag{3.11}$$

The term of order ε is canceled by requiring

$$\partial_t F_1 + v\cdot\nabla F_2 + \mathcal{L}F_3 - 2Q_B(F_1, F_2) = 0. \tag{3.12}$$

We use this to find F_3:

$$F_3 = -\mathcal{L}^{-1}\Big[\partial_t F_1 + v\cdot\nabla F_2 - 2Q_B(F_1, F_2)\Big] + \mu\left[\frac{\rho_3}{\rho} + \frac{|v|^2 - 3T}{2\rho T^2}T_3\right], \tag{3.13}$$

where ρ_3 and T_3 are functions of x, t yet undetermined, provided that

$$\int \mathrm{d}v(\partial_t F_1 + v\cdot\nabla F_2) = 0, \qquad \int \mathrm{d}v|v|^2(\partial_t F_1 + v\cdot\nabla F_2) = 0. \tag{3.14}$$

Finally, we are left with an equation for f which will be discussed later.

Now we proceed by finding the explicit expression of the hydrodynamic equations (3.10) and (3.11).

We have:

$$\partial_t\rho - \sum_{i,j}\partial_i\big[\int \mathrm{d}v v_i\mathcal{L}^{-1}(v_j\mu)\frac{\partial_j\rho}{\rho}\big] - \sum_{i,j}\partial_i\big[\int \mathrm{d}v v_i\mathcal{L}^{-1}(v_j\mu(\frac{|v|^2}{2T}-\frac{3}{2}))\frac{\partial_j T}{T}\big] = 0 \tag{3.15}$$

$$\begin{aligned}\frac{3}{2}\partial_t(\rho T) &- \sum_{i,j}\partial_i\big[T\int \mathrm{d}v(\frac{|v|^2}{2T}-\frac{3}{2})v_i\mathcal{L}^{-1}(v_j\mu)\frac{\partial_j\rho}{\rho}\big] - \frac{3}{2}\sum_{i,j}\partial_i\big[T\int \mathrm{d}v v_i\mathcal{L}^{-1}(v_j\mu)\frac{\partial_j\rho}{\rho}\big]\\ &- \sum_{i,j}\partial_i\big[T\int \mathrm{d}v v_i(\frac{|v|^2}{2T}-\frac{3}{2})\mathcal{L}^{-1}(v_j\mu(\frac{|v|^2}{2T}-\frac{3}{2}))\frac{\partial_j T}{T}\big]\\ &- \frac{3}{2}\sum_{i,j}\partial_i\big[T\int \mathrm{d}v v_i\mathcal{L}^{-1}(v_j\mu(\frac{|v|^2}{2T}-\frac{3}{2}))\frac{\partial_j T}{T}\big] = 0.\end{aligned} \tag{3.16}$$

We can use that $\mathcal{L}^{-1}$ is self-adjoint with respect to the L^2 scalar product with weight μ^{-1} to write

$$\int \mathrm{d}v(\frac{|v|^2}{2T}-\frac{3}{2})v_j\mathcal{L}^{-1}(v_i\mu) = \int \mathrm{d}v v_i\mathcal{L}^{-1}(v_j\mu(\frac{|v|^2}{2T}-\frac{3}{2})). \tag{3.17}$$

Define the transport coefficients:

$$H_{ij} = \int \mathrm{d}v v_j\mathcal{L}^{-1}(\mu v_i);; \quad H'_{ij} = \int \mathrm{d}v v_j(\frac{|v|^2}{2T}-\frac{3}{2})\mathcal{L}^{-1}(\mu v_i); \tag{3.18}$$

$$H'_{1_{ij}} = \int \mathrm{d}v v_j(\frac{|v|^2}{2T}-\frac{3}{2})\mathcal{L}^{-1}(v_i(\frac{|v|^2}{2T}-\frac{3}{2})\mu). \tag{3.19}$$

For isotropy reasons $H_{ij} = \delta_{ij}H; H'_{1ij} = \delta_{ij}H'; H'_{ij} = \delta_{ij}H'_1$.

Then, the previous equations take the form:

$$\partial_t\rho = \nabla\cdot\big[H\frac{\nabla\rho}{\rho}\big] + \nabla\cdot\big[H'\frac{\nabla T}{T}\big] \tag{3.20}$$

$$\partial_t e = \nabla\cdot\big[(TH' + \frac{3}{2}TH)\frac{\nabla\rho}{\rho}\big] + \nabla\cdot\big[(TH'_1 + \frac{3}{2}TH')\frac{\nabla T}{T}\big] \tag{3.21}$$

where $e = \frac{3}{2}\rho T$ is the internal energy density.

To relate the above transport coefficients to the Onsager coefficients, we introduce the fugacity z (related to the chemical potential μ through $\frac{\mu}{T} = \log z$) given for the perfect gas in $3d$ by $\log z = \log\frac{\rho}{T^{3/2}}$ and write the equations in the form

$$\partial_t\rho = -\nabla\cdot J_\rho \tag{3.22}$$

$$\partial_t e = -\nabla\cdot J_e \tag{3.23}$$

where

$$J_\rho = -L_{\rho\rho}\nabla \log z + L_{\rho e}\nabla \frac{1}{T} \tag{3.24}$$

$$J_e = -L_{e\rho}\nabla \log z + L_{ee}\nabla \frac{1}{T} \tag{3.25}$$

so that

$$\partial_t \rho = \nabla \cdot (L_{\rho\rho}\frac{\nabla\rho}{\rho}) - \nabla \cdot [(\frac{3}{2}L_{\rho\rho} - \frac{L_{\rho e}}{T})\frac{\nabla T}{T}] \tag{3.26}$$

$$\partial_t e = \nabla \cdot (L_{e\rho}\frac{\nabla\rho}{\rho}) - \nabla \cdot [(\frac{3}{2}L_{e\rho} - \frac{L_{ee}}{T})\frac{\nabla T}{T}]. \tag{3.27}$$

By comparison,

$$H = -L_{\rho\rho}; \quad T(H' + \frac{3}{2}H) = -L_{\rho e} = -L_{e\rho}; \quad T^2(H_1' - \frac{9}{4}H) = L_{ee}.$$

In [11] the form of the transport coefficients for a different choice of the linear Boltzmann operator and in dimension 2 is discussed.

To get the regularity properties of the transport coefficients we can for example use the method in [3], where the eigenvalues of the linearized and the linear Boltzmann operators are studied by an expansion in spherical functions.

Plugging the expression for F_1 (3.3) and the one for F_2 (3.7) in the first equation of (3.14) we get

$$\begin{aligned}&\int \mathrm{d}v\big\{\partial_t(-\mathcal{L}^{-1}[v\cdot\nabla\mu] + \mu\left[\frac{\rho_1}{\rho} + \frac{|v|^2-3T}{2\rho T^2}T_1\right])\big\}\\ &+\int \mathrm{d}v v\cdot\nabla\big[\mathcal{L}^{-1}\big(\partial_t\mu + v\cdot\nabla F_1 - Q_B(F_1,F_1)\big) + \mu\left[\frac{\rho_2}{\rho} + \frac{|v|^2-3T}{2\rho T^2}T_2\right]\big] = 0.\end{aligned} \tag{3.28}$$

The first and the third term in the first integral and the first and last term in the second integral do not give contribution. We are left with

$$\partial_t\rho_1 + \partial_i \int \mathrm{d}v v_i \mathcal{L}^{-1}\big[v\cdot\nabla[\mu(\frac{\rho_1}{\rho} + \frac{|v|^2-3T}{2\rho T^2}T_1)] - Q_B(F_1,F_1)\big] = 0.$$

By using the relation $Q_B(h,h) = -\mathcal{L}_B(\mu^{-1}h^2)$ and noting that $\int \mathrm{d}v v_i \mathcal{L}_B^{-1}[\mathcal{L}_B\frac{F_1^2}{\mu}] = 0$ by oddness, we get

$$\partial_t\rho_1 + \partial_i\int \mathcal{L}^{-1}(\mu v_i)v_j\partial_j[\frac{\rho_1}{\rho} + \frac{|v|^2-3T}{2\rho T^2}T_1] + \partial_i\int \mathcal{L}^{-1}(\mu v_i)v_j[\frac{\rho_1}{\rho} + \frac{|v|^2-3T}{2\rho T^2}T_1]\frac{\partial_j\mu}{\mu} = 0$$

which is a linear diffusive non homogenous equation for ρ_1

By proceeding in the same way starting from the second compatibility condition, we get a linear diffusive non homogeneous equation for T_1.

The compatibility conditions (3.6), (3.14) thus turn out to be diffusion equations for ρ, T, ρ_1 and T_1. On the other hand ρ_2 and T_2 are determined by the orthogonality condition (4.19) discussed in section 4, which are also diffusion equations; the procedure to get the equations for them is the same and we omit it. ρ_3 and T_3 are determined by a different condition (4.20), also discussed in section 4.

4. Proofs

In this technical section we construct a solution for the equation for the remainder and prove theorem 1.1.

4.1. General setup

We need some notation. We denote

$$Lf = \mu^{-1/2}\mathcal{L}(\sqrt{\mu}f). \tag{4.1}$$

Since $\mu(v-2\omega(\omega\cdot v)) = \mu(v)$, we have

$$Lf = L_Bf + L_d(f), \tag{4.2}$$

where

$$L_Bf = -\mu^{-\frac{1}{2}}\mathcal{L}_B(\sqrt{\mu}f), \quad L_df = -\alpha Q_d(f). \tag{4.3}$$

Below, depending on the context, (f,g) denotes the standard L^2 inner product in $L^2(\mathbb{R}^3_v)$ or in $L^2(\Omega\times\mathbb{R}^3_v)$. As is well known (see e.g. [4]), the quadratic form (f,L_Bf) is non negative and strictly positive if f belongs to the orthogonal complement of the null space of $\mathcal{L}_B$, which is spanned by the orthonormal functions

$$\psi_m = \sqrt{\rho}^{-1}\sqrt{\mu}, \quad \psi_i = \sqrt{\rho T}^{-1}v_i\sqrt{\mu}, \quad i=1,2,3, \quad \psi_e = \sqrt{6\rho T^2}^{-1}(|v|^2-3T)\sqrt{\mu}. \tag{4.4}$$

On the other hand, a direct check shows that there is $\lambda_d>0$ such that

$$(\psi_i, L_d\psi_j) = \lambda_d\delta_{ij} \quad \text{for } i,j=1,2,3. \tag{4.5}$$

Let $\mathcal{N}$ be the null space of L. By the previous observations we immediately conclude that it is the linear subspace spanned by the normalized vectors

$$\psi_m = \sqrt{\rho}^{-1}\sqrt{\mu}, \quad \psi_e = \sqrt{6\rho T^2}^{-1}(|v|^2-3T)\sqrt{\mu}. \tag{4.6}$$

In fact, let us consider the quadratic form

$$(f,Lf) = (f,L_Bf)+(f,L_df). \tag{4.7}$$

If $f\in\mathcal{N}$ then $(f,Lf)=0$. Since the two terms in (4.7) are both non negative, then $(f,L_Bf)=0$ and $(f,L_df)=0$. Thus $f\in\mathcal{N}_B$ and $(f,\psi_i)=0$ for $i=1,2,3$. Hence $\mathcal{N}$ is spanned by (4.6).

Denote by $\mathbf{P}$ the projector on such a subspace and by $(\mathbf{I}-\mathbf{P})$ the projector on the orthogonal subspace. Then

$$\mathbf{P}L = L\mathbf{P} = 0. \tag{4.8}$$

For the Boltzmann collisions Grad proved (see [12]),

$$L_Bf = \nu_Bf - K_Bf \tag{4.9}$$

where K_B is a compact operator and ν_B satisfies the following bounds: there are positive $\tilde{\nu}_0$ and $\tilde{\nu}_1$ such that

$$0 < \tilde{\nu}_0\langle v\rangle \leqslant \nu_B(x,v) \leqslant \tilde{\nu}_1\langle v\rangle, \tag{4.10}$$

with $\langle v\rangle = (1+|v|^2)^{\frac{1}{2}}$. The statement can be easily extended to the present operator L which can thus be decomposed as

$$Lf = \nu f - Kf \tag{4.11}$$

where K is a compact operator and ν satisfies the following bounds which follow immediately from (1.8): there are positive ν_0 and ν_1 such that

$$0 < \nu_0\langle v\rangle \leqslant \nu(x,v) \leqslant \nu_1\langle v\rangle. \tag{4.12}$$

The following spectral inequality holds:

$$(f, Lf) \geqslant \lambda\|(\mathbf{I}-\mathbf{P})f\|_\nu^2, \tag{4.13}$$

for a positive λ and

$$\|f\|_\nu = \|\sqrt{\nu}f\|_2. \tag{4.14}$$

To prove (4.13), we note, as is well known (see e.g. [4]), that for the linear Boltzmann operator L_B the following spectral inequality holds:

$$(f, L_B f) \geqslant \lambda_B\|(\mathbf{I}-\mathbf{P}_B)f\|_\nu^2, \tag{4.15}$$

where $\mathbf{P}_B f = \mathbf{P}f + \sum_{j=1}^3 (f,\psi_j)\psi_j$ and $\psi_j = \sqrt{\frac{\mu}{\rho T}}v_j$. By (4.5) we obtain (4.13) with $\lambda = \min(\lambda_B, \lambda_d)$. Note that $\lambda_d \sim \alpha$. Thus $\lambda_d \to 0$ as $\alpha \to 0$.

The main core of the proof of theorem 1.1 is the control of the remainder $\sqrt{\mu}f$ which satisfies the following equation

$$\partial_t(\sqrt{\mu}f) + \varepsilon^{-1}v\cdot\nabla(\sqrt{\mu}f) \\ = -\varepsilon^{-2}\mathcal{L}(\sqrt{\mu}f) + \varepsilon^{-1}\mathcal{L}^1(\sqrt{\mu}f) + \varepsilon^{1/2}Q_B(\sqrt{\mu}f, \sqrt{\mu}f) + \varepsilon^{-1/2}\tilde{A}, \tag{4.16}$$

where

$$\mathcal{L}^1 f = 2Q_B(F_2 + \varepsilon F_2 + \varepsilon^2 F_3, f) \tag{4.17}$$

and

$$\tilde{A} = [Q_B(F_2,F_2) + 2Q_B(F_1,F_3) + 2\varepsilon Q_B(F_2,F_3) + \varepsilon^2 Q(F_3,F_3) - \partial_t F_2 - \varepsilon\partial_t F_3 - v\cdot\nabla F_3]. \tag{4.18}$$

For the estimate of f it will be essential that the part of $\tilde{A}$ in the null space of $\mathcal{L}$ vanishes. For this reason we also impose that

$$\int \mathrm{d}v(\partial_t F_2 + v\cdot\nabla F_3) = 0, \qquad \int \mathrm{d}v|v|^2(\partial_t F_2 + v\cdot\nabla F_3) = 0. \tag{4.19}$$

$$\int \mathrm{d}v\partial_t F_3 = 0, \qquad \int \mathrm{d}v|v|^2\partial_t F_3 = 0. \tag{4.20}$$

As before, (4.19) becomes a couple of linear non homogeneous parabolic equations.

Finally, we use the freedom of choice of ρ_3 and T_3 to ensure (4.20).

We have the following

Proposition 4.1. *Assume $0 < \rho_0(x)$, $0 < T_0(x)$ with finite L^∞ norms. Then, for any $\bar{t} > 0$ the equations (3.20) and (3.21) have smooth solutions. Moreover, if (1.19) is verified at time $t = 0$, then it stays true for $t \in [0, \bar{t}]$. The functions F_i satisfy the inequalities*

$$\|\mu^{-1/2}F_i\|_2 + \|\mu^{-1/2}F_i\|_\infty \leqslant C(\rho, T), \tag{4.21}$$

for $i = 1, 2, 3$.

Proposition 4.1 is a simple consequence of the parabolic regularity and we omit the proof (see e.g. [9]).

It is convenient to write

$$\tilde{A} = \sqrt{\mu}A, \quad \Gamma(f,g) = \sqrt{\mu}^{-1}Q_B(\sqrt{\mu}f, \sqrt{\mu}g). \tag{4.22}$$

We note that

$$\mathbf{P}\Gamma(f,g) = 0. \tag{4.23}$$

The equation for f then becomes

$$\partial_t f + \varepsilon^{-1}v\cdot\nabla f + \frac{1}{2}f[(\partial_t + \varepsilon^{-1}v\cdot\nabla)\log\mu] + \varepsilon^{-2}Lf = \varepsilon^{-1}L^1 f + \varepsilon^{\frac{1}{2}}\Gamma(f,f) + \varepsilon^{-\frac{1}{2}}A, \tag{4.24}$$

where

$$L^1 f = 2\Gamma(f_1 + \varepsilon f_2 + \varepsilon^2 f_3, f), \tag{4.25}$$

and

$$\begin{aligned} A = {} & [\Gamma(f_2,f_2) + 2\Gamma(f_1,f_3) - \mu^{-1/2}(\partial_t(\sqrt{\mu}f_2) + v\cdot\nabla(\sqrt{\mu}f_3)] \\ & + 2\varepsilon\Gamma(f_2,f_3) + \varepsilon^2\Gamma(f_3,f_3) - \mu^{-1/2}\varepsilon\partial_t(\sqrt{\mu}f_3), \end{aligned} \tag{4.26}$$

where we have set $f_i = \mu^{-1/2}F_i$. Equation (4.24) has to be solved with initial datum $f(0)$ such that

$$F(0) = \mu(0) + \varepsilon F_1(0) + \varepsilon^2 F_2(0) + \varepsilon^3 F_3(0) + \varepsilon^{5/2}\sqrt{\mu}(0)f(0) > 0. \tag{4.27}$$

It is standard to check (see e.g. [7]) that it is possible to construct $f(0)$ so that (4.27) is satisfied.

It will be essential to have $\mathbf{P}A = 0$. To this end we have required (4.19) and (4.20) which imply

$$\mathbf{P}[\mu^{-1/2}(\partial_t(\sqrt{\mu}f_2) + v\cdot\nabla(\sqrt{\mu}f_3)] = 0, \quad \mathbf{P}[\mu^{-1/2}\partial_t(\sqrt{\mu}f_3)] = 0 \tag{4.28}$$

and hence

$$\mathbf{P}A = 0. \tag{4.29}$$

We note the presence in (4.24) of the divergent term $\frac{1}{2}f[\varepsilon^{-1}v\cdot\nabla\log\mu]$, which is a source of extra difficulties. It is due to the use of the decomposition $F = \mu + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + \varepsilon^{5/2}\sqrt{\mu}f$ instead of $F + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + \varepsilon^{5/2}R$. However the first decomposition is useful to take advantage of the spectral properties of the operator L in $L^2(\mathbb{R}^3_v)$. Alternatively, we could use the second decomposition, but then the spectral properties we need should be sought for in $L^2(\mathbb{R}^3_v, \mu^{-1})$ and, when integrating by parts, the weight μ^{-1} would produce a similar term in the resulting equation for $\|\mu^{-1/2}R\|_2$. As we shall see, the spectral properties are crucial in our proof, so we need to deal with this extra term.

We will follow the approach in [10] and [7]. Instead of repeating all the proofs in these papers, we only outline and give explicit proofs when the previous approach has to be modified to adapt to the case we study. The difference with respect to [10] is that the scaling in this case is diffusive instead of hyperbolic. The main difference with respect to [7] is in getting a L^2 bound for the linear equation due to the presence of the third term in (4.35) below. This term appears because μ depends on x and t.

First of all we remark that in this term there will be a contribution including a higher power of velocity, $|v|^3 f$,which is not present in the case studied in [7]. Following [10], we introduce L^2 and L^∞ polynomial norms to control it.

By the assumption that $C(\rho, T)$ is sufficiently small, we see that there is $T_M > 0$ such that for any $0 \leqslant s \leqslant t$ and $x \in \emptyset$,

$$T_M < T(s,x) < 2T_M. \tag{4.30}$$

We define the global Maxwellian

$$\mu_M = \frac{1}{(2\pi T_M)^{3/2}} \exp\left\{\frac{-|v|^2}{2T_M}\right\}. \tag{4.31}$$

The inequalities (4.30) imply that there exist constants c_1 and c_2 such that for some $1/2 < \alpha < 1$ and for each (t, x, v)

$$c_1\mu_M \leqslant \mu \leqslant c_2\mu_M^\alpha. \tag{4.32}$$

We stress that the previous bounds are true under the assumption of a slowly varying $T(x, t)$. Furthermore, we introduce the polynomial $w_\sigma(v) = (1 + |v|^2)^\sigma$ for the control of the cubic power of velocity. We define h by the position

$$f = (1 + |v|^2)^{-\sigma}\sqrt{\frac{\mu_M}{\mu}}h, \tag{4.33}$$

and choose $\sigma > 0$ later. Note that in consequence of (4.32) we have

$$|f(x,t)| \leqslant C|h(x,t)|, \tag{4.34}$$

for some $C > 0$.

4.2. The linear problem

As usual, to deal with the non linear problem (4.24), we use an iterative procedure (see section 4.3) where, at the step n of the iteration the right hand side of (4.24) is computed at step $n - 1$ and one is left with the solution of a linear problem. Thus we start with the linear equation

$$\partial_t f + \varepsilon^{-1}v\cdot\nabla f + \frac{1}{2}f[(\partial_t + \varepsilon^{-1}v\cdot\nabla)\log\mu] + \varepsilon^{-2}Lf = g, \tag{4.35}$$

with some $g \in L^2(\Omega \times \mathbb{R}_v)$, such that,

$$\mathbf{P}g = 0. \tag{4.36}$$

In applying the result, we shall use g of the form

$$g = \varepsilon^{-1}L^1 f + \varepsilon^{\frac{1}{2}}\Gamma(f,f) + \varepsilon^{-\frac{1}{2}}A, \tag{4.37}$$

so that (4.36) is satisfied.

As we shall see, the use of the spectral inequality (4.13) in the energy inequality provides the control of the L^2-norm of $(\mathbf{I} - \mathbf{P})f$. What is missing is the control of the L^p-norm of $\mathbf{P}f$. This is achieved as in [7], which can be extended to the present setup. We have the following

Proposition 4.2. *Suppose* $\mathbf{P}g = 0$. *If* $\varepsilon > 0$ *is sufficiently small, there exists a function* $G(t)$ *such that, for all* $0 \leqslant s \leqslant t$, $|G(s)| \leqslant \|f(s)\|_2^2$ *and a constant* C *so that*

$$\int_s^t \|\mathbf{P}f(\tau)\|_\nu^2 \leqslant C\left[G(t) - G(s) + \varepsilon^2\int_s^t \|\nu^{-\frac{1}{2}}g(\tau)\|_2^2 + \varepsilon^{-2}\int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2\right]. \tag{4.38}$$

The proof of this proposition is postponed to the next section. Note that in [7] it is only requested that $\int \mathrm{d}x\mathrm{d}vg = 0$, clearly implied by the stronger condition $\mathbf{P}g = 0$.

As we shall see, the control of the cubic in v term appearing in the energy inequality requires an L^∞ estimate of the function h defined in (4.33). This will be achieved by using the proposition below. As a consequence of (4.35), h satisfies the equation

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla h + \frac{1}{\varepsilon^2} L_M h = \tilde{g}, \tag{4.39}$$

with

$$L_M h = \frac{w}{\sqrt{\mu}_M} \mathcal{L}(\frac{1}{w}\sqrt{\mu_M} h), \quad \tilde{g} = w\sqrt{\frac{\mu}{\mu_T}} g, \tag{4.40}$$

where $w = (1 + |v|^2)^\sigma$.

Proposition 4.3. *If f solves (4.35) with g given by (4.37) and ρ, T are smooth solutions to (3.22) and (3.23) with stricly positive lower bounds uniform in t, such that inequalities (4.32) are satisfied, then there is a constant C such that, for ε sufficiently small,*

$$\sup_{0\leqslant s\leqslant t} \|\varepsilon^{3/2} h(s)\|_\infty \leqslant C\Big[\varepsilon^{3/2}\|h_0\|_\infty + \sup_{0\leqslant s\leqslant t} \|f\|_2 + \varepsilon^{7/2}\|\frac{1}{1+|v|^2}\tilde{g}\|_\infty\Big]. \tag{4.41}$$

The proof is given in [7] and [10].

This proposition will be used within an iterative procedure where, at some step we know $\bar{h}$ and want to compute h using (4.39) with

$$\tilde{g} = \varepsilon^{-1} L_M^1 \bar{h} + \varepsilon^{-1/2} \frac{w}{\sqrt{\mu}_M} \tilde{A} + \varepsilon^{1/2} \frac{1}{2} \frac{w}{\sqrt{\mu}} Q_B(\frac{1}{w}\sqrt{\mu_M}\bar{h}, \frac{1}{w}\sqrt{\mu_M}\bar{h}), \tag{4.42}$$

where

$$L_M^1 \bar{h} = \frac{w}{\sqrt{\mu}_M} \mathcal{L}^1(\frac{1}{w}\sqrt{\mu_M}\bar{h}). \tag{4.43}$$

We shall use the bounds [10]:

$$|L_M^1 \bar{h}|_\infty \leqslant \nu_M \|\bar{h}\|_\infty \|\frac{w}{\sqrt{\mu}_M} \sum_{i=1}^{3} \varepsilon^{i-1} F^i\|_\infty \leqslant \nu_M C(\rho, T)\|\bar{h}\|_\infty, \tag{4.44}$$

$$\Big|\frac{w}{\sqrt{\mu}_M} Q_B(\frac{1}{w}\sqrt{\mu_M}\bar{h}, \frac{1}{w}\sqrt{\mu_M}\bar{h})\Big| \leqslant \nu(\mu)\|\bar{h}\|_\infty^2. \tag{4.45}$$

To get an L^2 bound on the linear equation we multiply it by f and integrate in x and v to obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_2^2 + \int \mathrm{d}x\mathrm{d}v f \partial_t \log\mu + \frac{1}{2}\varepsilon^{-1} \int \mathrm{d}x\mathrm{d}v f^2 v \cdot \nabla \log\mu + \varepsilon^{-2}(f, Lf) = (f, g). \tag{4.46}$$

Proposition 4.4. *Suppose that $C(\rho, T)$ is sufficiently small. If f solves (4.35) with g satisfying (4.36) and ρ, T are smooth solutions to (3.22) and (3.23) with stricly positive lower bounds uniform in t, then, fixed $t > 0$, there is a constant C such that, for ε sufficiently small,*

$$\|f(t)\|_2^2+\varepsilon^{-2}\int_0^t \mathrm{d}s\|(\mathbf{I}-\mathbf{P})f(s)\|_\nu^2$$
$$\leqslant \varepsilon^2\int_0^t \mathrm{d}s\|\nu^{-1/2}g(s)\|_2^2 + C_\kappa C(\rho,T)\varepsilon^4\int_0^t \|h(s)\|_\infty^2 + 2\|f(0)\|_2^2. \tag{4.47}$$

Proof. In equation (4.46) the essential difference w.r.t. [7] are the second and third terms. To bound the third term we split $f^2=[\mathbf{P}f]^2+[(\mathbf{I}-\mathbf{P})f]^2+2\mathbf{P}f(\mathbf{I}-\mathbf{P})f$ and compute the contributions separately. The most singular (in ε) one is due to $|\mathbf{P}f|^2$, whose size in L^2 is ε^{-2}-times larger than $\|(\mathbf{I}-\mathbf{P})f\|_2^2$ by proposition 4.2. Fortunately, with $\mathbf{P}f=\sqrt{\mu}[a+c(|v|^2-3T)$ we have

$$\int \mathrm{d}x\mathrm{d}v(\mathbf{P}f)^2 v\cdot\nabla\log\mu=\int \mathrm{d}x\mathrm{d}v[a+c(|v|^2-3T)]^2\mu v\cdot\nabla\log\mu=0, \tag{4.48}$$

because $v\cdot\nabla\log\mu$ is odd in v, while $a+c(|v|^2-3T)$ is even in v. Thus the largest term vanishes. As for the term $\varepsilon^{-1}\int \mathrm{d}x\mathrm{d}v\mathbf{P}f(\mathbf{I}-\mathbf{P})fv\cdot\nabla\log\mu$, we note that

$$\|[v\cdot\nabla\log\mu]\nu^{-1/2}\mathbf{P}f\|_2\leqslant C(\rho,T)\|\mathbf{P}f\|_2$$

because the factor $\sqrt{\mu}$ in $\mathbf{P}f$ controls the polynomial $\nu^{-1/2}v\cdot\nabla\log\mu$. Thus

$$\left|\varepsilon^{-1}\int \mathrm{d}x\mathrm{d}v\nu^{-1/2}v\cdot\nabla\log\mu\mathbf{P}f\nu^{1/2}(\mathbf{I}-\mathbf{P})f\right|\leqslant C\varepsilon^{-1}C(\rho,T)\|\mathbf{P}f\|_2\|(\mathbf{I}-\mathbf{P})f\|_\nu$$
$$\leqslant\frac12 C(\rho,T)\|\mathbf{P}f\|_2^2+\frac12\varepsilon^{-2}C(\rho,T)\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 \tag{4.49}$$

and the first term is controlled using proposition 4.2.

Now, as in [10] we introduce a cut-off on the velocity $\kappa\varepsilon^{-a}$ for some $a>0$ to be chosen, and estimate separately the term with low and high velocity.

We bound $\frac12\varepsilon^{-1}\int \mathrm{d}x\mathrm{d}v((\mathbf{I}-\mathbf{P})f)^2v\cdot\nabla\log\mu$. We have

$$\left|\varepsilon^{-1}\int \mathrm{d}x\mathrm{d}v((\mathbf{I}-\mathbf{P})f)^2v\cdot\nabla\log\mu\right|\leqslant\varepsilon^{-1}\int_{v\leqslant\kappa\varepsilon^{-a}}+\varepsilon^{-1}\int_{v\geqslant\kappa\varepsilon^{-a}}.$$

The first term is bounded as

$$\varepsilon^{-1}C(\rho,T)\|[(1+v^2)^{3/4}\nu^{-1/2}I_{v\leqslant\frac{k}{\varepsilon^a}}(\mathbf{I}-\mathbf{P})f\nu^{1/2}]^2\leqslant$$
$$C\varepsilon^{-1}\left(\frac{\kappa}{\varepsilon^a}\right)^2C(\rho,T)\|(\mathbf{I}-\mathbf{P})f\|_\nu^2. \tag{4.50}$$

The high velocity part is bounded as

$$\varepsilon^{-1}C(\rho,T)\|(1+v^2)^{3/2}\nu^{-1/2}fI_{v\geqslant\frac{\kappa}{\varepsilon^a}}\|_\infty\|(\mathbf{I}-\mathbf{P})f\|_\nu\leqslant C_\kappa C(\rho,T)\varepsilon\|h\|_\infty\|(\mathbf{I}-\mathbf{P})f\|_\nu,$$

where $C_k=\frac{1}{(\varepsilon^{2a}+\kappa^2)^2}$. Moreover we have used that $\mu_M\leqslant C\mu$ and $|(1+v^2)^{3/2}\nu^{-/2}f|\leqslant|(1+v^2)^{-2}h|$ for $\sigma>\frac54+2$ and $a=\frac12$. Now,

$$C_\kappa C(\rho,T)\varepsilon\|h\|_\infty\|(\mathbf{I}-\mathbf{P})f\|_2\leqslant C_\kappa C(\rho,T)\varepsilon[\delta\|h\|_\infty^2+\frac{1}{4\delta}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2]$$
$$\leqslant C_\kappa C(\rho,T)\varepsilon^4\|h\|_\infty^2+\frac14 C_\kappa C(\rho,T)\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2], \tag{4.51}$$

by choosing $\delta = \varepsilon^3$.

The term

$$\int_0^t \mathrm{d}s \frac{1}{2} \int \mathrm{d}x\mathrm{d}v f^2 \partial_t \log \mu$$

also contains a contribution involving $|\mathbf{P}f|^2$ which in this case is not zero. We have

$$\int_0^t \mathrm{d}s \frac{1}{2} \int \mathrm{d}x\mathrm{d}v (\mathbf{P}f)^2 \partial_t \log \mu \leqslant C(\rho, T) \int_0^t \mathrm{d}s (\|a(s)\|_2^2 + \|c(s)\|_2^2), \tag{4.52}$$

so that proposition 4.2 can be used. The other terms can be controlled as before. Note that there is no ε^{-1} factor.

Next, since $\mathbf{P}g = 0$, we have the bound

$$|(f, g)| \leqslant \gamma\varepsilon^{-2}\|(\mathbf{I} - \mathbf{P})f\|_\nu^2 + \frac{1}{4\gamma}\varepsilon^2\|\nu^{-1/2}g\|_2^2. \tag{4.53}$$

Summarizing, by using (4.13) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_2^2 + \varepsilon^{-2}\|(\mathbf{I} - \mathbf{P})f\|_\nu^2[\lambda - \gamma - C(\rho, T)(C\kappa^2 + C_\kappa)]$$
$$\leqslant \frac{\varepsilon^2}{4\gamma}\|\nu^{-1/2}g\|_2^2 + CC(\rho, T)\|\mathbf{P}f\|_2^2 + C_\kappa C(\rho, T)\varepsilon^4\|h\|_\infty^2. \tag{4.54}$$

Integrating on time between 0 and t we obtain:

$$\frac{1}{2}\|f(t)\|_2^2 + \varepsilon^{-2}\int_0^t \mathrm{d}s\|(\mathbf{I} - \mathbf{P})f(s)\|_\nu^2[\lambda - \gamma - C(\rho, T)(C\kappa^2 + C_\kappa)] \leqslant \frac{1}{2}\|f(0)\|_2^2$$
$$+ \int_0^t \mathrm{d}s\frac{\varepsilon^2}{4\gamma}\|\nu^{-1/2}g(s)\|_2^2 + CC(\rho, T)\int_0^t \mathrm{d}s\|\mathbf{P}f(s)\|_2^2 + C_\kappa C(\rho, T)\varepsilon^4\int_0^t \mathrm{d}s\|h(s)\|_\infty^2. \tag{4.55}$$

By using proposition 4.2 we can replace $\int_0^t \mathrm{d}s\|\mathbf{P}f(s)\|_2^2$ by the right hand side of (4.38) so that (4.55) becomes

$$[\frac{1}{2} - CC(\rho, T)]\|f(t)\|_2^2 + \varepsilon^{-2}\int_0^t \mathrm{d}s\|(\mathbf{I} - \mathbf{P})f(s)\|_\nu^2[\lambda - \gamma - C(\rho, T)(C\kappa^2 + C_\kappa) - CC(\rho, T)]$$
$$\leqslant [\frac{1}{2} - CC(\rho, T)]\|f(0)\|_2^2 + \varepsilon^2\int_0^t \mathrm{d}s[\frac{1}{4\gamma} + CC(\rho, T)]\|\nu^{-1/2}g(s)\|_2^2$$
$$+ C_\kappa C(\rho, T)\varepsilon^4\int_0^t \mathrm{d}s\|h(s)\|_\infty^2. \tag{4.56}$$

Now we choose the parameters κ, γ and $C(\rho, T)$ is such a way that

$$\frac{1}{2} - CC(\rho, T) > \frac{1}{4},$$
$$\lambda - \gamma - C(\rho, T)(C\kappa^2 + C_\kappa) - CC(\rho, T) > \frac{1}{4},$$
$$\frac{1}{4\gamma} + CC(\rho, T) < \frac{1}{4},$$

so that (4.57) becomes

$$\|f(t)\|_2^2+\varepsilon^{-2}\int_0^t \mathrm{d}s\|(\mathbf{I}-\mathbf{P})f(s)\|_\nu^2$$
$$\leqslant \varepsilon^2\int_0^t \mathrm{d}s\|\nu^{-1/2}g(s)\|_2^2+C_\kappa C(\rho,T)\varepsilon^4\int_0^t \|h(s)\|_\infty^2+2\|f(0)\|_2^2. \tag{4.57}$$

□

Proof of proposition 4.2.

Proof. It can be shown along the lines of [6]. We consider the following weak version of (4.35), obtained by multiplying (4.35) by a smooth function ψ, integrating for $(x,v,s)\in\Omega\times\mathbb{R}_v^3\times[0,t]$ and integrating by parts to move all the derivatives on ψ:

$$\begin{aligned}&\int_{\emptyset\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,t)\psi(x,v,t)-\int_{\emptyset\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,0)\psi(x,v,0)\\&-\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,s)\partial_t\psi(x,v,s)+\frac{1}{2}\int_0^t\mathrm{d}s\int_{\emptyset\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,s)\psi(x,v,s)\partial_t\log\mu\\&-\frac{1}{\varepsilon}\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,s)v\cdot\nabla\psi(x,v,s)\\&+\frac{1}{2\varepsilon}\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,s)\psi(x,v,s)v\cdot\nabla\log\mu\\&+\frac{1}{\varepsilon^2}\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v\psi(x,v,s)(Lf)(x,v,s)=\int_0^t\mathrm{d}s\int_{\emptyset\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v\psi(x,v,s)g(x,v,s),\end{aligned}\tag{4.58}$$

for smooth test function ψ. We apply this for $\psi=\sqrt{\mu}\zeta$. Then from (4.58) we get

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,t)\sqrt{\mu}\zeta(x,v,t)-\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,0)\sqrt{\mu}\zeta(x,v,0)\\&-\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,s)\sqrt{\mu}\partial_t\zeta(x,v,s)-\frac{1}{\varepsilon}\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v f(x,v,s)\sqrt{\mu}v\cdot\nabla\zeta(x,v,s)\\&+\frac{1}{\varepsilon^2}\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v\sqrt{\mu}\zeta(x,v,s)(Lf)(x,v,s)=\int_0^t\mathrm{d}s\int_{\Omega\times\mathbb{R}_v^3}\mathrm{d}x\mathrm{d}v g(x,v,s)\sqrt{\mu}\zeta(x,v,s).\end{aligned}\tag{4.59}$$

We remark that the *bad* term disappears due to a cancellation. We can write

$$\mathbf{P}f=[a+c(|v|^2-3T)]\sqrt{\mu},\tag{4.60}$$

for some funtions $a(x,t)$, $c(x,t)$ such that

$$\int_\emptyset \mathrm{d}x a(x,t)=0,\qquad \int_\emptyset \mathrm{d}x c(x,t)=0.\tag{4.61}$$

The conditions (4.61) are satisfied in consequence of the assumption (1.12).

To get bounds on a and c we use some particular functions ζ_a and ζ_c. In fact we choose

$$\zeta_a=(|v|^2-\beta_a)v\cdot\nabla\phi_a,\tag{4.62}$$

$$\zeta_c=(|v|^2-\beta_c)v\cdot\nabla\phi_c,\tag{4.63}$$

where ϕ_a solves

$$-\Delta\phi_a = a, \tag{4.64}$$

and ϕ_c solves

$$-\Delta\phi_c = c, \tag{4.65}$$

and β_a and β_c are constants to be chosen as in [6, 7]. The zero average conditions for a and c (4.61) are essential to ensure the solvability of (4.62) and (4.63) and they are compatible with the equation as a consequence of the (1.12). The estimates (4.38) of $\|a\|_2$ and $\|c\|_2$ are obtained as in [6, 7]. □

4.3. The non linear problem

Now we remind that g is given by (4.37). The strategy to construct the solution to (4.24) is to define a sequence $\{f^{(n)}\}_{n=0}^{n=\infty}$ of solutions to the following *linear* problems: $f^{(0)} = 0$ and, for $n > 0$

$$\partial_t f^{(n)} + \varepsilon^{-1} v\cdot\nabla f^{(n)} + \frac{1}{2} f^{(n)}[(\partial_t + \varepsilon^{-1} v\cdot\nabla)\log\mu] + \varepsilon^{-2} L f^{(n)} = g^{(n)}, \tag{4.66}$$

with given $g^{(n)} \in L^2(\Omega\times\mathbb{R}_v)$. In view of (4.24) the choice of $g^{(n)}$ will be

$$g^{(n)} = \varepsilon^{-1} L^1 f^{(n-1)} + \varepsilon^{\frac{1}{2}}\Gamma(f^{(n-1)}, f^{(n-1)}) + \varepsilon^{-\frac{1}{2}} A, \tag{4.67}$$

for $n \geqslant 1$. In consequence, we also define $h^{(n)}$ so that

$$f^{(n)} = (1+|v|^2)^{-\sigma}\sqrt{\frac{\mu_M}{\mu}}\, h^{(n)}. \tag{4.68}$$

By (4.67), we have

$$\mathbf{P} g^{(n)} = 0. \tag{4.69}$$

We have the following

Lemma 4.5.

$$\|\nu^{-1/2}\Gamma(f,f)\|_2 \leqslant C\|f\|_2\|h\|_\infty, \|\nu^{-1/2}\Gamma(f,f)\|_2 \leqslant C\|h\|_\infty^2; \tag{4.70}$$

$$\|L^1 f\|_2 \leqslant C(\rho,T)\|f\|_2, \qquad \|L^1 f\|_\infty \leqslant C(\rho,T)\|h\|_\infty; \tag{4.71}$$

$$\|A\|_2 \leqslant C(\rho,T), \qquad \|A\|_\infty \leqslant C(\rho,T). \tag{4.72}$$

The proof of the lemma follows as in [7], using also proposition 4.1 for (4.71) and (4.72) and we do not repeat it.

As a consequence, reminding (4.67), and using $\|f^{n-1}\|_\infty \leqslant C\|h^{n-1}\|_\infty$, we have

$$\varepsilon^2\int_0^t \mathrm{d}s\|g^{(n-1)}(s)\|_2^2$$
$$\leqslant C(\sup_{0\leqslant s\leqslant t}\varepsilon^{3/2}\|h^{(n-1)}(s)\|_\infty)^2\int_0^t \mathrm{d}s\|f^{(n-1)}(s)\|_2^2 + C(\rho,T)^2\int_0^t \mathrm{d}s\|f^{(n-1)}(s)\|_2^2 + tC(\rho,T)^2\varepsilon, \tag{4.73}$$

$$\varepsilon^{7/2}\sup_{0\leqslant s\leqslant t}\|\tilde{g}^{n-1}\|_\infty\leqslant\varepsilon C(\varepsilon^{3/2}\sup_{0\leqslant s\leqslant t}\|h^{n-1}(s)\|_\infty)^2+\varepsilon C(\rho,T)\varepsilon^{3/2}\sup_{0\leqslant s\leqslant t}\|h^{n-1}(s)\|_\infty+\varepsilon^2C(\rho,t). \tag{4.74}$$

By using (4.73) and (4.47) we have

$$\|f^{(n)}(t)\|_2^2+\varepsilon^{-2}\int_0^t\mathrm{d}s\|(\mathbf{I}-\mathbf{P})f^{(n)}(s)\|_\nu^2\leqslant(\sup_{0\leqslant s\leqslant t}\varepsilon^{3/2}\|h^{(n-1)}(s)\|_\infty)^2\int_0^t\mathrm{d}s\|f^{(n-1)}(s)\|_2^2$$
$$+C(\rho,T)^2\int_0^t\mathrm{d}s\|f^{(n-1)}(s)\|_2^2+\varepsilon tC_\kappa C(\rho,T)(\sup_{0\leqslant s\leqslant t}\varepsilon^{3/2}\|h^{(n-1)}(s)\|_\infty)^2+2\|f(0)\|_2^2+tC(\rho,T)^2. \tag{4.75}$$

By using (4.74) and (4.41) we have

$$\sup_{0\leqslant s\leqslant t}\|\varepsilon^{3/2}h^{(n)}(s)\|_\infty^2\leqslant C^2\Big[\varepsilon^{3/2}\|h_0\|_\infty+\sup_{0\leqslant s\leqslant t}\|f^{(n)}\|_2$$
$$+\varepsilon C(\varepsilon^{3/2}\sup_{0\leqslant s\leqslant t}\|h^{(n-1)}(s)\|_\infty)^2+\varepsilon C(\rho,T)\varepsilon^{3/2}\sup_{0\leqslant s\leqslant t}\|h^{(n-1)}(s)\|_\infty+\varepsilon^2C(\rho,t)\Big]^2. \tag{4.76}$$

We assume that

$$\|f(0)\|_2^2\leqslant c_0\ll 1,\quad \varepsilon^{3/2}\|h(0)\|_\infty^2\leqslant c_1\ll 1. \tag{4.77}$$

Inductive hypothesis: Fixed $\bar{t}>0$, assume

$$\sup_{0\leqslant\ell\leqslant n-1}\sup_{0\leqslant t\leqslant\bar{t}}\|f^{(\ell)}(t)\|_2^2\leqslant\eta_0\ll 1,\qquad \sup_{0\leqslant\ell\leqslant n-1}\sup_{0\leqslant t\leqslant\bar{t}}(\varepsilon^{3/2}\|h^{(\ell)}(t)\|_\infty)^2\leqslant\eta_1\ll 1. \tag{4.78}$$

By using this assumption we have

$$\sup_{0\leqslant t\leqslant\bar{t}}\|f^{(n)}(t)\|_2^2\leqslant\eta_1\eta_0\bar{t}+C(\rho,T)^2\bar{t}\eta_0+\varepsilon\bar{t}C_\kappa C(\rho,T)\eta_1^2+2c_0+\bar{t}C(\rho,T). \tag{4.79}$$

We choose ε, c_0, $C(\rho,T)$, C_κ so that

$$2c_0+tC(\rho,T)<\frac{\eta_0}{3},\quad \varepsilon\bar{t}C_\kappa C(\rho,T)\eta_1^2<\frac{\eta_0}{3},\quad \eta_1\bar{t}+C(\rho,T)^2\bar{t}<\frac{1}{3},$$

so that

$$\sup_{0\leqslant t\leqslant\bar{t}}\|f^{(n)}(t)\|_2^2<\eta_0. \tag{4.80}$$

Then

$$\sup_{0\leqslant t\leqslant\bar{t}}(\|\varepsilon^{3/2}h^{(n)}(t)\|_\infty)^2\leqslant C^2c_1+C\eta_0+\varepsilon^2C^2\eta_1^2+\varepsilon C(\rho,T)\eta_1+\varepsilon^4C(\rho,t)^2\Big]. \tag{4.81}$$

We choose ε, c_1, and η_0 so that

$$C^2c_1+C\eta_0+\varepsilon^4C(\rho,t)^2<\frac{\eta_1}{2},\quad \varepsilon^2C^2\eta_1+\varepsilon C(\rho,T)<\frac{\eta_1}{2},$$

so that

$$\sup_{0\leqslant t\leqslant\bar{t}}\|h^{(n)}\|_\infty^2<\eta_1. \tag{4.82}$$

Therefore the inductive hypothesis is verified up to n. This shows that the sequence $\{f^{(n)}\}_{n=0}^{n=\infty}$ is uniformly bounded in L^2 by η_0 and in L^∞ by η_1. By similar arguments one can show that

$\|f^{(n)} - f^{(n-1)}\|_2 \leqslant \theta \|f^{(n-1)} - f^{(n-2)}\|_2$ for some $\theta < 1$ and hence the sequence is convergent and the limit solves uniquely (4.24). The proof of the positivity of F is standard [7]. This concludes the proof of theorem 1.1.

Acknowledgments

This work was supported in part by AFOSR (Grant FA-9550-16-1- 0037). PLG was supported also by the Spanish governement project FIS2013-43201P.

ORCID iDs

R Esposito https://orcid.org/0000-0003-1744-4161

References

[1] Arkeryd L, Esposito R, Marra R and Nourri A 2011 Ghost effect by curvature in planar Couette flow *Kinetic Relat. Models* **4** 109–38
[2] Bobylev A V 1995 Quasistationary hydrodynamics for the Boltzmann equation *J. Stat. Phys.* **80** 1063–83
[3] Bobylev A V and Mossberg E 2008 On some properties of linear and linearized Boltzmann collision operators for hard spheres *Kinetic Relat. Models* **1** 521–55
[4] Cercignami C, Illner R and Pulvirenti M 1994 *The Mathematical Theory of Dilute Gases* (Berlin: Springer)
[5] De Masi A, Esposito R and Lebowitz J L 1989 Incompressible Navier–Stokes and Euler limits of the Boltzmann equation *Commun. Pure Appl. Math.* **42** 1189–214
[6] Esposito R, Guo Y, Kim C and Marra R 2013 Non-Isothermal boundary in the Boltzmann theory and Fourier law *Commun. Math. Phys.* **323** 177–239
[7] Esposito R, Guo Y, Kim C and Marra R 2018 Non-Isothermal stationary solutions to the Boltzmann equation in the Hydrodynamic limit *Ann. PDE* **4** 1–119
[8] Esposito R, Guo Y and Marra R 2018 Hydrodynamic limit of a kinetic gas flow past an obstacle *Commun. Math. Phys.* **364** 765–823
[9] Friedman A 1964 *Partial Differential Equations of Parabolic Type* (Englewood Cliffs, NJ: Prentice-Hall)
[10] Guo Y, Jang J and Jiang N 2010 Acoustic limit for the Boltzmann equation in optimal scaling *Commun. Pure Appl. Math.* **63** 337–61
[11] Garrido P L and Lebowitz J L 2018 Diffusion equations from kinetic models with non-conserved momentum *Nonlinearity* **32** 5441–62
[12] Grad A 1963 Asymptotic theory of the Boltzmann equation *Phys. Fluids* **6** 147–81
Grad A 1962 Asymptotic theory of the Boltzmann equation II *Rarefied Gas Dynamics II* (Paris) pp 26–59
[13] Sone Y 2002 *Kinetic Theory and Fluid Dynamics* (Boston, MA: Birkhäuser)
Sone Y 2007 *Molecular Gas Dynamics, Theory, Techniques, and Applications* (Boston, MA: Birkhäuser)