Integrability and Random Interface Growth

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Integrable probability in a nutshell

Study scaling and statistics of complex random systems through exactly solvable examples which predict larger universality class.

These special systems come from algebraic structures:

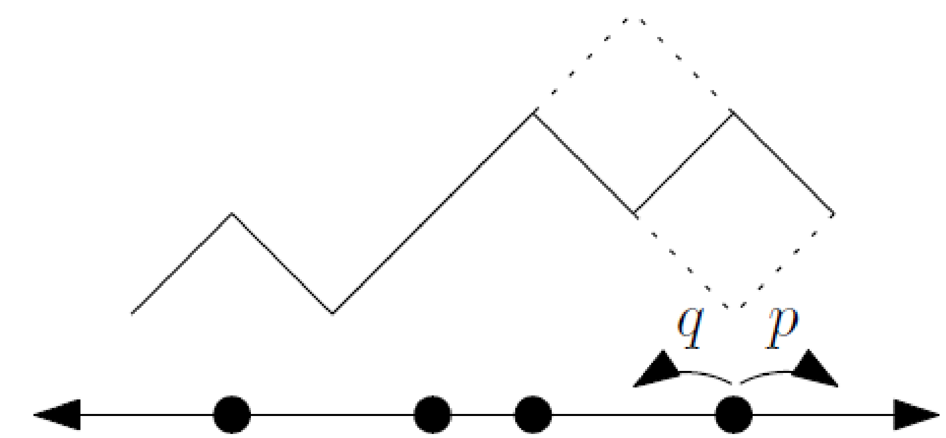
Representation theory (Schur/Macdonald processes) $\xleftrightarrow{\textbf{today}}$ Quantum integrable systems (stochastic vertex models)

Integrable probabilistic systems

Connecting these two sides yields new tools in studying models such as tilings, stochastic six vertex model and ASEP in full/half space.

But first, what does this bridge yield?

<u>Theorem</u> [C-Dimitrov '17]: For $q < p$, in long time ASEP with step initial data has a transversal scaling exponent 2/3 with limiting spatial process which is absolutely continuous w.r.t. to Brownian motion.

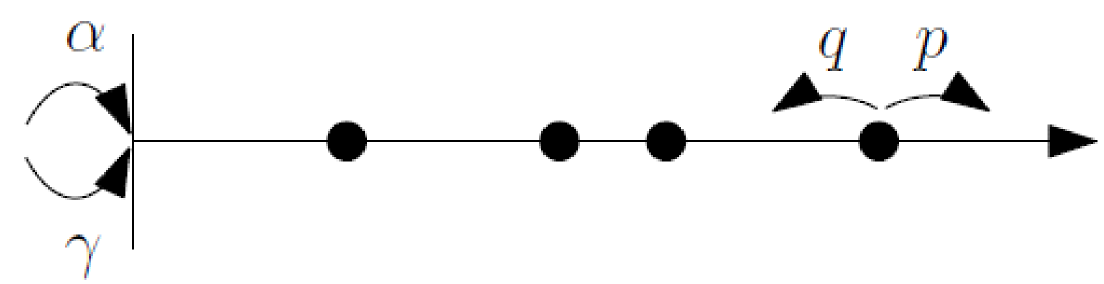


- We have similar results for the stochastic six vertex model (S6V).
- The 1/3 exponent for step initial data was established by [Tracy-Widom '09] for ASEP and [Borodin-C-Gorin '15] for S6V.
- For determinantal models (e.g. TASEP), the limiting spatial process is well understood under KPZ scaling (e.g. [Prahofer-Spohn '01]).
- The only other non-determinantal model with proved 2/3 exponent the KPZ equation itself [C-Hammond '13].

But first, what does this bridge yield?

Theorem [Barraquand-Borodin-C-Wheeler '17]:

For ASEP with $q < p, \alpha = p/2, \gamma = q/2$



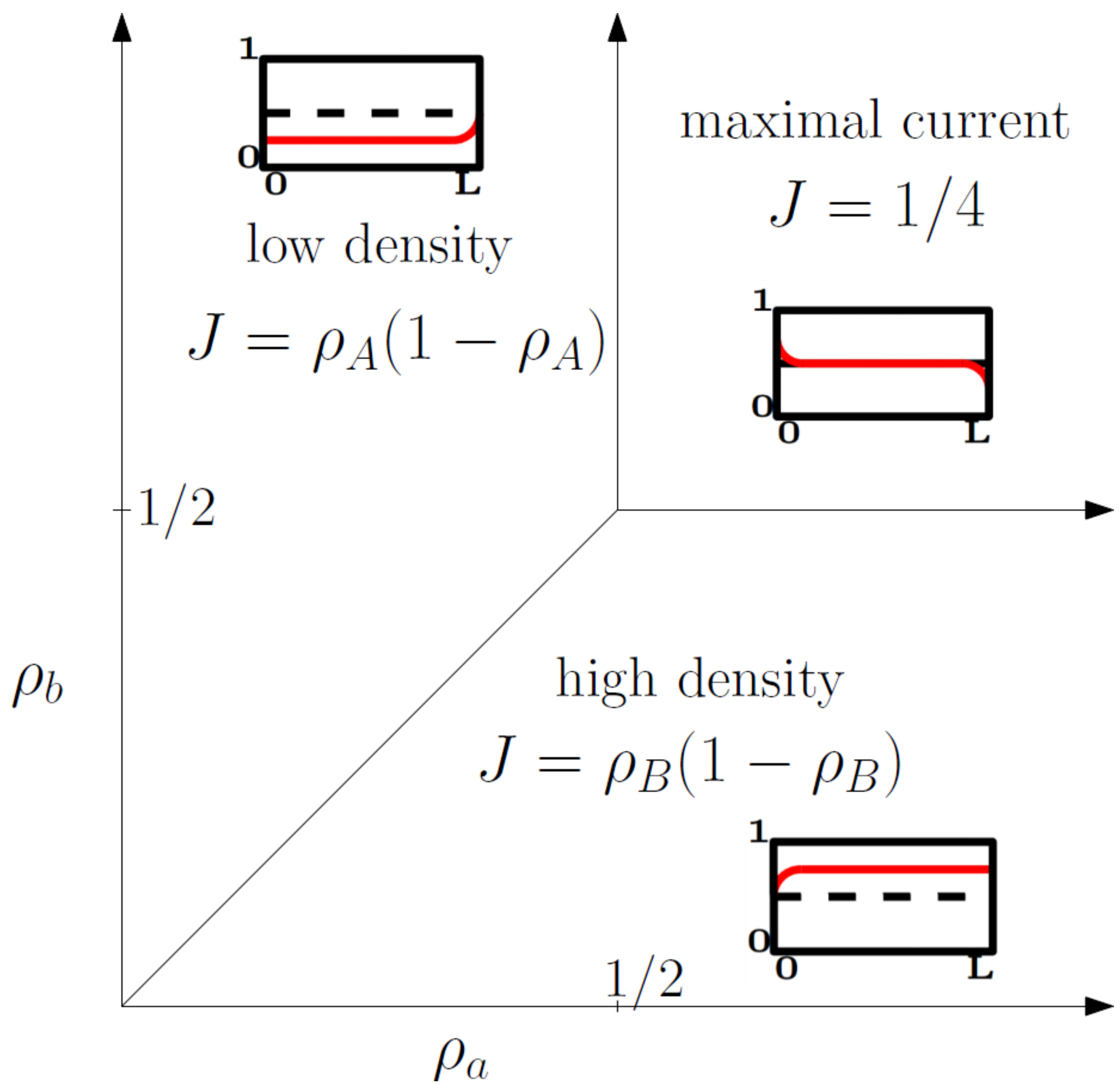
$$\mathbb{P}\left(\frac{N(T/(p-q)) - T/4}{2^{-4/3}T^{1/3}} \geq -x\right) \to F_{\mathrm{GOE}}(x)$$

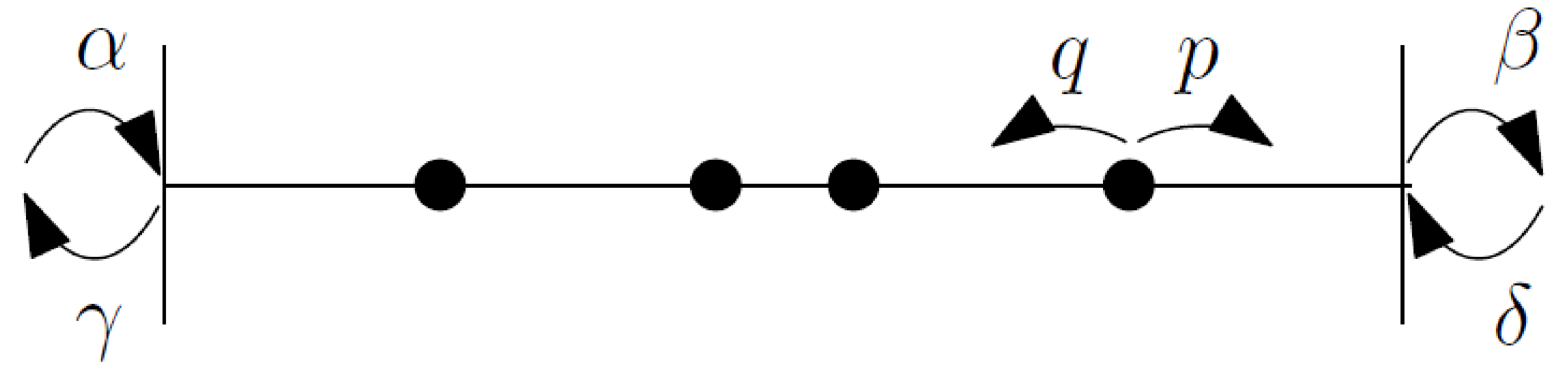
where $N(T)$ is the number of particles in the system when started with from empty initial data and $F_{\mathrm{GOE}}(x)$ is the GOE Tracy-Widom distribution.

- First KPZ class limit for a half-space non-determinantal model.
- [Baik-Rains '01] established analogous one-point TASEP result and [Imamura-Sasamoto '05], [Baik-Barraquand-C-Suidan '16] established 2/3 transversal exponent and limit process for TASEP.
- [C-Dimitrov '17] method should be able to prove 2/3 for ASEP.

Open ASEP phase diagram

J=(Steady-state average current)/(p-q) through very large system:





$\rho_A = \frac{1}{1+a}, \ \rho_B = \frac{b}{1+b}$

$a = \frac{p-q-\alpha+\gamma+\sqrt{(p-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}$

$b = \frac{p-q-\beta+\delta+\sqrt{(p-q-\beta+\delta)^2+4\beta\delta}}{2\beta}$

Complete ***half space*** *diagram proved for TASEP:*

low density — $J = \rho_A(1-\rho_A)$ | maximal current — $J = 1/4$ | ρ_a | $1/2$

$\mathbb{P}\left(\frac{N(T)-\alpha T/2}{c_\alpha T^{1/2}} \geq -x\right) \to G(x)$

$\mathbb{P}\left(\frac{N(T)-T/4}{2^{-4/3} T^{1/3}} \geq -x\right) \to F_{\mathrm{GOE}}(x)$

$\mathbb{P}\left(\frac{N(T)-T/4}{2^{-4/3} T^{1/3}} \geq -x\right) \to F_{\mathrm{GSE}}(x)$

Half space results suggest different fluctuation behavior in each phase.

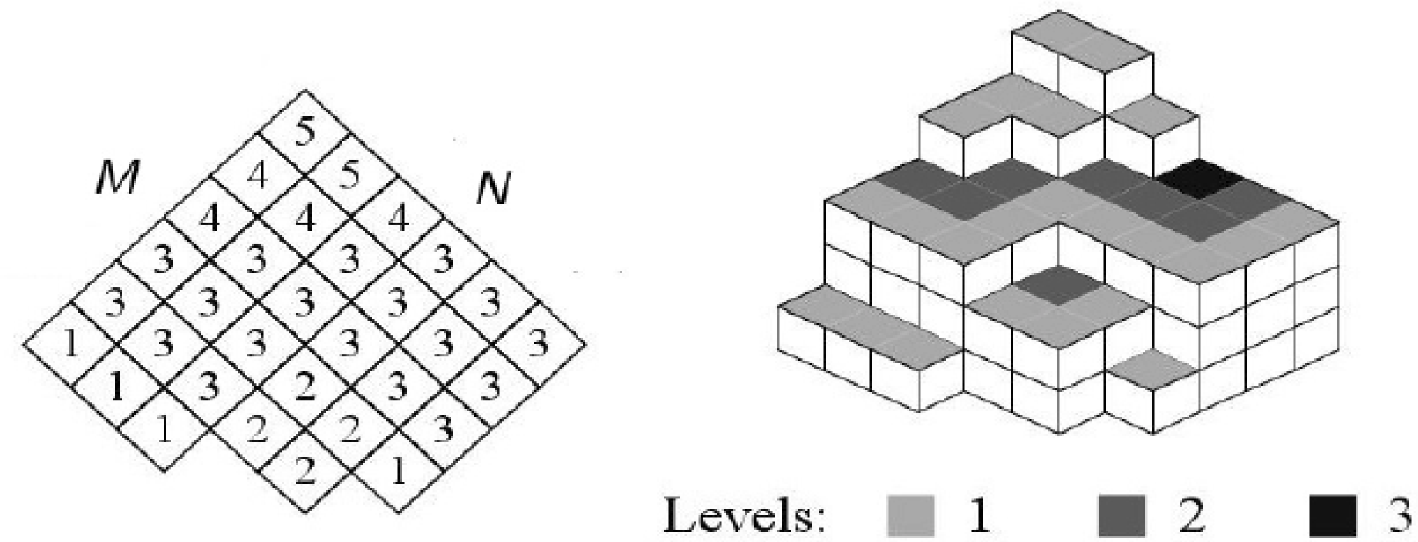
Tiling

We consider a measure on plane partitions (equivalently rhombus tilings, dimers, or 3d Young diagrams) determined by ζ and t as:

$$\mathrm{Prob}(\pi) = \zeta^{\mathrm{diag}(\pi)} A_\pi(t)$$

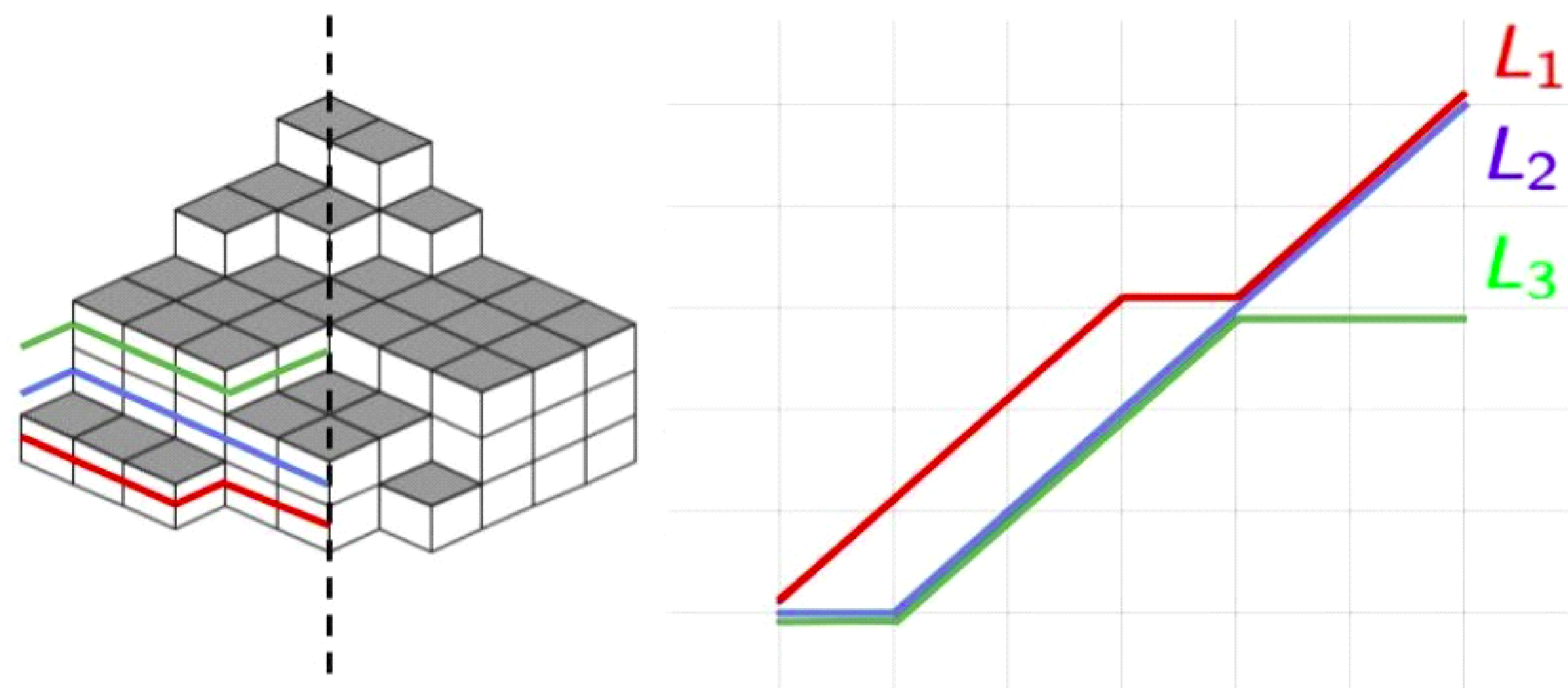
where $\mathrm{diag}(\pi) = \sum_i \pi_{i,i}$ and

$$A_\pi(t) = \prod_{(i,j)\in \mathrm{supp}(\pi)} (1 - t^{\mathrm{level}}).$$



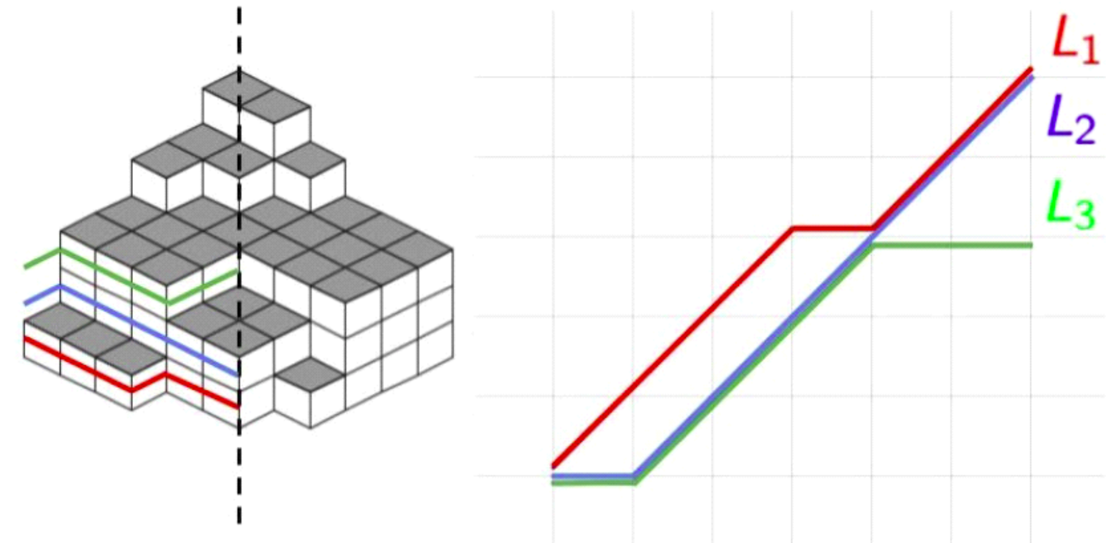
Eg. $\mathrm{diag}(\pi) = 16$ $\quad A_\pi(t) = (1-t)^7(1-t^2)^3(1-t^3)$

We associate an ensemble of non-crossing level lines which we call the Hall-Littlewood line ensemble.

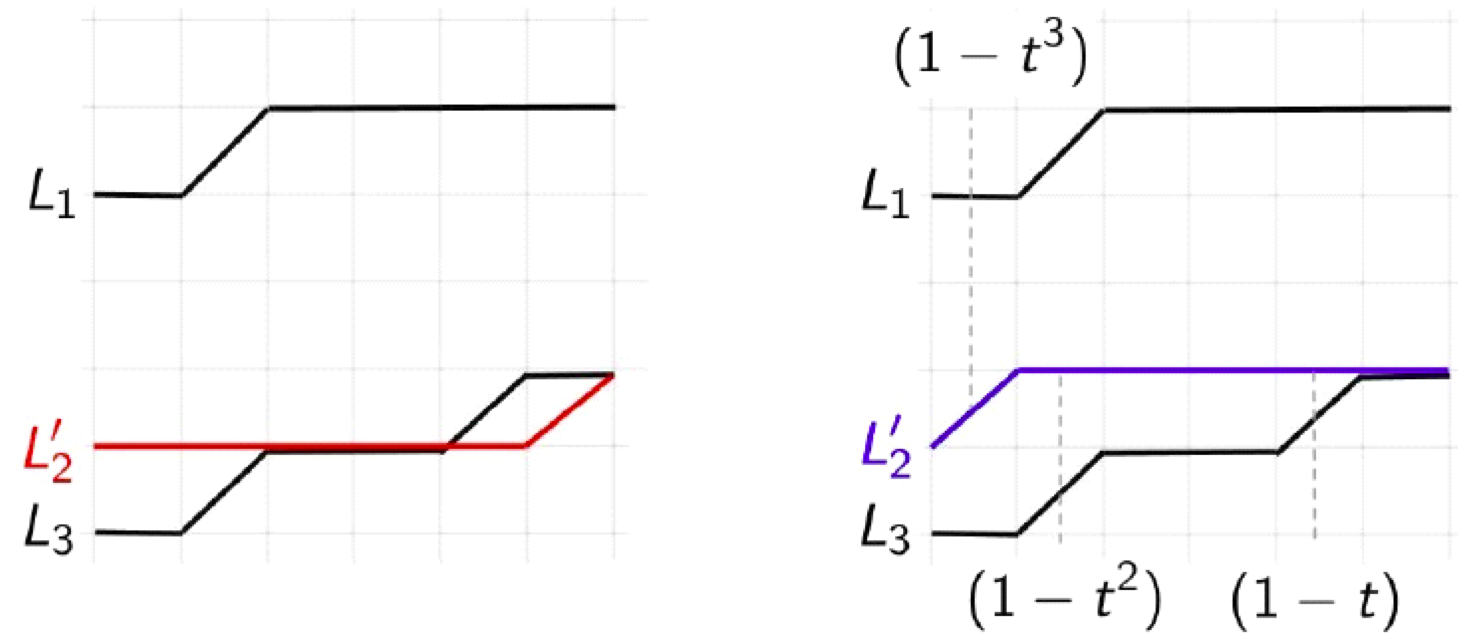


Tiling

The Hall-Littlewood line ensemble enjoys a Gibbs resampling property.



Given the curve above and below, the law of the middle curve is uniform times a local weight as shown.



$W_t(L_2', L_1, L_3) = 0$

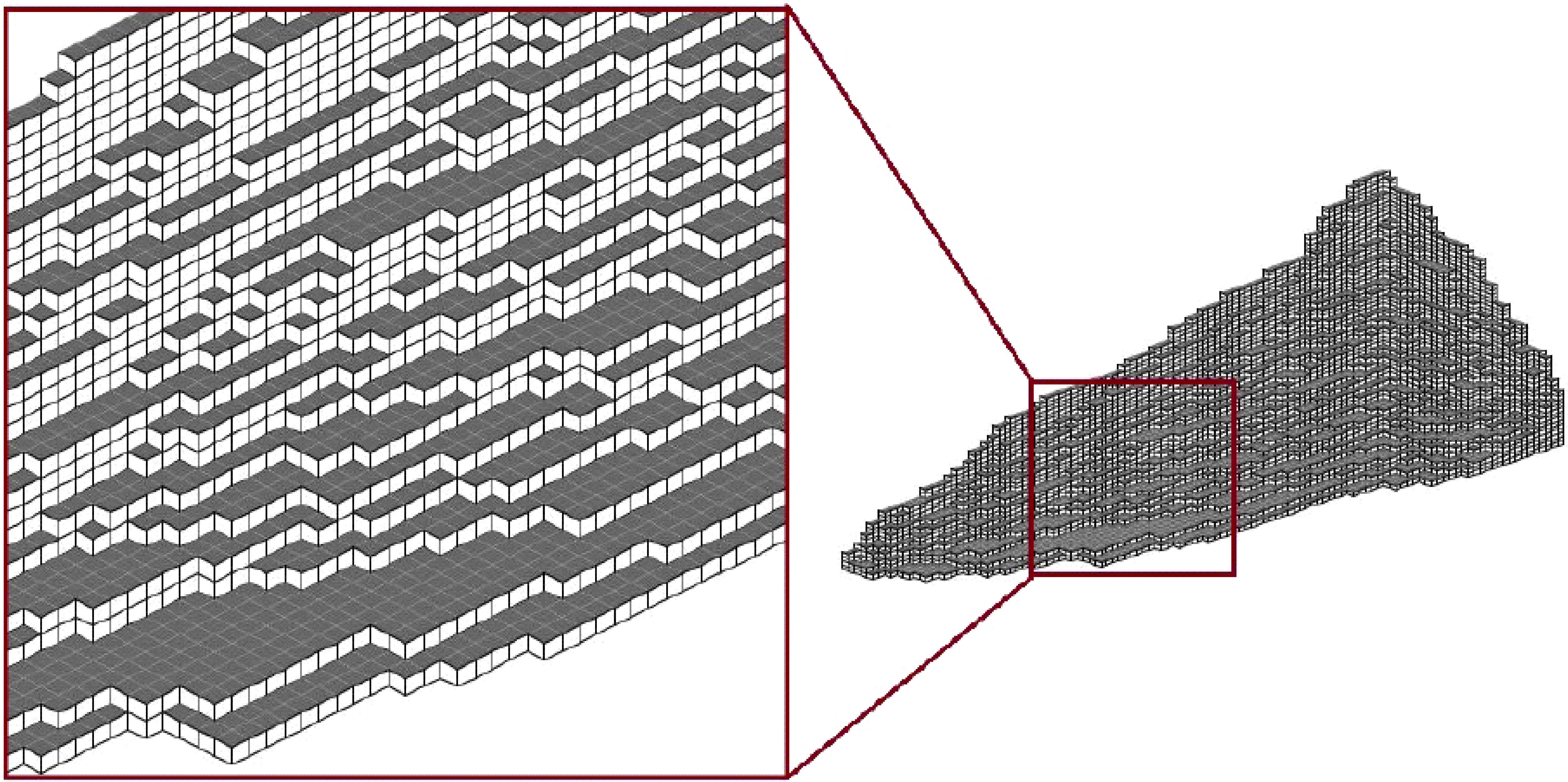
$W_t(L_2', L_1, L_3) = (1-t)(1-t^2)(1-t^3)$

[C-Dimitrov '17] (building on [C-Hammond '11,'13]) develop machinery to show spatial tightness from such Gibbs properties.

Tiling

Taking M, N large seems to yields a limit shape -- what is it?.

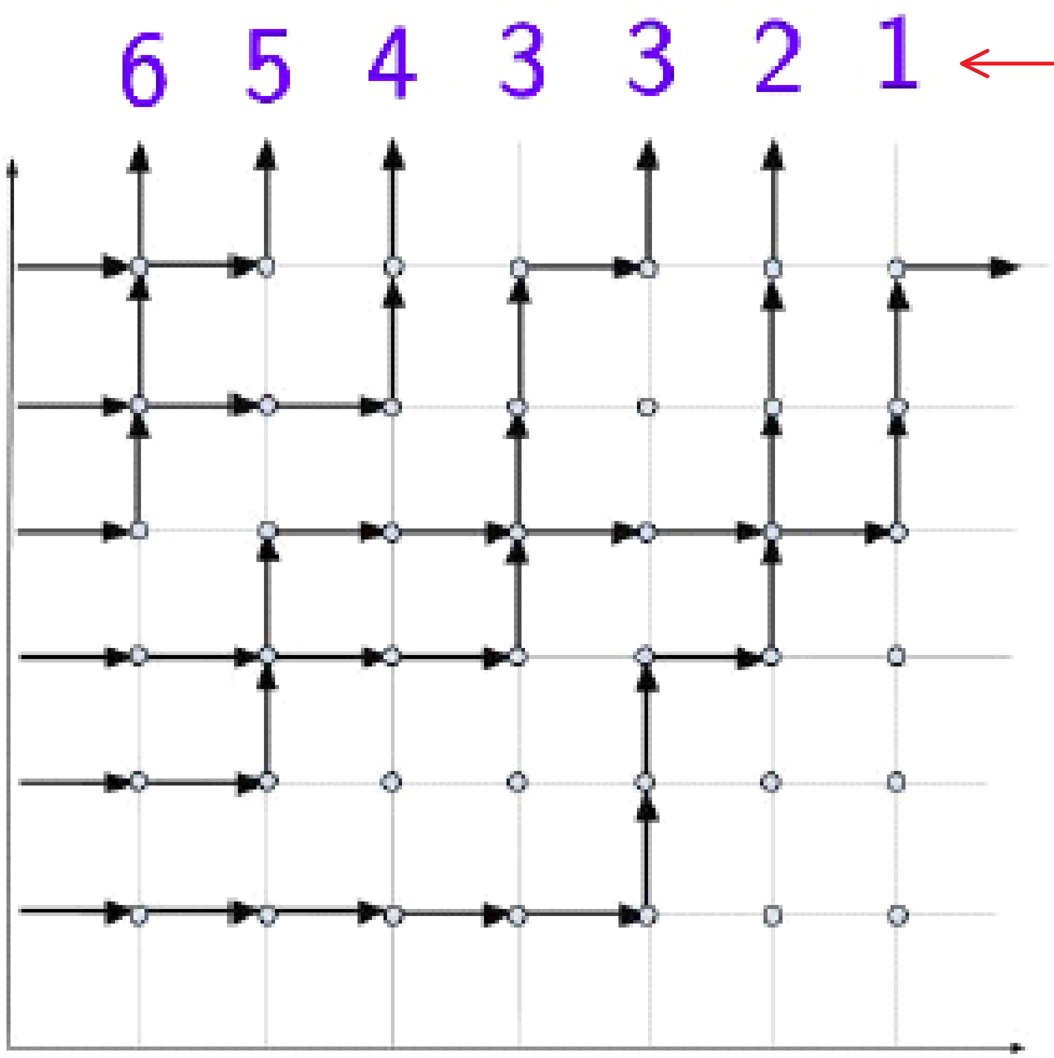
We prove edge fluctuation exponent 1/3, transversal exponent 2/3.



S6V

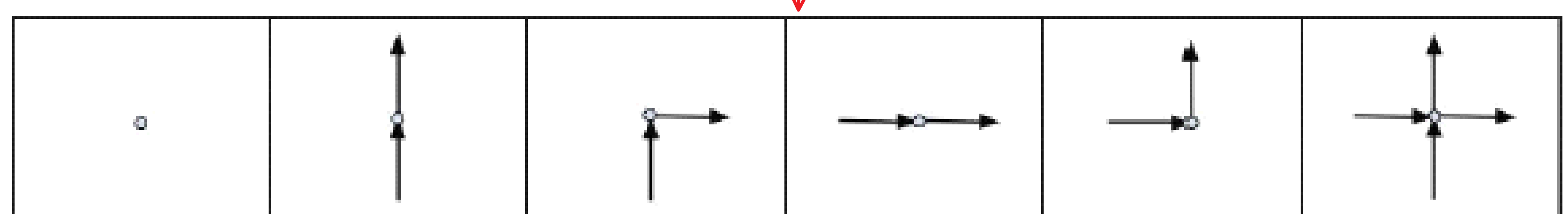
Stochastic six vertex model [Gwa-Spohn '93], [Borodin-C-Gorin '15]

(Gauge-transform of the a,b,c model where weights sum for fixed input to 1.)



Height function $h(x, N)$ records number of arrows at or to the right of a given location.

Stochastic weights

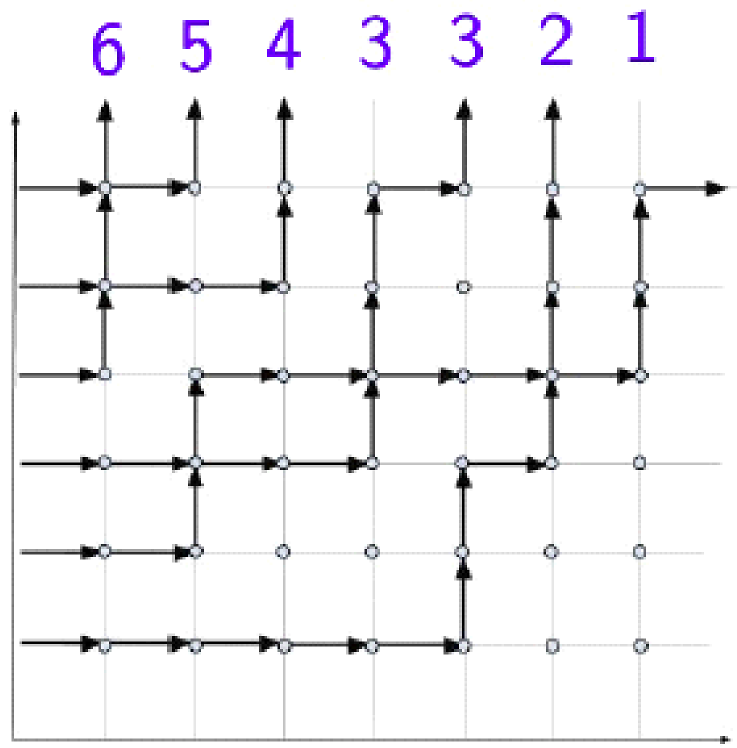


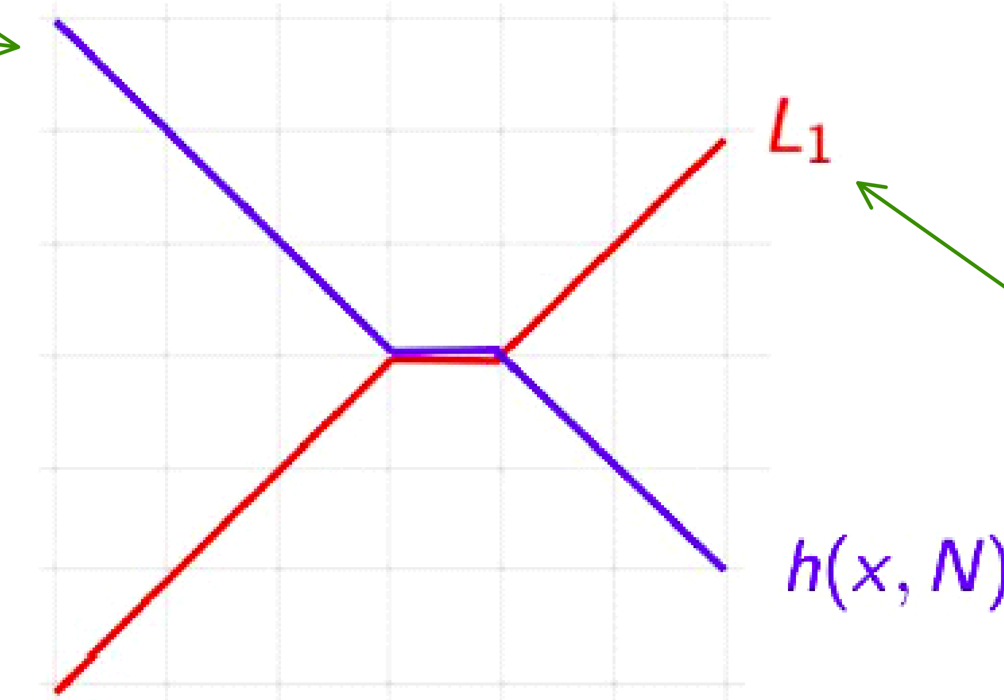
1	b_1	$1-b_1$	b_2	$1-b_2$	1

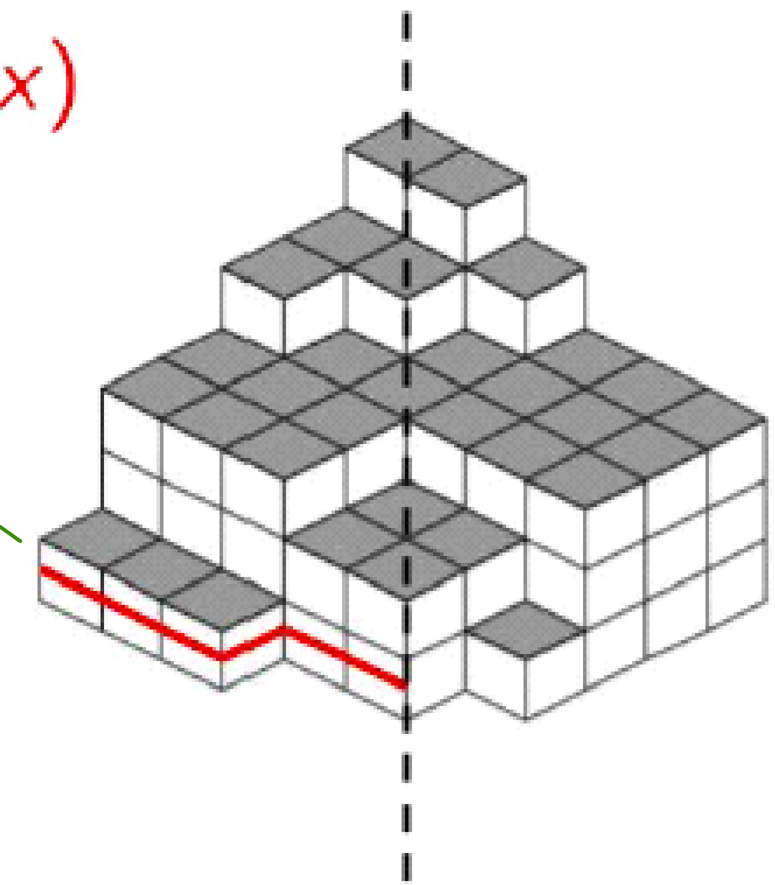
Tiling <--> S6V

[Borodin-Bufetov-Wheeler '17] relate these two models so that

$h(x,N)$ equals in law $N - L_1(x)$







With $b_1 = t \cdot \frac{1-\zeta}{1-t\zeta}$

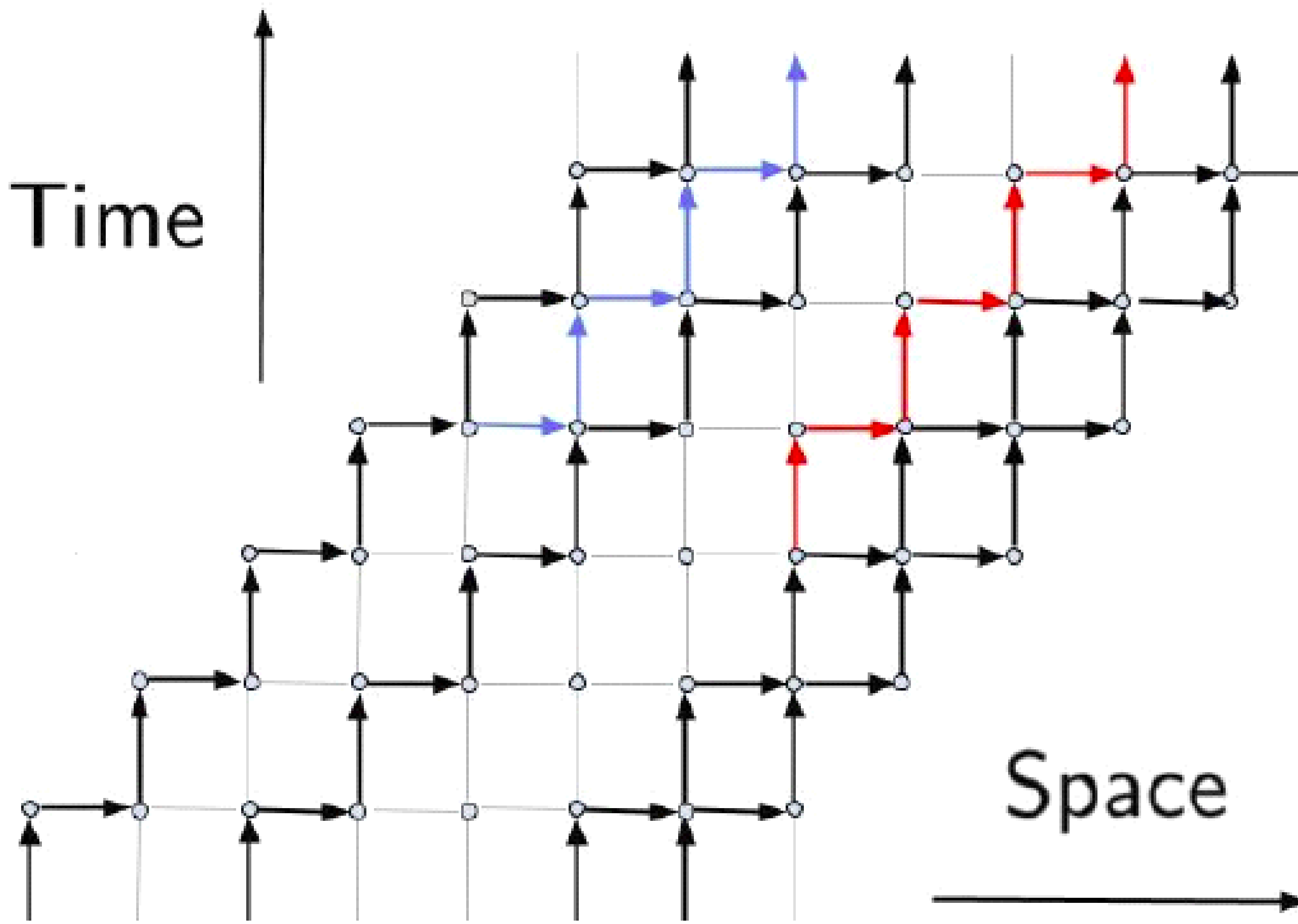
and $b_2 = \frac{1-\zeta}{1-t\zeta}$

With $\mathrm{Prob}(\pi) = \zeta^{\mathrm{diag}(\pi)} A_\pi(t)$

Proved by relating tiling to a vertex model and using Yang-Baxter.

S6V -> ASEP

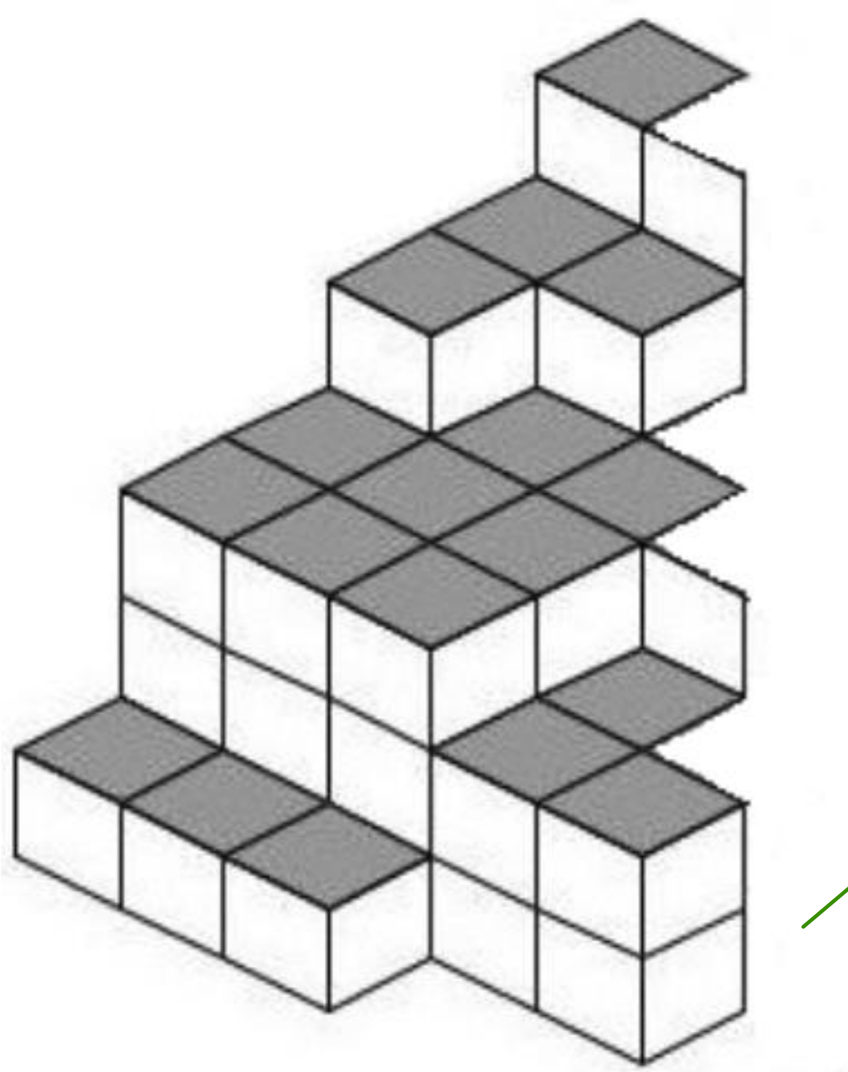
Taking $b_1 = \epsilon q$, $b_2 = \epsilon p$, $N = \epsilon^{-1}T$, $x = \epsilon^{-1}T + \tilde{x}$, and ϵ to 0 the S6V height function converges to that of ASEP.

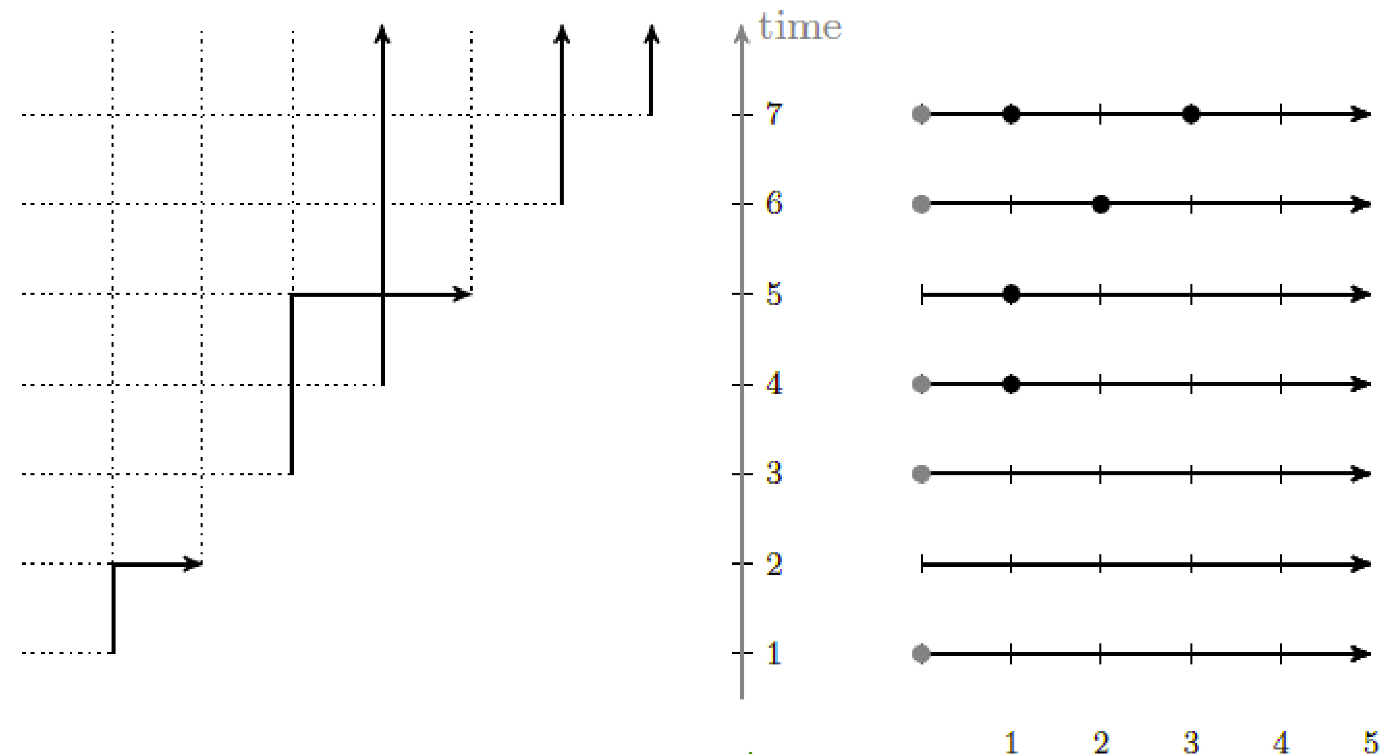


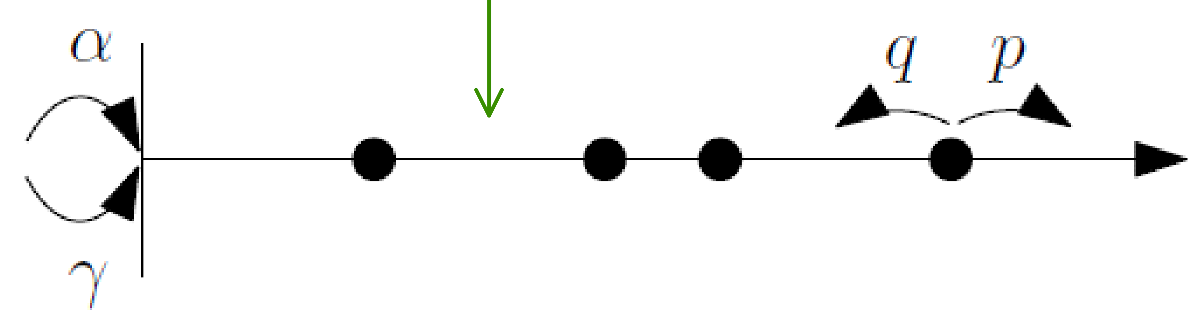
This is just like how the a,b,c 6 vertex model goes to XXZ spin chain

Half space tiling -> S6V -> ASEP

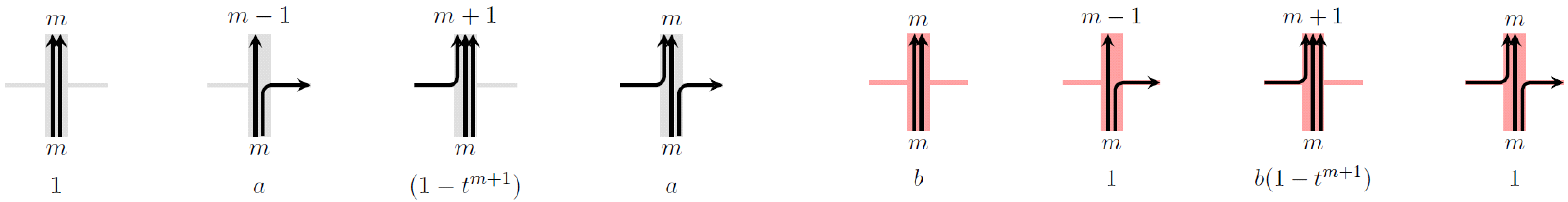
There is a similar story for a half space tiling model and a half space S6V / ASEP proved in [Barraquand-Borodin-C-Wheeler '17] also using Yang-Baxter, plus reflection equations.





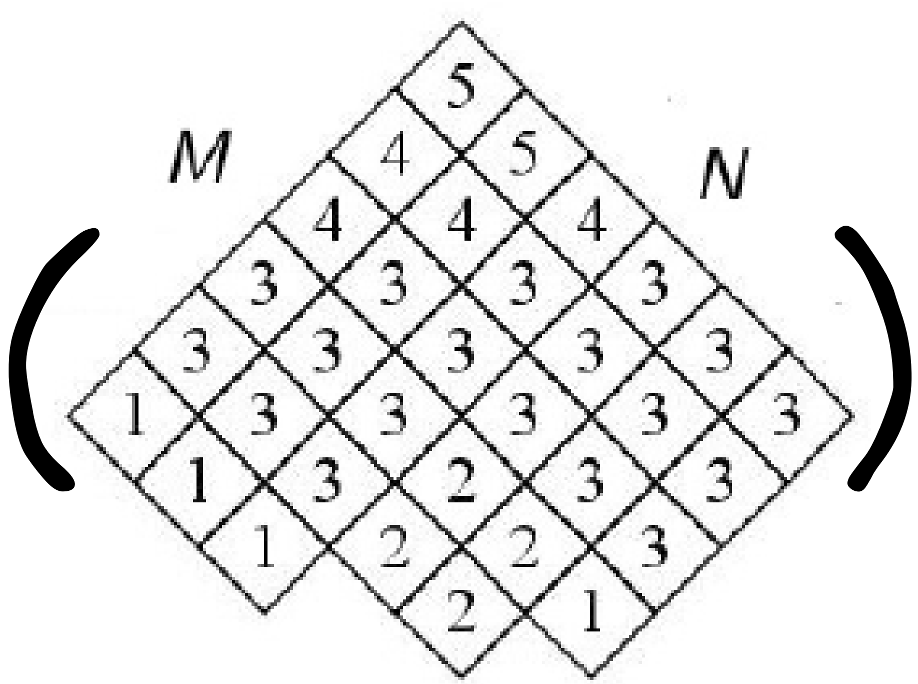


t-Boson vertex model

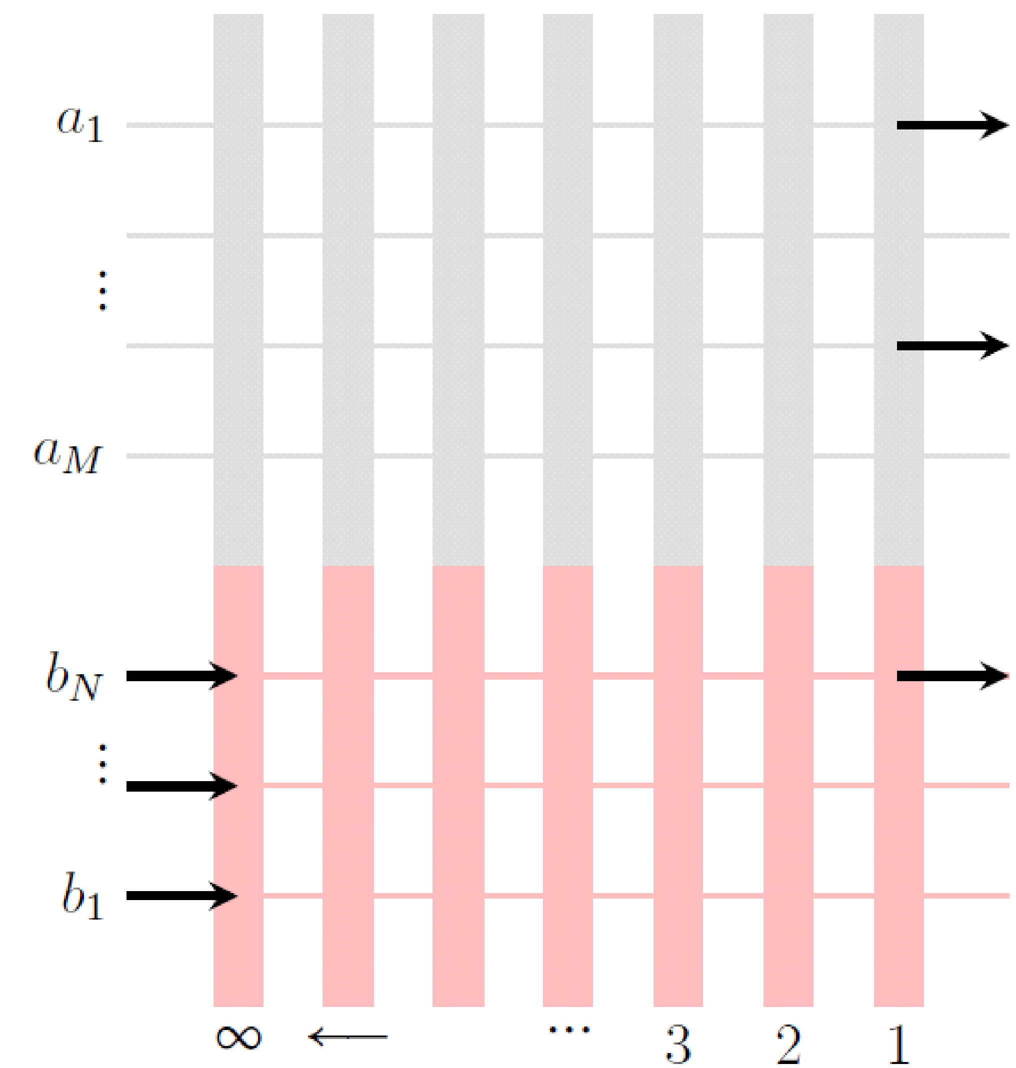


Plane partition (tiling) a formed by increasing, then decreasing interlacing partitions. t-Boson weights induce a measure on such a sequence. Setting $a_i b_j = \zeta$ we get back our original measure.

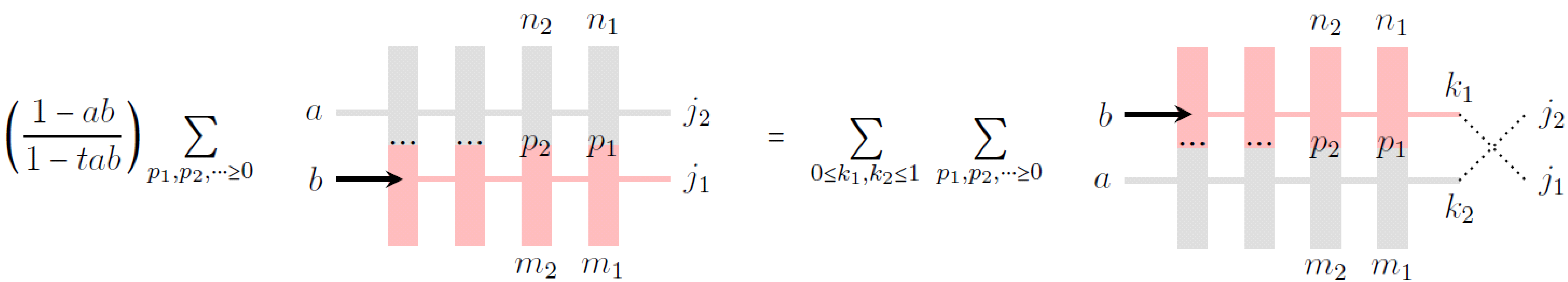
Law of



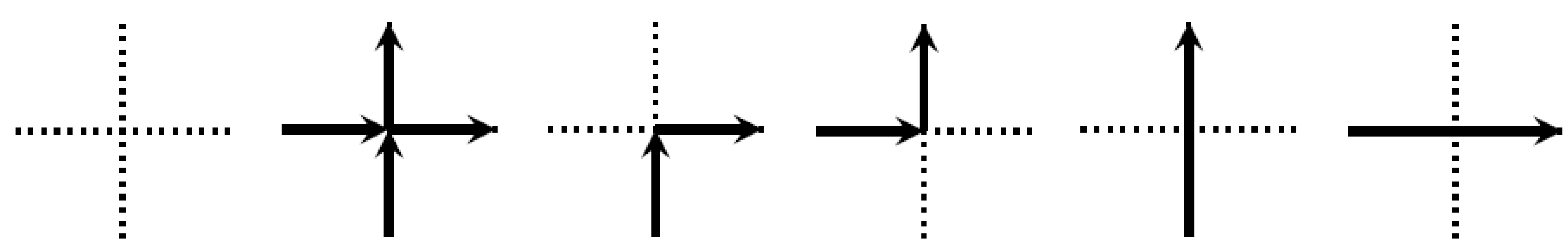
$$= \prod_{i=1}^{M} \prod_{j=1}^{N} \left(\frac{1 - a_i b_j}{1 - t a_i b_j} \right) \times$$



Yang-Baxter equation



The sum is over all internal vertices and on the right is a vertex from the S6V model (rotated 45 degrees) with weights:



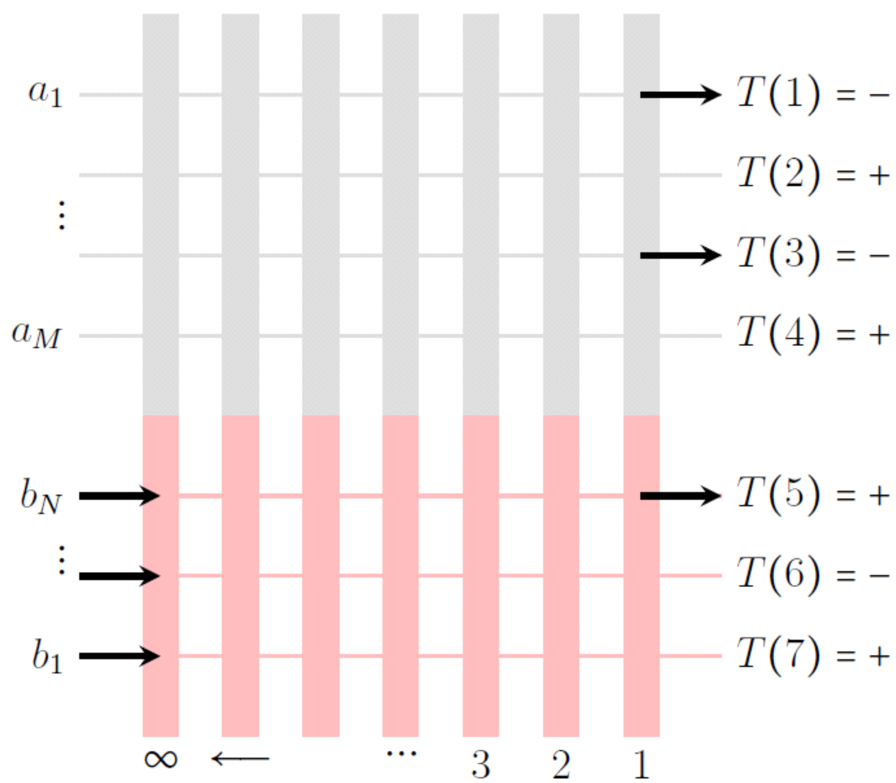
$$1 \qquad 1 \qquad \frac{1-t}{1-tab} \qquad \frac{(1-t)ab}{1-tab} \qquad \frac{t(1-ab)}{1-tab} \qquad \frac{1-ab}{1-tab}$$

Follows single vertex t-Boson YBE by tensoring and taking a limit.

Yang-Baxter equation

Using the YBE to switch the red and grey rows

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\left(\frac{1-a_ib_j}{1-ta_ib_j}\right)\times$$



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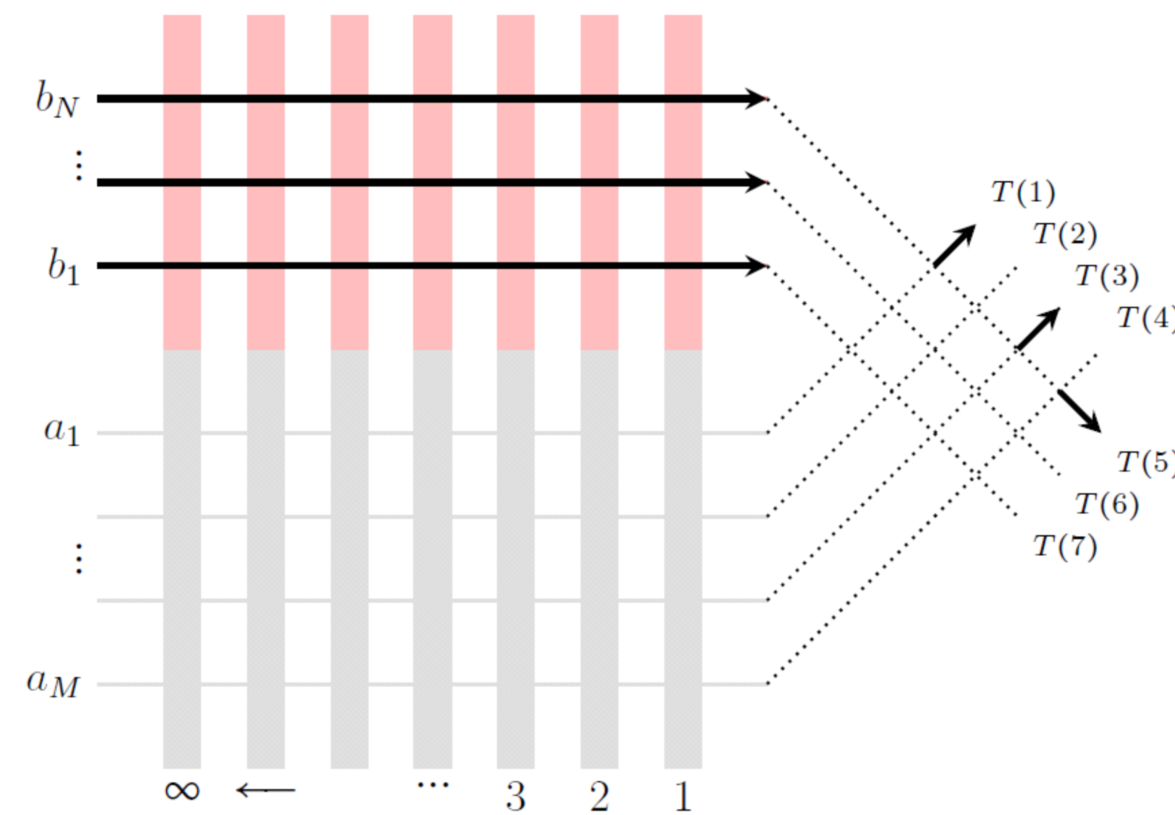
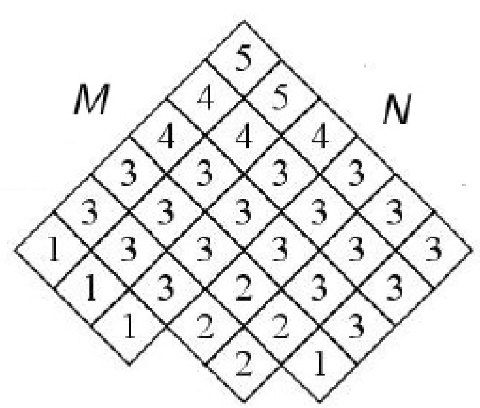
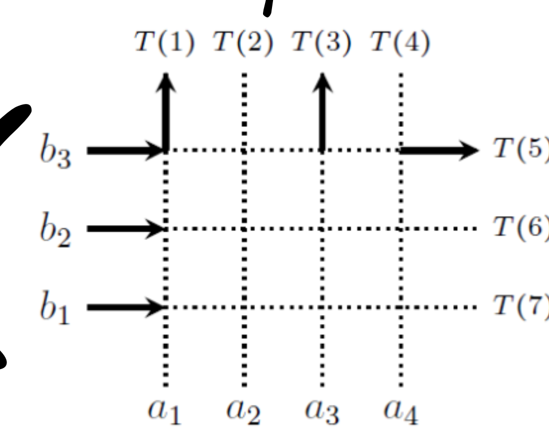


relates law of the tiling base to that of the S6V output arrows.

Law of the base () = Law of output ()

In half space case, have to additionally use "reflection equations".

Summary

- ✧ Relate S6V height function to "Hall-Littlewood" tiling base. The tiling is a special case of Macdonald processes at $q=0$.
- ✧ Using properties of Macdonald / Hall-Littlewood / Schur symmetric functions we compute certain expectations explicitly and perform one-point asymptotics (not explained in this talk).
- ✧ Using the tiling's Gibbs property, we can extend the one-point 1/3 exponent tightness to the transversal 2/3 exponent.
- ✧ Both models admit limits to ASEP and the KPZ equation and hence this provides a means to study those models too.
- ✧ **Some questions: Tiling limit shape? Asymptotics for more general boundary rates? Two-sided open ASEP? Higher spin models?**