J Stat Phys (2017) 166:1016–1027
DOI 10.1007/s10955-016-1633-6

Number Rigidity in Superhomogeneous Random Point Fields

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Received: 17 August 2016 / Accepted: 24 September 2016 / Published online: 21 October 2016

Abstract We give sufficient conditions for the number rigidity of a large class of point processes in dimension $d = 1$ and 2, based on the decay of correlations. Number rigidity implies that the probability distribution of the number of particles in a bounded domain $\Lambda \subset \mathbb{R}^d$, conditional on the configuration on $\Lambda^{\complement}$, is concentrated on a single integer N_Λ. Our conditions are: (a) $\rho_1(x) = -\int_{\mathbb{R}^d} \rho_{\text{tr}}^{(2)}(x, y)\mathrm{d}y$ for all x, where ρ_1 and $\rho_{\text{tr}}^{(2)}$ are the intensity and the truncated pair correlation function resp., and (b)$|\rho_{\text{tr}}^{(2)}(x, y)|$ is bounded by $C_1[|x - y| + 1]^{-2}$ in $d = 1$ and by $C_2[|x - y| + 1]^{-(4+\varepsilon)}$ in $d = 2$. Condition (a) covers a wide class of processes, including translation invariant or periodic point process on $\mathbb{R}^d$, $d = 1, 2$, that are superhomogeneous or hyperuniform (that is the variance of the number of particles in a bounded domain $\Omega \subset \mathbb{R}^d$ grows slower than the volume of Ω). It also covers determinantal point processes having a projection kernel. Our conditions for number rigidity are satisfied by all known processes with number rigidity in $d = 1, 2$. We also observe, in the light of the results of [26], that no such criteria exist in $d > 2$.

Dedicated to the masters David Ruelle and Yasha Sinai.

1 Introduction

Point processes on $\mathbb{R}^d$ (or $\boldsymbol{Z}^d$) are measures $\mu(\mathrm{d}X)$ giving rise to consistent probability measures μ_Λ on configurations of particles X_Λ on all regions $\Lambda \subset \mathbb{R}^d$ (or $\boldsymbol{Z}^d$). When the volume of Λ, denoted $|\Lambda|$, is finite, this probability is concentrated on configurations $X_\Lambda = \{x_1, \ldots, x_N; x_l \in \Lambda\}$, N finite. From $\mu(\mathrm{d}X)$, we can find the conditional probabilities

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$\mu(\mathrm{d}X_\Lambda|\Lambda^\complement)$ of configurations in Λ, given the configuration in $\Lambda^\complement = \mathbb{R}^d \setminus \Lambda$. We shall generally assume that μ is ergodic and either translation invariant or periodic, but our results also hold for significant classes of inhomogeneous systems .

The simplest example of a point process on $\mathbb{R}^d$ is the translation invariant Poisson process with density (intensity) $\rho > 0$. For this process, there is no information gained about X_Λ from knowing about $X_{\Lambda^\complement}$, so $\mu(\mathrm{d}X_\Lambda|X_{\Lambda^\complement}) = \mu(\mathrm{d}X_\Lambda)$. In general, for equilibrium systems with sufficiently rapidly decaying (tempered) interaction potentials $U(X)$, the infinite volume Gibbs measures, μ_{eq}, describe a point process whose conditional probabilities satisfy the DLR (Dobrushin, Lanford, Ruelle) equation [10,27]

$$\mu_{\mathrm{eq}}(\mathrm{d}X_\Lambda|X_{\Lambda^\complement}) = \exp[-\beta U(X_\Lambda|X_{\Lambda^\complement})]\mathrm{d}X_\Lambda \bigg/ \left(\int_\Lambda \exp(-\beta U(X_\Lambda|X_{\Lambda^\complement}))\mathrm{d}X_\Lambda\right), \quad (1)$$

where β is the inverse temperature [10,27]. This gives a probability distribution for the number of particles in Λ given $X_{\Lambda^\complement}$, whose variance is, in general, bounded below by $c|\Lambda|$ [11].

In contrast to the above situation, we shall be interested here in the case, called "[number] rigidity" by Ghosh and Peres [16], where $X_{\Lambda^\complement}$ determines the precise number of particles in Λ. An early example of such a property (presumably the earliest) was proven by Aizenman and Martin [1] for one dimensional Coulomb systems in which U is a sum of interactions between particles which increases linearly with distance. For such systems the DLR equations are not well defined for the limiting infinite volume Gibbs measure. Aizenman and Martin [1] considered a description of the infinite system using the local electric field for two 1D cases of charge neutral systems:

1. A one component plasma (OCP), a.k.a. jellium, in which there is a uniform background of negative charge with density ρ and charge one positive point particles of average density ρ. In this case the extremal states are periodic with period ρ^{-1}.
2. A multicomponent charged system with both positively and negatively charged particles, for which $\mu(\mathrm{d}X)$ is translation invariant.

In the second case, number rigidity is replaced by net charge rigidity, i.e. $X_{\Lambda^\complement}$ determines the total net charge in Λ. For the OCP, net charge is just $N_\Lambda - \langle N_\Lambda\rangle = N_\Lambda - \rho|\Lambda|$.

The system considered by [16] is the determinantal point process (with projection kernel) corresponding to the OCP in $d = 2$, at a value of the reciprocal temperature $\beta = 2$ and uniform background density ρ. This measure is known to be identical, after some rescaling of the density, to that of the eigenvalue distribution of the Ginibre ensemble (the weak limit of eigenvalues of matrices with i.i.d. complex Gaussian entries with mean 0 and variance 1), [12]. For the 2D OCP (or the multicomponent Coulomb system) the interaction between the charges grows like the logarithm of the distance between them. They proved number rigidity for this system as well as for the zeroes of the standard planar Gaussian analytic function (the latter process exhibits rigidity of quantities other than the particle number also, but this is beyond the scope of our discussion in this paper). Ghosh and Peres [16] showed, in addition, that despite the number rigidity in the Ginibre ensemble, the measure $\mu(\mathrm{d}\boldsymbol{X}_\Lambda|\boldsymbol{X}_\Lambda^\complement)$ is absolutely continuous with respect to the Lebesgue measure. This is also the case for the 1D Coulomb system.

A similar result was proven by Ghosh [14] for the determinantal point process (with projection kernel) corresponding to the eigenvalue distribution of the Gaussian Unitary Ensemble (GUE) or the Circular Unitary Ensemble (CUE). This distribution is the same as that of the equilibrim 1-D "Dyson logarithmic gas" (DLG), again at $\beta = 2$ (see [7]). This system also has a uniform negative charge background, but the interaction between the positive charges only grows logarithmically, rather than linearly, with distance.

Other examples of "number rigidity" and related phenomena were studied by [4–6,22–26]. A different concept of "rigidity" (distinct from our notion of number rigidity) has recently been studied by [2] in the context of 2D OCP; see also [29].

In this note we give sufficient conditions for number rigidity which covers the above cases, as well as the [26] case in $d = 1, 2$, and extends to a large class of other point processes in $d = 1, 2$, e.g. the G-process and related self-correcting queuing processes described in [13] as well as to the Conway–Radin tilings [28].

To state our results precisely, we need some definitions. For a point process Π, we denote by $\rho_1(\cdot)$ and $\rho_2(\cdot, \cdot)$ the first and second intensity (correlation) functions of Π respectively.

We define

$$\rho_{\text{tr}}^{(2)}(x, y) = \rho_2(x, y) - \rho_1(x)\rho_1(y)$$

to be the truncated pair correlation function of Π.

Let

$$G(x, y) = \rho_1(x)\delta(x, y) + \rho_{\text{tr}}^{(2)}(x, y)$$

denote the truncated total pair correlation function of Π [17], where δ is the Dirac delta function.

Finally, we introduce superhomogeneous or hyperuniform processes [9,18,20,21,30].

Definition 1 A point process Π on $\mathbb{R}^d$ is said to be superhomogeneous or hyperuniform if

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \frac{\text{Var}(N(\Lambda))}{|\Lambda|} = 0, \tag{2}$$

where $N(\mathfrak{D})$ is the number of points of Π inside the domain $\mathfrak{D}$, and $\Lambda \uparrow \mathbb{R}^d$ is a sequence of self-similar domains, e.g. balls of increasing radius centred at the origin.

We are now ready to state:

Theorem 1.1 *Let Π be a point process on $\mathbb{R}^d$, $d = 1, 2$, with the property that*

$$\int_{\mathbb{R}^d} G(x, y)\mathrm{d}y = 0 \tag{3}$$

for each $x \in \mathbb{R}^d$. Then Π exhibits number rigidity if:

- *In $d = 1$, if $\rho_{\text{tr}}^{(2)}(x, y) \le C_1[1 + |x - y|]^{-2}$*
- *In $d = 2$, if $\rho_{\text{tr}}^{(2)}(x, y) \le C_2[1 + |x - y|]^{-(4+\varepsilon)}$.*

Remark 1.1 Condition (3) is satisfied, in particular, by the following important classes of point processes:

- Translation invariant hyperuniform processes
- Determinantal processes with a projection kernel.

Remark 1.2 It is easy to see the (3) is equivalent to $\rho_1(x) = -\int_{\mathbb{R}^d} \rho_{\text{tr}}^{(2)}(x, y)\mathrm{d}y$ for each x.

As a corollary to Theorem 1.1, we can deduce

Corollary 1.2 *Translation invariant point processes in 1D with bounded variance of particle numbers, that is,* $\text{Var}(N(\Lambda))$ *being bounded uniformly in the size of the interval Λ, exhibit number rigidity.*

The proofs of Theorem 1.1 and Corollary 1.2 are taken up in Section 5. Remark 1.1 will be established in Sections 2 and 3.

2 Hyperuniform Systems

In this section, we show that for translation invariant point processes (or periodic point processes averaged over a random shift inside the period) having sub-volume growth of number variance (i.e., variance of the number of particles in a ball of radius R is $o(R^d)$), the variance of linear statistics assumes a particularly simple form.

To this end, we consider the truncated total pair correlation function [21]

$$G(x, y) = \rho_1(x)\delta(x, y) + \rho_{\mathrm{tr}}^{(2)}(x, y),$$

where $\rho_{\mathrm{tr}}^{(2)} = \rho_2(x, y) - \rho_1(x)\rho_1(y)$ is the truncated pair correlation function, $\rho_1(x)$ is the one particle density, and $\delta(x, y)$ is the Dirac delta function. Clearly, we have $G(x, y) = G(y, x)$. Moreover, for a translation invariant process, $G(x, y) = G(x - y)$.

Consider now a sequence of domains $\Lambda \uparrow \mathbb{R}^d$ in a regular (e.g. self-similar) way. In a translation invariant setting, we have (see [21])

$$\mathrm{Var}(N_\Lambda) \equiv \int_\Lambda \int_\Lambda G(x, y)\mathrm{d}x\mathrm{d}y = |\Lambda| \int_{\mathbb{R}^d} G(x, y)\mathrm{d}y + o(|\Lambda|). \tag{4}$$

For processes with sub-extensive growth of the variance of N_Λ, that is, hyperuniform systems [refer to Definition 1 and, in particular, Eq. (2)], this gives

$$\int_{\mathbb{R}^d} G(x, y)\mathrm{d}x = \int_{\mathbb{R}^d} G(x, y)\mathrm{d}y = 0. \tag{5}$$

This verifies the condition (3) for translation invariant hyperuniform systems. Conversely, using (4), one can see that (5) implies hyperuniformity for translation invariant systems.

Averaging $\alpha_\Lambda/|\partial\Lambda|$ over rotations we obtain

$$\lim_{|\Lambda|\to\infty} \frac{\alpha_\Lambda(r)}{|\partial\Lambda|} = \alpha_d|r|,$$

where α_d is a constant ([21]).

For hyperuniform systems we thus have that the spherically averaged $G(r)$ has the property

$$\int_0^\infty r^{d-1}G(r)\mathrm{d}r = 0 \tag{6}$$

and

$$\lim_{\Lambda\uparrow\mathbb{R}^d} \frac{\mathrm{Var}(N(\Lambda))}{|\partial\Lambda|} = -\alpha_d \int_0^\infty r^d G(r)\mathrm{d}r \geq 0. \tag{7}$$

In the translation invariant setting, it is natural to consider the Fourier transform of $G(x - y)$. Usually denoted as $S(k)$, this Fourier transform is non-negative, and is referred to as the "structure function" in the physics literature (e.g. see [17]). Hyperuniform systems are those for which $S(k) \to 0$ as $k \to 0$. The rate at which $S(k) \to 0$ is related to how the variance will grow in a hyperuniform system.

Recall that (3) is equivalent to

$$\rho_1(x) = \int_{\mathbb{R}^d} -\rho_{\mathrm{tr}}^{(2)}(x, y)\mathrm{d}y.$$

This implies, in particular, that systems satisfying (3) for which $\rho_{\mathrm{tr}}^{(2)}(x, y) \geq 0$, e.g. those satisfying the FKG inequalities (see [10,27]), cannot be hyperuniform.

Note that for point processes satisying the DLR equation with tempered potentials (and some hard-core like conditions), the variance of the number of particles in Λ (denoted N_Λ) is, in general, bounded below by $c|\Lambda|$, $c > 0$, for β finite [11,19]. Such processes are therefore not hyperuniform and also not rigid for finite β.

3 Determinantal Processes

In this section we discuss determinantal processes in the context of the criterion (3). We can rewrite condition (3) as

$$\rho_1(x) = \int_{\mathbb{R}^d} -\rho_{\mathrm{tr}}^{(2)}(x, y)\mathrm{d}y. \tag{8}$$

For any determinantal point process with Hermitian kernel K, we have $\rho_1(x) = K(x, x)$ and

$$\rho_2(x, y) = \mathrm{Det}\begin{bmatrix} K(x, x) & K(x, y) \\ K(y, x) & K(y, y) \end{bmatrix},$$

which, in turn, implies $-\rho_{\mathrm{tr}}^{(2)}(x, y) = |K(x, y)|^2$.

The most important class of determinantal processes are those for which K is a projection kernel, which implies that

$$K(x, x) = \int_{\mathbb{R}^d} |K(x, y)|^2 \mathrm{d}y$$

for each x. This is equivalent to (8), and therefore we deduce that determinantal processes with projection kernels satisfy criterion (3).

For more on the connection between determinantal processes, projection kernels, hyperuniformity and rigidity, we refer to Remark 4.1.

4 Linear Statistics

For a test function φ and a point process Π on $\mathbb{R}^d$, we define the linear statistic

$$\int \varphi \mathrm{d}[\Pi] = \sum_{x \in \Pi} \varphi(x).$$

We then have

$$\begin{aligned} &\mathrm{Var}\left(\int \varphi \mathrm{d}[\Pi]\right) \\ &= \int\int \varphi(x)\varphi(y)G(x, y)\mathrm{d}x\mathrm{d}y \\ &= -\frac{1}{2}\int\int |\varphi(x) - \varphi(y)|^2 G(x, y)\mathrm{d}x\mathrm{d}y + \int\int |\varphi(y)|^2 G(x, y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{9}$$

But for systems satisfying (3) we have $\int |\varphi(x)|^2 G(x, y)\mathrm{d}x\mathrm{d}y = 0$. This implies that (9) can be rewritten as

$$\mathrm{Var}\left(\int \varphi \mathrm{d}[\Pi]\right) = -\frac{1}{2}\int\int |\varphi(x) - \varphi(y)|^2 G(x, y)\mathrm{d}x\mathrm{d}y. \tag{10}$$

However, due the presence of $|\varphi(x) - \varphi(y)|^2$, the $\delta(x - y)$ component of $G(x, y)$ contribute 0 to the integral on the right hand side of (10). This enables us to write

$$\operatorname{Var}\left(\int \varphi \mathrm{d}[\Pi]\right) = -\frac{1}{2}\int\int |\varphi(x) - \varphi(y)|^2 \rho_{\mathrm{tr}}^{(2)}(x, y)\mathrm{d}x\mathrm{d}y. \tag{11}$$

We now briefly outline our approach to the proof of our main theorem, the details of which will be taken up in the following section. Our goal is to construct, for any $\varepsilon > 0$, a function φ^ε such that $\varphi^\varepsilon \equiv 1$ on the unit ball in $\mathbb{R}^d$, $d = 1, 2$, and $\operatorname{Var}\left(\int \varphi^\varepsilon \mathrm{d}[\Pi]\right) < \varepsilon$. Using the general strategy in [16] or [14], we can then deduce number rigidity of Π.

Remark 4.1 Note that for any point process for which we have $\rho_{\mathrm{tr}}^{(2)}(x, y) \leq 0$, we can conclude, using (9),

$$\operatorname{Var}\left(\int \varphi \mathrm{d}[\Pi]\right) \geq \kappa \int |\varphi(x)|^2,$$

where

$$\kappa = \int G(x, y)\mathrm{d}y = \lim_{|\Lambda|\uparrow\mathbb{R}^d} \frac{\operatorname{Var}(N(\Lambda))}{|\Lambda|} \geq 0.$$

This negativity of $\rho_{\mathrm{tr}}^{(2)}(x, y)$ holds for all determinantal processes, so this argument shows that this strategy will not yield number rigidity for determinantal processes unless the kernel is a projection kernel (whence $\kappa = 0$). In fact, one can prove that, if the kernel is not a projection, one does not have number rigidity [15].

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 and Corollary 1.2. We focus on number rigidity for balls (or intervals) centered at the origin of $\mathbb{R}^d$. In case of inhomogeneous systems, this could be a potential limitation. However, we note that any bounded domain D can be thought of as a subset of a large enough ball B centered at the origin, and therefore number rigidity with respect to D can be deduced from number rigidity with respect to B.

5.1 Exponential Decay of Correlations

To illustrate our method, we consider first the case where, in $d = 2$, we have an exponential decay of the truncated pair correlation function:

$$|\rho_{\mathrm{tr}}^{(2)}(x, y)| \leq C\exp(-\gamma|x - y|).$$

We begin with a non-negative C_c^2 function Φ supported on a disk of radius K such that $\|\Phi\|_\infty \leq 1$ and $\Phi \equiv 1$ on the unit disk.

We set $\Phi_R(x) = \Phi(x/R)$. The variance of the linear statistic corresponding to Φ_R has the bound

$$\operatorname{Var}\left(\int \Phi_R \mathrm{d}[\Pi]\right) \leq C\int\int_{\mathbb{R}^2\times\mathbb{R}^2} \exp(-\gamma|x - y|)|\Phi_R(x) - \Phi_R(y)|^2 dm(x)dm(y)$$

where $\mathrm{d}m(x)$ is the Lebesgue measure on $\mathbb{R}^2$. We upper bound the above by integrals over two regions, the point being that due to the support properties of Φ the integrand vanishes outside $A_1 \cup A_2$ defined below:

$$A_1 := \{|x| \le KR\},$$
$$A_2 := \{|y| \le KR\}.$$

Due to symmetry between x and y, it suffices to estimate from above

$$\int\int_{A_1} \exp(-\gamma|x-y|)|\Phi_R(x)-\Phi_R(y)|^2 dm(x)dm(y).$$

For the integral over A_1, we expand Φ_R in a Taylor series in y as

$$\Phi_R(y) = \Phi_R(x) + \frac{\langle \nabla\Phi(\frac{x}{R}), y-x\rangle}{R} + h(x,y),$$

where the error term $h(x,y)$ is bounded as

$$|h(x,y)| \le \frac{C_2(\Phi)|y-x|^2}{R^2},$$

and the quantity $C_2(\Phi)$ is given by $C_2(\Phi) = A\sup_z \|D^2\Phi(z)\|_2$ for some universal constant A; in other words the supremum of the 2-norm of the Hessian matrix $D^2\Phi(z)$ of Φ. Analogously, define $C_1(\Phi) = A\sup_z \|\nabla\Phi(z)\|_2$. The upshot of this is that

$$|\Phi_R(x)-\Phi_R(y)|^2 \le \left\|\nabla\Phi\left(\frac{x}{R}\right)\right\|^2 \frac{|y-x|^2}{R^2} + \frac{2C_1(\Phi)C_2(\Phi)|y-x|^3}{R^3} + \frac{C_2(\Phi)^2|y-x|^4}{R^4}.$$

Observe that

$$\begin{aligned}&\int\int_{|x|\le KR} \frac{2C_1(\Phi)C_2(\Phi)|y-x|^3}{R^3}\exp(-\gamma|y-x|)dm(y)dm(x)\\ &\le \frac{2C_1(\Phi)C_2(\Phi)}{R^3}\left(\int_{|x|\le KR} dm(x)\right)\left(\int |y-x|^3\exp(-\gamma|y-x|)dm(y)\right)\\ &= K^2C_1(\Phi)C_2(\Phi)C_3(\gamma)/R.\end{aligned}$$

Similarly, we can deduce that

$$\int_{|x|\le KR} \frac{C_2(\Phi)^2|y-x|^4}{R^4}\exp(-\gamma|x-y|)dm(y)dm(x) \le K^2C_2(\Phi)^2C_4(\gamma)/R^2.$$

We are thus left with the term

$$\begin{aligned}&\int\int_{|x|\le KR} \left\|\nabla\Phi\left(\frac{x}{R}\right)\right\|^2 \frac{|y-x|^2}{R^2}\exp(-\gamma|y-x|)dm(y)dm(x)\\ &= C_5(\gamma)\int_{|x|\le KR} \frac{\|\nabla\Phi(x/R)\|^2}{R^2}dm(x)\\ &= C_5(\gamma)\int_{|u|\le K} \|\nabla\Phi(u)\|^2 dm(u)\\ &= C_5(\gamma)\|\nabla\Phi\|_2^2.\end{aligned}$$

Thus, for Φ as described above, we have the estimate

$$\mathrm{Var}\left(\int \Phi_R \mathrm{d}[\Pi]\right) \le 2C_5(\gamma)\|\nabla\Phi\|_2^2 + 2K^2C_1(\Phi)C_2(\Phi)C_3(\gamma)/R + 2K^2C_2(\Phi)^2C_4(\gamma)/R^2.$$

where C is a universal constant.

Now we select Φ such that $\|\nabla\Phi\|_2^2 \le \varepsilon/6C_5(\gamma)$ (this can be done as in [16] for the case of the Ginibre ensemble). Depending on Φ, we choose R so large that $\max\{2K^2C_1(\Phi)C_2(\Phi)C_3(\gamma)/R, 2K^2C_2(\Phi)^2C_4(\gamma)/R^2\} \le \varepsilon/3$. For such a choice of Φ and R, we can take $\phi^\varepsilon = \Phi_R$ and we shall have $\mathrm{Var}\left(\int \phi^\varepsilon \mathrm{d}[\Pi]\right) < \varepsilon$, as desired.

5.2 Power-Law Decay of Correlations

In this section, we present a refinement of the argument in the previous section, which implies a similar variance bound (and hence, rigidity of numbers) when we have a power law decay of correlations. We will see that a power law decay of the truncated pair correlation function

$$|\rho_{\mathrm{tr}}^{(2)}(x,y)| \le C(1+|x-y|^{4+\varepsilon})^{-1} \tag{12}$$

will suffice.

To this end, we once again consider the variance of linear statistics, and consider a non-negative C_c^2 function Φ supported on a disk of radius K such that $\|\Phi\|_\infty \le 1$ and $\Phi \equiv 1$ on the unit disk. With notations as before, the variance is given by

$$\mathrm{Var}\left(\int \Phi_R \mathrm{d}[\Pi]\right) = \frac{1}{2}\int\int |\rho_{\mathrm{tr}}^{(2)}(x,y)||\Phi_R(x)-\Phi_R(y)|^2 dm(x)dm(y).$$

As before, we consider the integral only in the region $\{(x,y)\in A_1\cup A_2\}$, where $A_1 := \{|x|\le KR\}$, $A_2 := \{|y|\le KR\}$.

By symmetry, it suffices to obtain an upper bound on the integral only over the region A_1. We will work with the decomposition $A_1 = (A_1\cap A_2)\cup(A_1\cap A_2^{\complement})$.

For integration over $A_1\cap A_2$, as in the previous section, we start with the estimate

$$|\Phi_R(x)-\Phi_R(y)|^2 \le \left\|\nabla\Phi\left(\frac{x}{R}\right)\right\|^2\frac{|y-x|^2}{R^2} + \frac{2C_1(\Phi)C_2(\Phi)|y-x|^3}{R^3} + \frac{C_2(\Phi)^2|y-x|^4}{R^4}. \tag{13}$$

Of the various quantities on the right hand side of (13), we estimate

$$\int\int_{A_1\cap A_2}\left\|\Phi\left(\frac{x}{R}\right)\right\|^2\frac{|y-x|^2}{R^2}\rho_{\mathrm{tr}}^{(2)}(x,y)\mathrm{d}m(x)\mathrm{d}m(y)$$

from above by

$$\int_{|x|\le KR}\left(\int |y-x|^2\rho_{\mathrm{tr}}^{(2)}(x,y)\mathrm{d}m(y)\right)\left\|\nabla\Phi\left(\frac{x}{R}\right)\right\|^2\frac{\mathrm{d}m(x)}{R^2} = B\|\nabla\Phi\|_2^2,$$

where $B = \int |y-x|^2\rho_{\mathrm{tr}}^{(2)}(x,y)\mathrm{d}m(y) < \infty$. It is easy to see that B is independent of x, and its finiteness follows from the power law decay (12) assumed on $\rho_{\mathrm{tr}}^{(2)}$.

Our task now boils down to showing that $\frac{1}{R^3}\int_{A_1\cap A_2}|y-x|^3|\rho_{\mathrm{tr}}^{(2)}(x,y)|\mathrm{d}m(x)\mathrm{d}m(y)$ and $\frac{1}{R^4}\int_{A_1\cap A_2}|y-x|^4|\rho_{\mathrm{tr}}^{(2)}(x,y)|\mathrm{d}m(x)\mathrm{d}m(y)$ are $o(1)$ as $R\to\infty$. We will show this in detail for the first integral; the argument is similar for the second one.

To upper bound $\frac{1}{R^3}\int_{A_1\cap A_2}|y-x|^3|\rho_{\mathrm{tr}}^{(2)}(x,y)|\mathrm{d}m(x)\mathrm{d}m(y)$, we proceed as:

$$\frac{1}{R^3}\int_{A_1\cap A_2}|y-x|^3|\rho_{\mathrm{tr}}^{(2)}(x,y)|\mathrm{d}m(y)\mathrm{d}m(x) \le \frac{C}{R^3}\int_{|x|\le KR}R^{1-\varepsilon}\mathrm{d}m(x) = o(1),$$

where in the last inequality we have used (12).

This leaves us with the integral over $A_1 \cap A_2^{\complement}$. To handle this, we notice that on $A_1 \cap A_2^{\complement}$, we have $\Phi_R(y) = 0$ due to the support properties of Φ. Consequently,

$$\begin{aligned}
&\int\int_{A_1 \cap A_2^{\complement}} |\Phi_R(x) - \Phi_R(y)|^2 |\rho_{\mathrm{tr}}^{(2)}(x, y)| \mathrm{d}m(x) \mathrm{d}m(y) \\
&= \int_{A_1} |\Phi_R(x)|^2 \left(|\rho_{\mathrm{tr}}^{(2)}(x, y)| \mathrm{d}m(y) \right) \mathrm{d}m(x) \\
&\leq C \int_{|x| \leq KR} |\Phi_R(x)|^2 \cdot \frac{1}{R^{2+\varepsilon}} \mathrm{d}m(x) \\
&= \frac{1}{R^{\varepsilon}} \int |\Phi_R(x)|^2 \frac{\mathrm{d}m(x)}{R^2} \\
&= \frac{1}{R^{\varepsilon}} \|\Phi\|_2^2 \\
&= o(1).
\end{aligned}$$

Remark 5.1 A careful accounting in the above argument shows that, in fact,

$$\lim_{R \to \infty} \mathrm{Var} \left(\int \Phi_R \mathrm{d}[\Pi] \right) = C \|\nabla \Phi\|_2^2$$

for point processes with $4 + \varepsilon$ (or faster) correlation decay.

Remark 5.2 It is also clear that a power law $4 + \varepsilon$ is not strictly necessary, a milder growth faster than $|x - y|^{-4}$, like $|x - y|^{-4}(\log |x - y|)^{-1}$, should also be enough for the method to work.

Remark 5.3 The analysis of the case $d = 1$ is simpler than the case $d = 2$. The method and the result is the same as in [14]. All that is required for the rigidity of hyperuniform systems in $d = 1$ is $|\rho_{\mathrm{tr}}^{(2)}(x, y)| \leq C[1 + |x - y|]^{-2}$.

5.3 Proof of Corollary 1.2

We begin by recalling (7) for translation invariant systems. $\mathrm{Var}(N(\Lambda))$ will grow like $|\partial \Lambda|$ when the right hand side of (7), corresponding to the first moment of G, exists. This implies in particular that $G(r)$ must decay faster than $1/r^{d+1}$. It follows that in $d = 1$, bounded variance $\mathrm{Var}(N(\Lambda)) \leq C$ implies that

$$|\rho_{\mathrm{tr}}^{(2)}(r)| \leq \frac{K}{1 + r^2}.$$

Using Theorem 1.1, this completes the proof of Corollary 1.2.

6 Rigidity in Perturbed Lattice Models

Coulomb systems and determinantal point processes with projection kernels, which were already discussed before, are key models exhibiting superhomogeneity. Another class of examples are given by perturbed lattice models, i.e. the processes given by i.i.d. perturbations of a lattice, where each lattice point $z \in \boldsymbol{Z}^d$ is shifted to $z + x \in \mathbb{R}^d$ with a probability distribution $h(x)\mathrm{d}x$. The superhomogeneity of this system (see e.g. [8]) can be seen from the fact that the particle density $\rho_1(x)$ and the truncated pair correlation function $\rho_{\text{tr}}^{(2)}(x, y)$ are given by

$$\rho_1(x) = \sum_{\mathbf{z} \in \mathbf{Z}^d} h(x - z)$$

and

$$\rho_{\text{tr}}^{(2)}(x, y) = -\sum_{z \in \mathbf{Z}^d} h(x - z)h(y - z).$$

It follows immediately from the definition of G that $\int_{\mathbb{R}^d} G(x, y)\mathrm{d}y = 0$, implying that these systems are hyperuniform. When h decays fast enough, $|\rho_{\text{tr}}^{(2)}(x, y)|$ is bounded above in absolute value by $C_h h(x - y)$, where C_h is a quantity that depends on the function h. More generally this would be true when $h(x)$ decays faster than $C|x|^{-\gamma}$ with $\gamma > d$, as $|x| \to \infty$. The main idea is that when $\int_{\mathbb{R}^d} h(x)\mathrm{d}x$ is finite, the sum $\sum_{z \in \mathbf{Z}^d} h(x-z)h(z-y)$ is dominated by the terms corresponding to $z = x$ and $z = y$.

The number rigidity of this system has recently been studied by Peres and Sly [26]. They prove that for $d = 1(2)$, a sufficient condition for number rigidity is that the first (second) moment of $h(\mathrm{d}x)$ exist. This is consistent with the conditions given above. They have further shown that for Gaussian lattice perturbations (i.e., $h(x) = (2\pi)^{-d} \exp(-\frac{-x^2}{2\sigma^2})$), there is, remarkably, a phase transition in the rigidity behaviour in dimension $d \geq 3$. Namely, there is a critical $\sigma_c > 0$ such that if the standard deviation σ of the perturbations satisfies $\sigma < \sigma_c$, then there is rigidity of the number of particles, whereas there is no rigidity when $\sigma > \sigma_c$. In the case of i.i.d. Gaussian perturbations of $\mathbf{Z}^d$, the truncated pair correlation function decays as a Gaussian.

For $d \geq 3$, this provides us a concrete counterexample to any conjectured relationship between rigidity and decay of correlations, even under the assumption of superhomogeneity. Therefore, to understand the precise relationship between rigidity, decay of correlations and superhomogeneity in higher dimensions remains a delicate open question.

7 Concluding Remarks

All the point processes for $d = 1, 2$ for which rigidity has been proven satisfy our conditions. For 2-D systems like the Ginibre ensemble, the zeroes of the standard planar Gaussian analytic function on the complex plane and the processes considered by Peres and Sly [26], the variance of the number of points grows like the surface area of the domain. This is the slowest possible rate of growth of variance for isotropic point processes [3]. This is in fact expected (and in many cases proven) for the variance of particle number, e.g. for perturbed lattices [8] and for Coulomb systems in $d > 1$. For the Coulomb system in $d = 1$, the variance is bounded, $|\partial\Lambda| = 2$ [18,21]. For the Dyson log gas, the variance grows like $\log|\Lambda|$. It satisfies, for $\beta \leq 2$, the bound $|\rho_{\text{tr}}^{(2)}(x, y)| \leq C[1+|x-y|]^{-2}$ [7]. Note that the integrability of $|y|\rho_{\text{tr}}^{(2)}(y)$ is not required.

Another family of hyperuniform processes is given by determinantal point processes on Euclidean spaces whose kernels are projections when thought of as an integral operator. This includes the systems studied by Ghosh [14] and by Bufetov [4] and Bufetov et al. [5]. A third family of examples of such processes is given by i.i.d. perturbations of the lattice $\mathbf{Z}^d$ studied by Peres and Sly [26]. This class of examples is perhaps the one for which superhomogeneity is the easiest to verify. For other interesting examples of hyperuniform systems, we refer the reader to [9, 13, 30].

We expect our results to extend to point processes on lattice systems, in $d = 1, 2$, under the same conditions of superhomogeneity and decay of correlations. It is an interesting question whether superhomogeneity (possibly coupled with good decay) is necessary to guarantee rigidity behaviour in a point process. A heuristic reason for such a conjecture is that when the variance grows like $|\Lambda|$, it behaves in an additive way for two adjacent domains which seems to suggest that surface effects become negligible for large $|\Lambda|$, which is inconsistent with number rigidity. This heuristic can, in fact, be made rigorous to establish absence of number rigidity in some important examples, like that of Gibbs measures with tempered potentials.

Acknowledgements We thank Michael Aizenman and Peter Forrester for many helpful discussions on the rigidity of Coulomb systems. The work of J.L.L. was supported in part by the NSF Gant DMR1104500 and AFOSR Grant FA9550-16-1-0037. The work of S.G. was supported in part by the ARO Grant W911NF-14-1-0094.

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