Spatial networks with random connections

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Outline

- Introduction to spatial networks
- Random geometric graphs
- Random connection models: Phys. Rev. E **93**, 032313 (2016)
- K-connectivity: EPL **103**, 28006 (2013)
- Anisotropy: Trans. Wireless Commun. **13**, 4534 (2014)

Networks

A graph $G = (V, E)$ is a set of vertices V ("nodes") together with a subset of unordered pairs ("edges" or "links," E).

- Node degree k_i: Number of nodes linked to node i.

- Mean degree $\mathcal{K} = \frac{\sum_i k_i}{\sum_i 1}$.

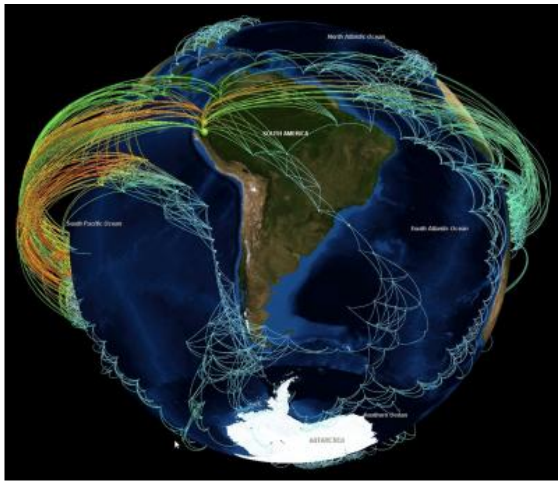
- Path: Sequence of nodes such that adjacent nodes are linked by an edge.

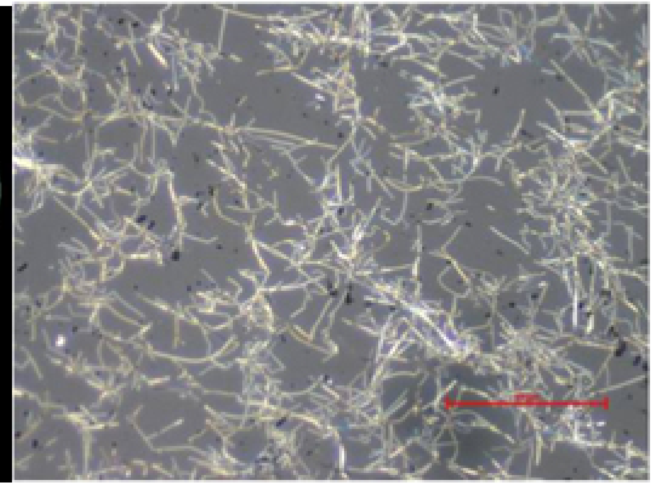
- (Fully) connected: A graph for which there is a path between any pair of nodes. Implies that all $k_i > 0$.

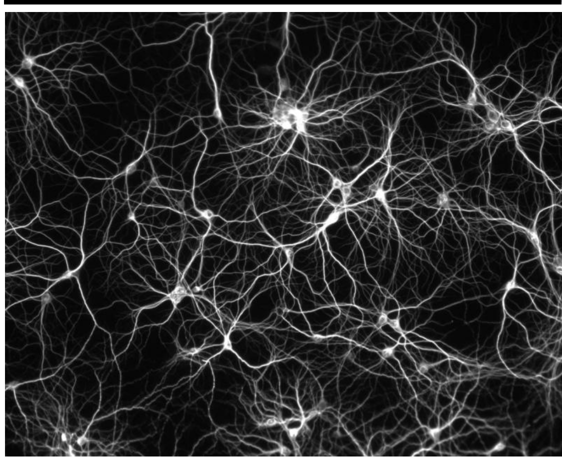
- (Fully) connection probability P_{fc}: Probability of (full) connection in a random graph model.

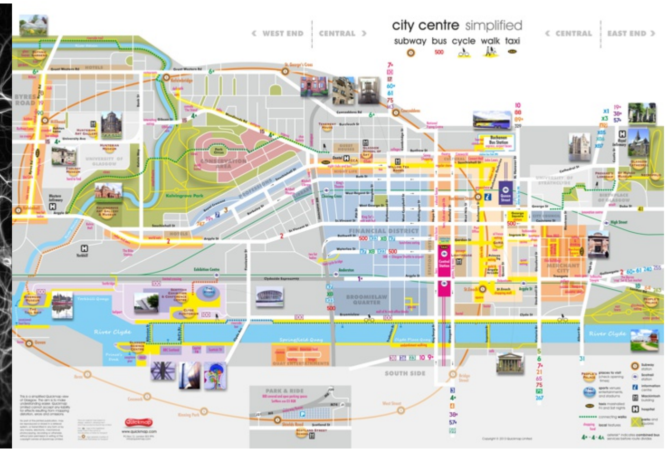
Spatial networks

Nodes (and sometimes links) have a location in space.









Random geometric graphs

Introduced in 1961 by E. N. Gilbert:

> *Recently random graphs have been studied as models of communications networks. Points (vertices) of a graph represent stations; lines of a graph represent two-way channels. ... To construct a random plane network, first pick points from the infinite plane by a Poisson process with density D points per unit area. Next join each pair of points by a line if the pair is separated by distance less than R.*

Then:

Communications networks Many authors, since 1980s

Connectivity threshold Penrose (1997), Gupta & Kumar (1999)

Books/reviews:

- **Meester & Roy (1996)** Continuum percolation
- **Penrose (2003)** Random geometric graphs
- **Franceschetti & Meester (2008)** Random networks for communication
- **Walters (2011)** Random geometric graphs
- **Barthélemy (2011)** Spatial networks
- **Haenggi (2012)** Stochastic geometry for wireless networks

Wireless network considerations

Mesh architectures Multihop connections rather than direct to a base station: Reduces power requirements, interference, single points of failure.

Random node locations In many applications (sensor, vehicular, swarm robotics, disaster recovery, ...) device locations are unplanned and/or mobile.

Network characteristics Full connectivity, k-connectivity (resilience), algebraic connectivity (synchronisation), betweenness centrality (importance, overload).

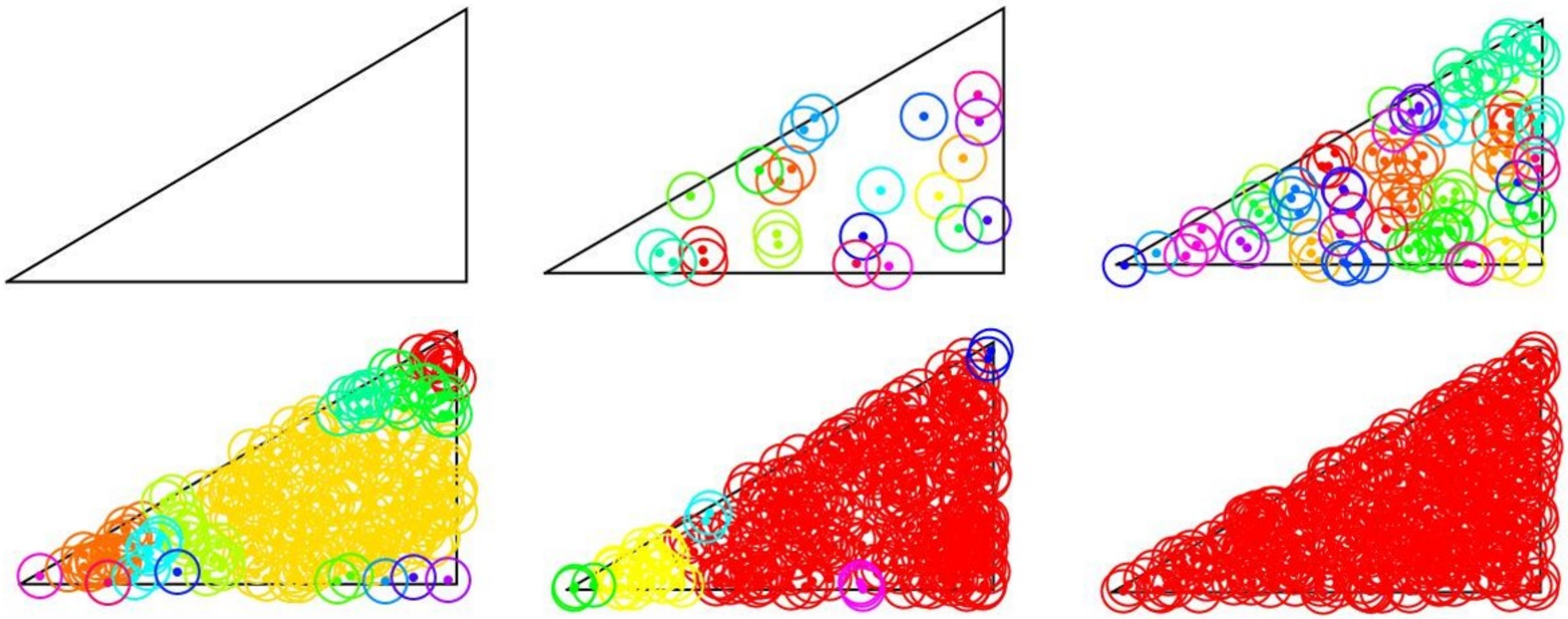
Useful extensions:

Random connection models Extra randomness: Link with (iid) probability $H(r) \in [0, 1]$, a function of mutual distance r.

Anisotropy Orientations as well as positions.

Line of sight condition Impenetrable and/or reflecting boundaries: Particular relevance to millimetre waves.

Example: A triangle



Isolated nodes occur mostly near the corners...

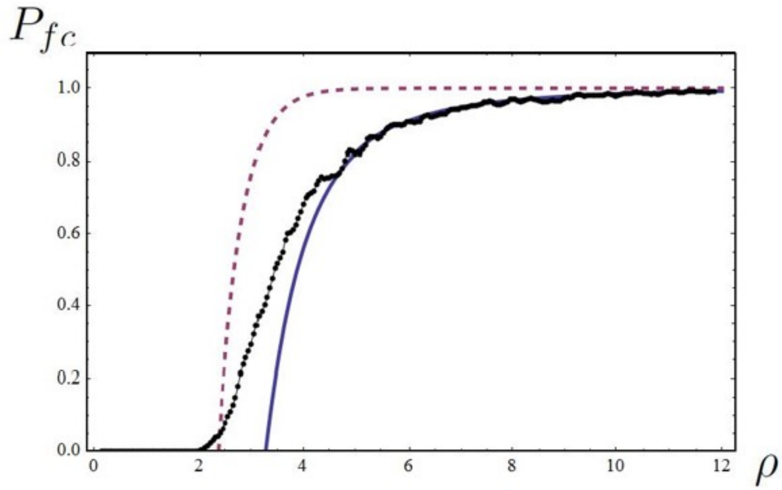
Dependence on density and geometry

We see two main transitions as density increases:

Percolation Formation of a cluster comparable to system size:
Largely independent of geometry. $\mathcal{K} = 4.5122\ldots$

Connectivity All nodes connected in multi-hop fashion:
Strongly dependent on geometry. $\mathcal{K} \approx \ln N$.

What is the full connection probability as a function of density and geometry?



Previous results

Mathematically rigorous results are for $N \to \infty$, with an appropriate scaling of at least two of r_0, ρ and the system size L.

For the random geometric graph in dimension $d \geq 2$, it was shown by Penrose, and by Gupta & Kumar, that the r_0 threshold for **connectivity** is almost always the same as for **isolated nodes**.

In turn, isolated nodes are local events, so described by a limiting Poisson process: The probability of a node having degree k is given by

$$P(k) = \frac{\mathcal{K}^k}{k!} e^{-\mathcal{K}}$$

where $\mathcal{K}$ is the mean degree, equal to $\rho \pi r_0^2$ for the 2D RGG. This leads to

$$P_{fc} \approx \exp\left[-\rho V e^{-\rho \pi r_0^2}\right]$$

where V is the "volume" (ie area) of the domain.

At fixed probability and connection range, V increases exponentially with ρ

Random connection model

Penrose (2015) gives proofs for many $H(r)$ of compact support as $N \to \infty$; we assume true more generally

$$P_{fc} \approx \exp\left[-\int \rho e^{-\rho \int H(r_{12}) d\mathbf{r}_1} d\mathbf{r}_2\right]$$

where ρ is the density, $H(r)$ is the iid probability of connection between nodes with mutual distance r and the integrals are over the domain $\mathcal{V} \subset \mathbb{R}^d$.

We want to approximate P_{fc} for finite ρ, taking into account boundaries.

Open problem: 1D, eg vehicular networks!

Specific random connection models

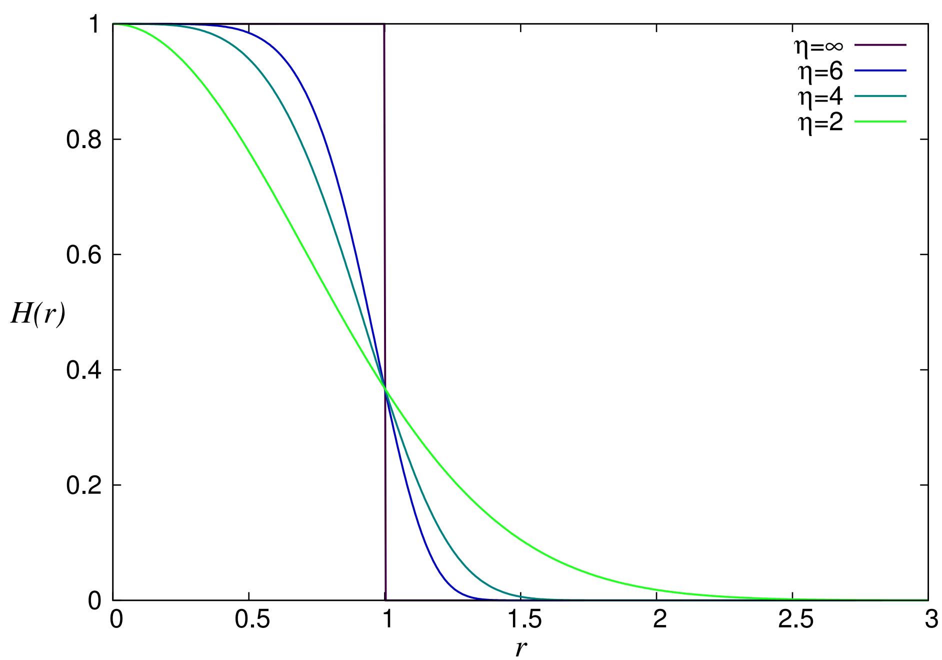
The connection function is the complement of the outage probability,

$$H(r) = \mathbb{P}(\log_2(1 + SNR\ |h|^2) > R_0)$$

neglecting interference, with $SNR \propto r^{-\eta}$, path loss exponent $\eta \in [2, 6]$, rate R_0. Simplest is Rayleigh fading (diffuse signal), for which the channel gain $|h|^2$ is exponentially distributed, giving

$$H(r) = \exp[-(r/r_0)^\eta]$$

Similar, though more involved: MIMO, Rician (specular plus diffuse), ...



Connectivity and boundaries

For large ρ, dominated by the regions of small connectivity mass

$$M(\mathbf{r}_2) = \int H(r_{12}) d\mathbf{r}_1$$

Exactly on the boundary, this is given by

$$M_B = H_{d-1}\omega_B$$

where

$$H_m = \int_0^\infty H(r) r^m dr$$

is the mth moment, and ω_B is the (solid) angle associated with the boundary component B, eg $\pi/2$ for a right angled corner, π for an edge.

Analysing the vicinity of boundaries more carefully...

General formula

$$P_{fc} = \exp\left[-\sum_B \rho^{1-i_B} G_B V_B e^{-\rho\omega_B H_{d-1}}\right]$$

where i_B is the boundary codimension, V_B is its $d-i$ dimensional volume, and G_B is the geometrical factor

G_B	$i=0$	$i=1$	$i=2$	$i=3$
$d=2$	1	$\frac{1}{2H_0}$	$\frac{1}{H_0^2 \sin\omega}$	
$d=3$	1	$\frac{1}{2\pi H_1}$	$\frac{1}{\pi^2 H_1^2 \sin(\omega/2)}$	$\frac{4}{\pi^2 H_1^3 \omega \sin\omega}$

where the 3D corner has a right angle.

Curved boundaries? To leading order, modification of the exponential but not the geometrical factor:

$$P_{2,1} = \ldots e^{-\rho(\pi H_1 - \kappa H_2)}$$

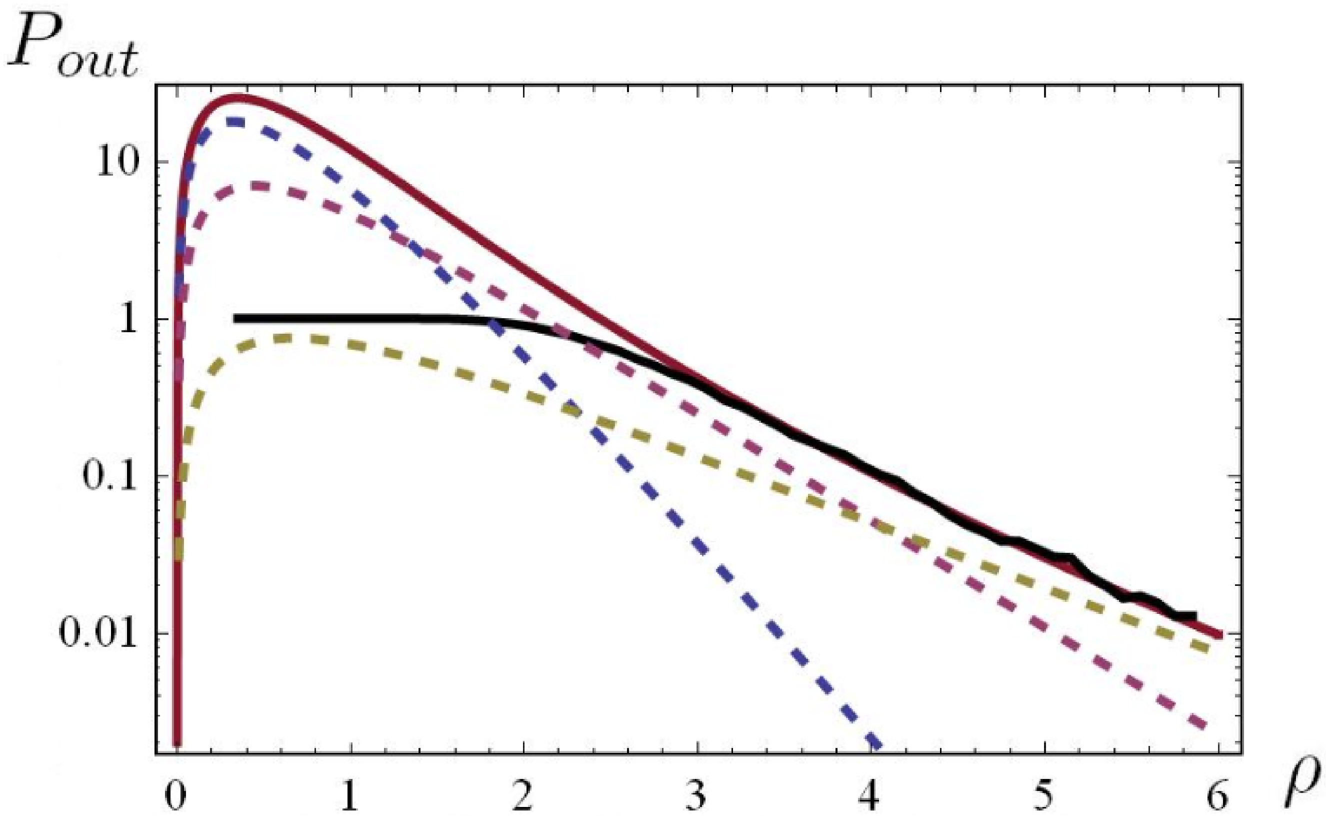
$$P_{3,1} = \ldots e^{-\pi\rho(2H_2 - \kappa H_3)}$$

where κ is (mean) curvature.

Example: A square

The previous formula gives

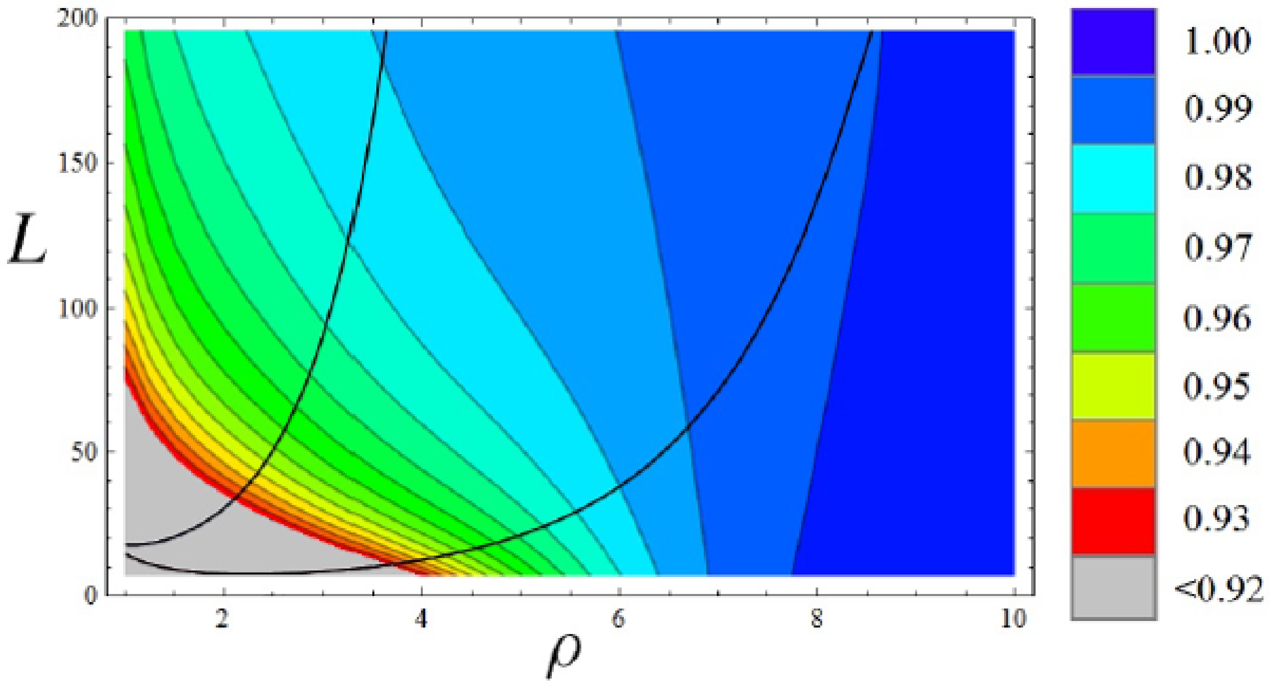
$$1 - P_{fc} \approx L^2 \rho e^{-\pi\rho} + \frac{4L}{\sqrt{\pi}} e^{-\frac{\pi\rho}{2}} + \frac{16}{\pi\rho} e^{-\frac{\pi\rho}{4}}$$



Phase diagram

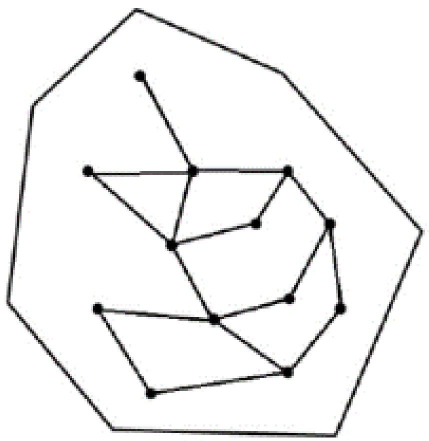
Testing convergence of

$$\frac{1 - P_{fc}}{\sum_B \cdots}$$

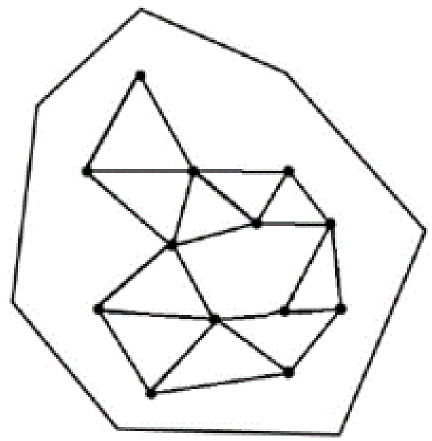


K-connectivity

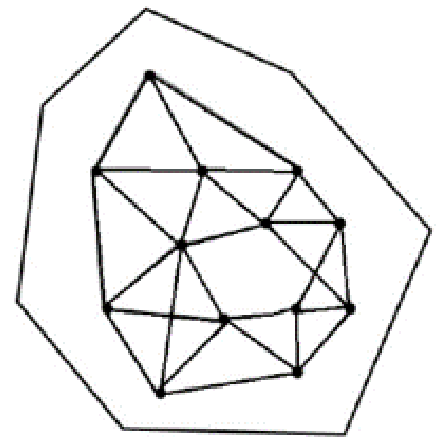
A network is (vertex) k-connected if any $k-1$ nodes can be removed and it remains connected. It is a useful measure of network resilience.



1-connected



2-connected



3-connected

Vertex connectivity $\leq$ Edge connectivity $\leq$ Minimum degree

Minimum degree

Assume independence ...

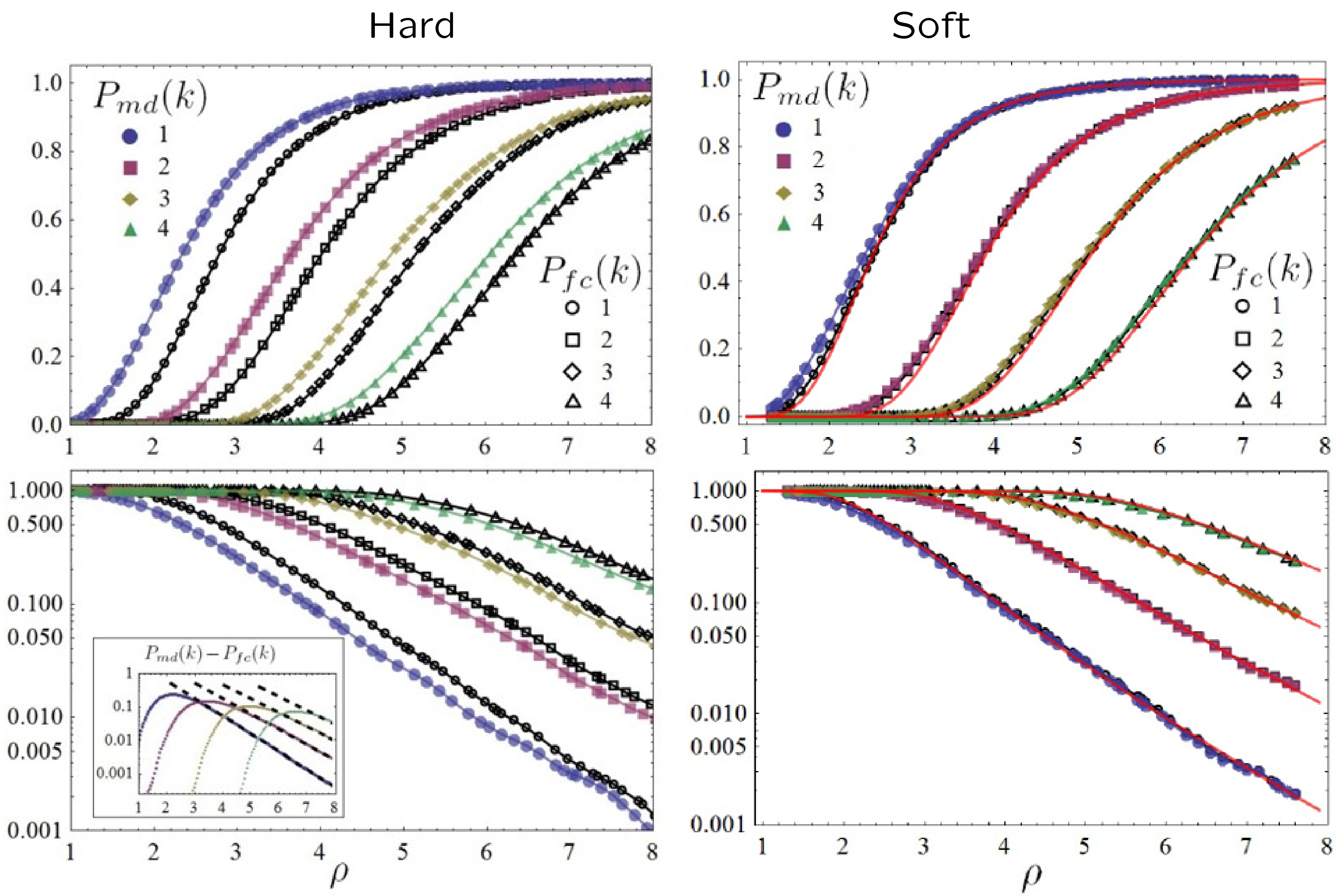
- For each node, degree is Poisson:

$$P_i(k) \approx \frac{\mathcal{K}_i^k}{k!} e^{-\mathcal{K}_i}$$

- Node degrees are independent:

$$P_{md}(k) \approx \left[1 - \sum_{m=0}^{k-1} \frac{\rho^m}{m!} \frac{1}{V} \int_{\mathcal{V}} M_H^m(\mathbf{r}_i) e^{-\rho M_H(\mathbf{r}_i)} d\mathbf{r}_i \right]^N$$

Numerical results



Random connections: Minimum degree is a better proxy for k-connectivity.

Why? Connections are less correlated in the random model.

Anisotropic connections

- Angle-dependent transmit and receive gains:

$$H(r,\phi,\theta_T,\theta_R) = \exp\left(-\frac{\beta r^\eta}{G_T(\phi-\theta_T)G_R(\phi+\pi-\theta_R)}\right)$$

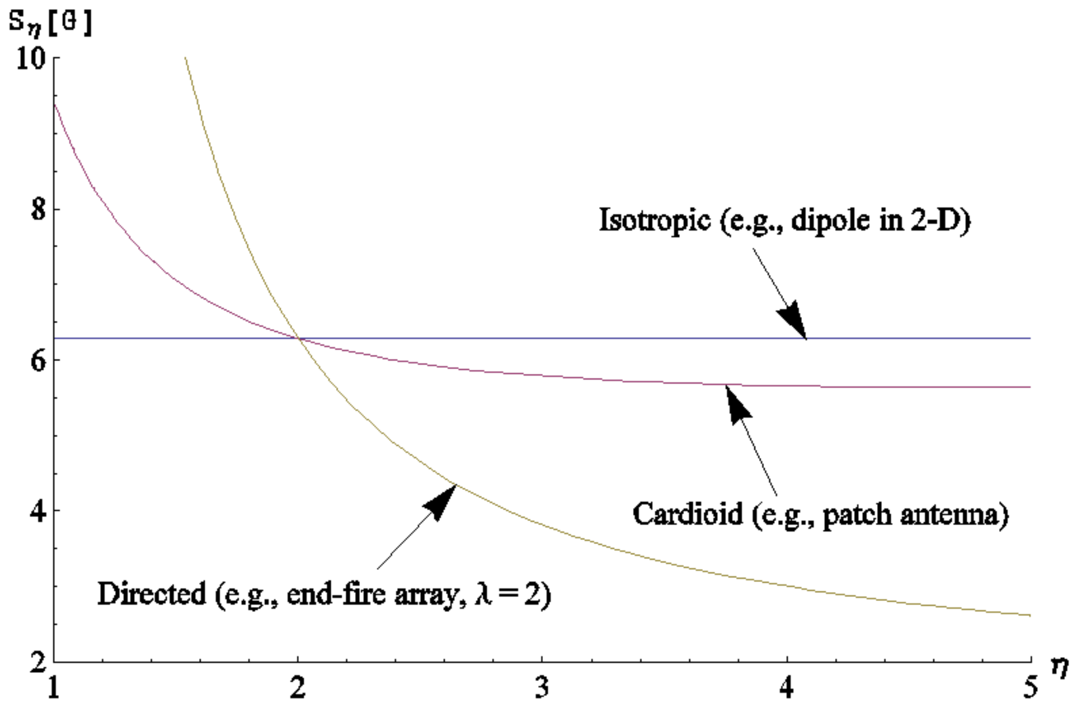
- Fix total power per node

$$\int_0^{2\pi} G_T(\phi)d\phi = \int_0^{2\pi} G_R(\phi)d\phi = 2\pi$$

- Connectivity mass is now

$$M = \frac{1}{2\pi}\int rH(r,\phi,\theta_T,\theta_R)drd\phi d\theta_R$$

Transition at $\eta = d$

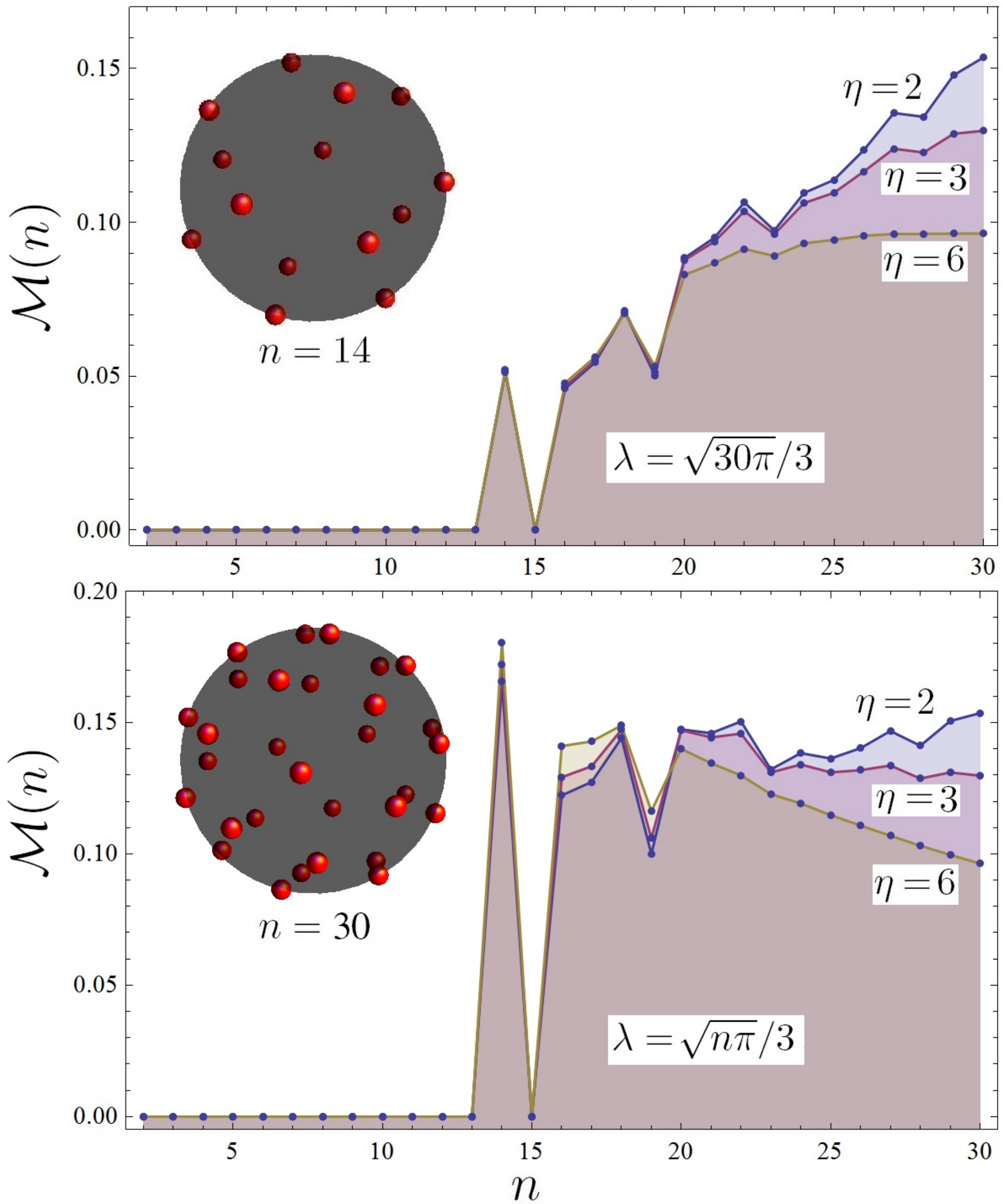


Anisotropy and boundaries

- Homogeneous case:
  - Path loss exponent $\eta > d$: Isotropic optimal
  - Path loss exponent $\eta < d$: Delta spike(s) optimal
- With boundaries, for $\eta < d$, trade-off between system size/shape and number/width of spikes. Examples:
  - Square, best to have a multiple of 4 spikes.
  - Cube . . .

Cube optimal pattern

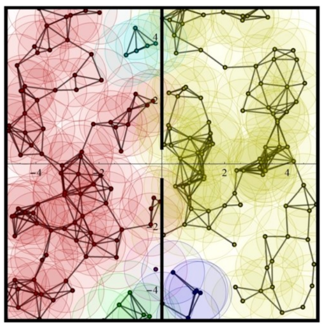
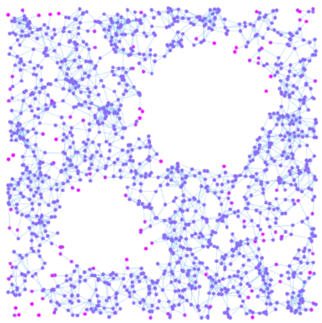
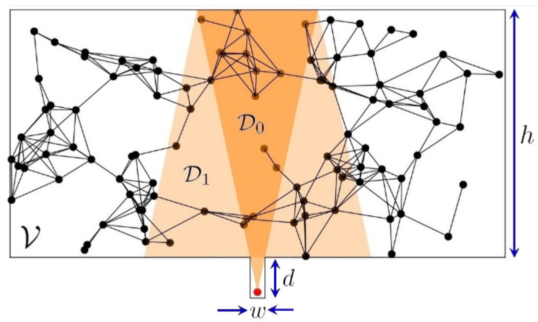
14 spikes: Gyroelongated hexagonal bipyramid!

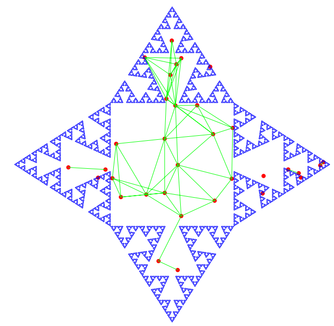


Outlook

Random connection models are more realistic and have smoother properties.

Other results/in progress: Non-convex domains, betweenness, interference, nonuniform, mobility, spectrum ...



Connection functions for other spatial networks?

Very long range connections?

Spatially Embedded Networks

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