Exact computation of the critical exponents of the jamming transition

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Collaborators

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Special thanks Edan Lerner and Matthieu Wyart

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Outline

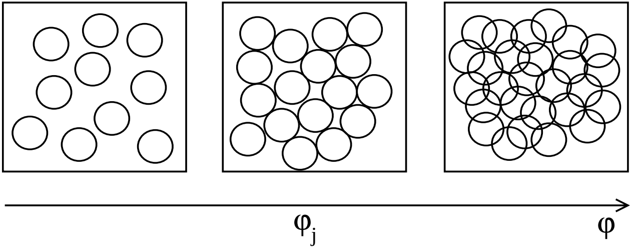
1. The jamming transition
2. A theory of the jamming transition: large d expansion
3. The Gardner transition and the critical exponents

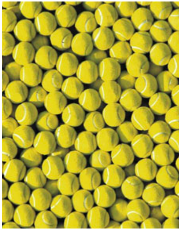
Outline

The jamming transition

A transition that is observed in everyday experience

An *athermal* assembly of repulsive particles
Transition from a loose, floppy state to a mechanically rigid state
Above jamming a mechanically stable network of particles in contact is formed



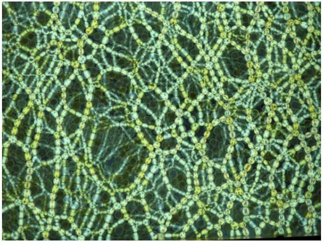


For hard spheres, φ_j is also known as *random close packing*: $\varphi_j(d = 3) \approx 0.64$

[Bernal, Mason, Nature 188, 910 (1960)]
[Liu, Nagel, Nature 396, 21 (1998)]
[O'Hern, Langer, Liu, Nagel, PRL 88, 075507 (2002)]

The jamming transition

Granular materials, emulsion droplets, colloidal suspensions, powders, . . .

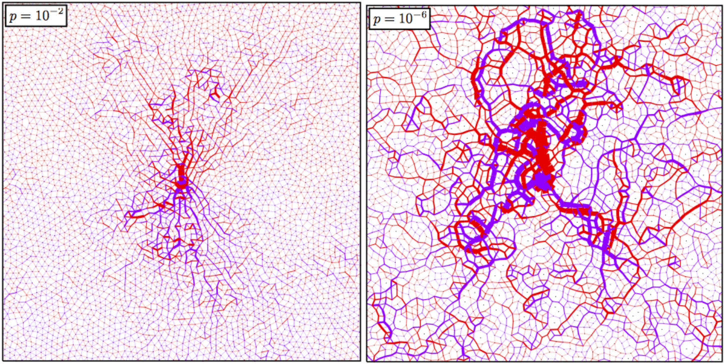


. . . , board games, . . .

Photoelastic disks from B.Behringer's group
ZipZap courtesy of O.Dauchot

The jamming transition

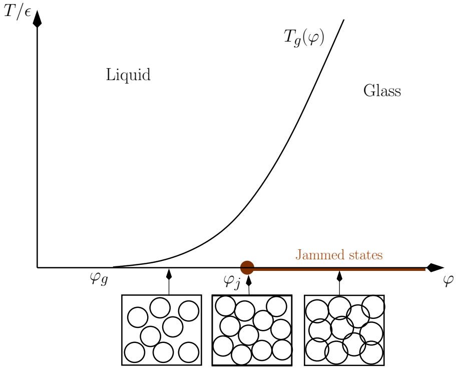
Anomalous "soft modes" associated to a diverging correlation length of the force network



[Wyart, Silbert, Nagel, Witten, PRE 72, 051306 (2005)]
[Van Hecke, J.Phys.: Cond.Mat. 22, 033101 (2010)]

Glass/jamming phase diagram

- Statistical mechanics: introduce temperature T and eventually send $T \to 0$
- The soft sphere model: $v(r) = \epsilon(1 - r/\sigma)^2\theta(r - \sigma)$
- Two control parameters: T/ϵ and $\varphi = v_\sigma N/V$
- $T/\epsilon = 0$ & $\varphi < \varphi_j \leftrightarrow$ hard spheres



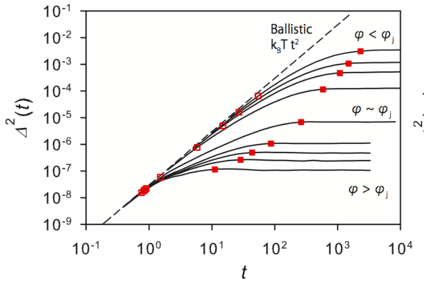
Jamming is a transition from "entropic" rigidity to "mechanical" rigidity
A theoretical description of the glass transition is difficult; and jamming happens inside the glass!

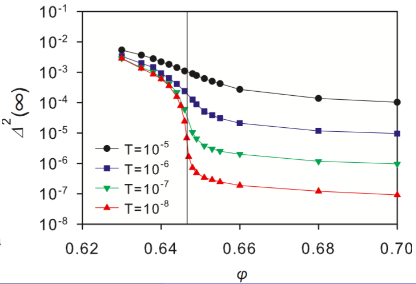
[Berthier, Jacquin, FZ, PRE 84, 051103 (2011)]
[Ikeda, Berthier, Sollich, PRL 109, 018301 (2012)]

Criticality around jamming

- In the glass the MSD has a plateau: diffusion is arrested, only vibrations
- The plateau value $\Delta_{\rm EA}$ is the Debye-Waller factor
- Scaling $\Delta_{\rm EA} \sim T^{\kappa/2}\mathcal{D}[(\varphi-\varphi_j)/\sqrt{T}]$
- Shear modulus of the glass $\mu \sim T/\Delta_{\rm EA}$ has a similar scaling
- At $\varphi = \varphi_j$ & $T = 0$, pair correlation $g(r) \sim (r-\sigma)^{-\alpha}$ and force distribution $P(f) \sim f^{\theta}$

[Donev, Torquato, Stillinger, PRE 71, 011105 (2005)]
[Wyart, PRL 109, 125502 (2012)]
[Charbonneau, Corwin, Parisi, FZ, PRL 109, 205501 (2012)]
[Ikeda, Berthier, Biroli, JCP 138, 12A507 (2013)]





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- Three critical exponents κ, α, θ
- Scaling relations based on *marginal mechanical stability of the packing*
- $\alpha = 1/(2 + \theta)$ and $\kappa = 2 - 2/(3 + \theta)$
- Only one exponent remains undetermined
- Numerically $\alpha \approx 0.4$ in all dimensions, which implies $\theta \approx 0.5$ and $\kappa \approx 1.4$

[DeGiuli, Lerner, Brito, Wyart, arXiv:1402.3834]

The jamming transition is a new kind of zero-temperature critical point, characterized by scaling and non-trivial critical exponents

Outline

Expansion around $d = \infty$ in statistical mechanics

Many fields of physics (QCD, turbulence, critical phenomena, non-equilibrium ... liquids&glasses!) struggle because of the absence of a small parameter

In $d = \infty$, exact solution using mean-field theory

Proposal: use $1/d$ as a small parameter [E.Witten, Physics Today 33, 38 (1980)]

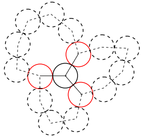
Question: which features of the $d = \infty$ solution translate smoothly to finite d?

For the glass transition, the answer is very debated!

For the jamming transition, numerical simulations show that the properties of the transition and the values of κ, α, θ are *very weakly dependent on d*

[Goodrich, Liu, Nagel, PRL 109, 095704 (2012)]
[Charbonneau, Corwin, Parisi, FZ, PRL 109, 205501 (2012)]

$1/d$ as a small parameter – amorphous hard spheres



- Geometric argument:
  kissing number $e^d \gg$ coordination at jamming $2d$
  $\Rightarrow$ uncorrelated neighbors
  Uncorrelated neighbors correspond to a mean field situation
  (like Ising model in large d)
- Statistical mechanics argument:
  third virial (three body terms) $\ll$ second virial (two-body term).
  Rigorously true for $2^d\varphi \lesssim 1$
  Re-summation of virial series (in the metastable liquid state) gives a pole at $2^d\varphi \sim e^d$.
  Glass transition is around $2^d\varphi \sim d$ Percus, Kirkwood

Keep only ideal gas + second virial term (as in TAP equations of spin glasses):

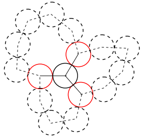
$-\beta F[\rho(x)] = \int dx \rho(x)[1 - \log \rho(x)] + \frac{1}{2} \int dxdy \rho(x)\rho(y)[e^{-\beta v(x-y)} - 1]$

Solve $\frac{\delta F[\rho(x)]}{\delta \rho(x)} = 0$ to find minima of $F[\rho(x)]$

Exact* solution for $d = \infty$ is possible, using your favorite method (we used replicas)

* *Exact for theoretical physics, not rigorous for the moment*

$1/d$ as a small parameter – amorphous hard spheres



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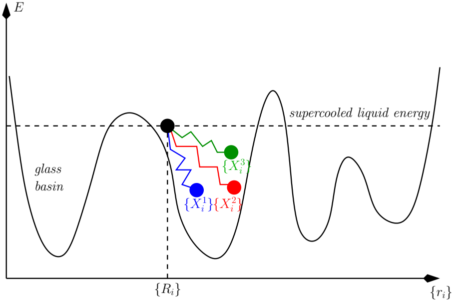
Why replicas? (no quenched disorder!)



Gibbs measure split in many glass states

$$F_g = -k_B T \int dR \frac{e^{-\beta H[R]}}{Z} \log Z[X|R] \qquad Z[X|R] = \int dX e^{-\beta' H[X] + \beta' \varepsilon \sum_i (X_i - R_i)^2}$$

Need replicas to average the log, self-induced disorder

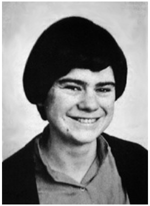
[Franz, Parisi, J. de Physique I 5, 1401 (1995)]
[Monasson, PRL 75, 2847 (1995)]

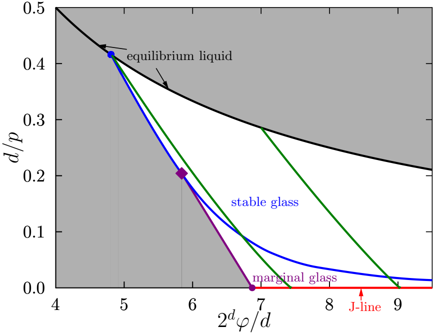
Outline

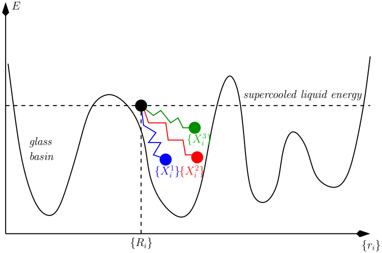
The phase diagram

Crucial result:

- A *Gardner transition* inside the glass phase
- Stable → marginally stable glass *in phase space*
  [Gardner, Nucl.Phys.B 257, 747 (1985)]
- The jamming line falls inside the marginal phase







[Charbonneau, Kurchan, Parisi, Urbani, FZ, Nature Comm. 5, 3725 (2014)]

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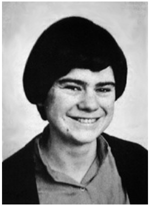

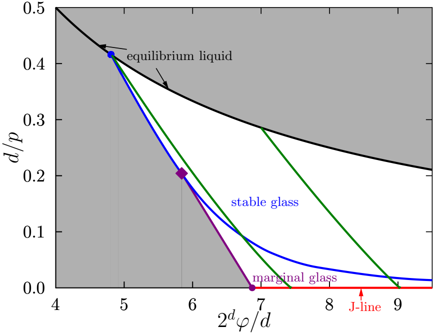

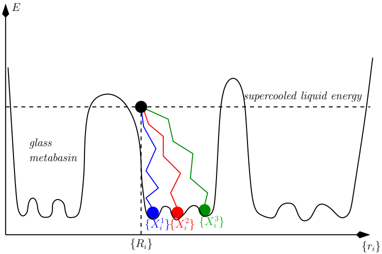

[Charbonneau, Kurchan, Parisi, Urbani, FZ, Nature Comm. 5, 3725 (2014)]

Critical exponents of jamming

- Neglecting the Gardner transition gives $\theta = 0$ and $\alpha = 1$: plain wrong
- Taking into account the Gardner transition gives correct values: $\kappa = 1.41574\ldots$, $\alpha = 0.41269\ldots$, $\theta = 0.42311\ldots$
- Consistent with scaling relations $\alpha = 1/(2+\theta)$ and $\kappa = 2 - 2/(3+\theta)$
- α and κ are perfectly compatible with the numerical values
- Some debate on θ in low dimensions
- Marginal stability in phase space and marginal mechanical stability are intimately connected

[Charbonneau, Kurchan, Parisi, Urbani, FZ, Nature Comm. 5, 3725 (2014)]

Critical exponents of jamming

A short technical detour on the computation of exponents:

- In the replica language the Gardner phase is decribed by the Parisi fullRSB structure
unexpected analogy between HS in $d \to \infty$ and the SK model!
[Wyart, PRL 109, 125502 (2012)]
- Order parameter is $\Delta(y)$ for $y \in [1, 1/m]$, the overlap probability distribution
- Coupled Parisi equation for $\Delta(y)$ and a function $P(y, f)$, probability of the forces
- At jamming, $m \to 0$, $y \in [1, \infty)$
- Scaling solution at large y: $\Delta(y) \sim y^{-1-c}$ and $P(y, f) \sim y^a p(f\, y^b)$
- a, b and c are related to κ, α and θ
- Equation for $p(t)$ in scaling limit: boundary conditions give scaling relations for a, b, c
- One free exponent is fixed by the condition of marginal stability of the fullRSB solution
[Charbonneau, Kurchan, Parisi, Urbani, FZ, arXiv:1310.2549]

Summary

- The jamming transition is a new kind of zero-temperature critical point, characterized by scaling and non-trivial critical exponents

- The $d = \infty$ phase diagram is qualitatively realized in finite d
  Quantitative computations in finite d are possible, in progress

- Critical properties of jamming are obtained only by taking into account the Gardner transition to a marginal phase
  Analytic computation of the non-trivial critical exponents α, θ, κ

- An unexpected connection between hard spheres in $d \to \infty$ and the SK model

THANK YOU FOR YOUR ATTENTION