Interacting electrons in a random medium

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A simple one-dimensional random model

The pieces (or Luttinger-Sy) model

- On $\mathbb{R}$, consider the points of a Poisson process, say, $(x_k(\omega))_{k\in\mathbb{Z}}$.
- For $\Lambda = [-L/2, L/2]$, on $L^2(\Lambda)$, define

$$H_\omega(L) = \bigoplus_{k\in\mathbb{Z}} -\frac{d^2}{dx^2}\Big|^{D}_{\Delta_k\cap\Lambda}$$

  where $\Delta_k = [x_k, x_{k+1}]$

- Integrated density of states

$$N(E) = \frac{\exp(-\ell_E)}{1-\exp(-\ell_E)} 1_{E\geq 0}$$

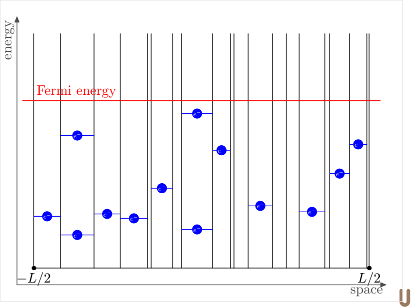
  where $\ell_E := \dfrac{\pi}{\sqrt{E}}$.

The n particle system

- On $\bigwedge_{j=1}^n L^2([-L/2,L/2])$, $H^0_\omega(L,n) = \sum_{i=1}^n \underbrace{1\otimes\ldots\otimes 1}_{i-1 \text{ times}} \otimes H_\omega(L) \otimes \underbrace{1\otimes\ldots\otimes 1}_{n-i \text{ times}}$.
- Pick $U: \mathbb{R}\to\mathbb{R}^+$ even s.t. $U\in L^p(\mathbb{R})$ $(p>1)$ and $x^3\cdot\int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} 0$.
- Define $H^U_\omega(L,n) = H^0_\omega(L,n) + W_n$ where $W_n(x^1,\cdots,x^n) := \sum_{i<j} U(x^i - x^j)$.
- $\Psi^U_\omega(\Lambda,n)$ and $E^U_\omega(\Lambda,n)$: ground state and ground state energy.

The non interacting ground state

1. Fermi energy: $N(E_\rho)=\rho$;
2. Pick all the pieces $\Delta_k = [x_k(\omega), x_{k+1}(\omega)]$ of length larger than $\ell_\rho = \pi/\sqrt{E_\rho}$.
3. For each piece, take all the states associated to levels below E_ρ.
4. Form the Slater determinant to get the non interacting ground state.



The reduced one-particle density matrix for the non interacting ground state

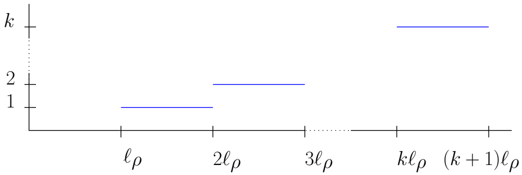
$$\gamma^{(1)}_{\Psi^0_\omega(L,n)} = \sum_{k\geq 1}\left(\sum_{k\ell_\rho\leq|\Delta_k|<(k+1)\ell_\rho}\left(\sum_{j=1}^{k}\gamma^{(j)}_{\varphi^j_{\Delta_k}}\right)\right)$$

where φ^j_I is j-th normalized eigenvector of $-\Delta^D_{|I}$.

The non interacting system: the ground state energy per particle

$$\mathscr{E}^0(\rho) = \lim_{\substack{L\to+\infty\\ n/L\to\rho}} = \frac{E^0_\omega(L,n)}{n}\,\frac{1}{\rho}\int_{-\infty}^{E_\rho} E\,dN(E) \sim E_\rho \sim \pi^2\,|\log\rho|^{-2}\,.$$

Another representation for the ground state:



Existence of the ground state energy per particle

Theorem

Under our assumptions on U, ω−almost surely, the following limit exists and is independent of ω

$$\mathcal{E}^U(\rho) := \lim_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{E^U_\omega(L,n)}{n}.$$

Ground state energy asymptotic expansion

Theorem

Under our assumptions on U, one has

$$\mathcal{E}^U(\rho) = \mathcal{E}^0(\rho) + \frac{\pi^2\gamma_*}{|\log\rho|^3}\rho + o\left(\frac{\rho}{|\log\rho|^3}\right),$$

where $\gamma_* = 1 - \exp\left(-\frac{\gamma}{8\pi^2}\right)$.

Systems of two fermions: within the same piece:

Lemma

Assume that $U \in L^p(\mathbb{R})$ *for some* $p \in (1,+\infty]$ *and that* $\int_{\mathbb{R}} x^2 U(x)dx < +\infty$. *Consider two fermions in* $[0,\ell]$ *interacting via the pair potential* U*, i.e., on* $L^2([0,\ell]) \wedge L^2([0,\ell])$*, consider the Hamiltonian*

$$-\frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} + U(x_1 - x_2). \tag{1}$$

Then, for large ℓ*,* $E^{2,U}(\ell)$*, its ground state energy admits the following expansion*

$$E^{2,U}(\ell) = \frac{5\pi^2}{\ell^2} + \frac{\gamma}{\ell^3} + o\left(\ell^{-3}\right)$$

where $\gamma := \frac{5\pi^2}{2}\left\langle u\sqrt{U(u)}, \left(Id + \frac{1}{2}U^{1/2}(-\Delta_1)^{-1}U^{1/2}\right)^{-1} u\sqrt{U(u)}\right\rangle.$

Uniqueness of the ground state:

Theorem

Assume U *is analytic. Then, for any* L *and* n*,* $H^U_\omega(L,n)$ *has a unique ground state* ω*-almost surely.*

Interacting ground state: "optimal" approximation

Let ζ_I^1 be the ground state of $-\Delta\big|_{I^2}^{D} + U$ acting on $L^2_-(I^2)$. Define

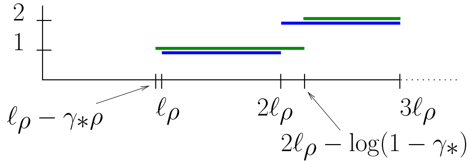
$$\gamma^{(1)}_{\Psi^{\rm opt}_{L,n}} = \sum_{\ell_\rho - \rho\gamma_* \le |\Delta_k| \le 2\ell_\rho - \log(1-\gamma_*)} \gamma^{(1)}_{\varphi^1_{\Delta_k}} + \sum_{2\ell_\rho - \log(1-\gamma_*) \le |\Delta_k|} \gamma^{(1)}_{\zeta^1_{\Delta_k}},$$

Theorem

We assume U compact support. There exists $\rho_0 > 0$ s.t. for $\rho \in (0,\rho_0)$, ω-a.s., one has

$$\limsup_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n} \left\| \gamma^{(1)}_{\Psi^U_\omega(L,n)} - \gamma^{(1)}_{\Psi^{opt}_{L,n}} \right\|_1 \lesssim \frac{\rho}{|\log\rho|},$$

$$\limsup_{\substack{L\to+\infty \\ n/L\to\rho}} \frac{1}{n^2} \left\| \gamma^{(2)}_{\Psi^U_\omega(L,n)} - \frac{1}{2}(Id - Ex)\left[\gamma^{(1)}_{\Psi^{opt}_{L,n}} \otimes \gamma^{(1)}_{\Psi^{opt}_{L,n}}\right] \right\|_1 \lesssim \frac{\rho}{|\log\rho|}.$$



When U is "long" range:

Assume $Z(x) := x^3 \cdot \int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} 0$. For $\ell > 0$, let $\mathbf{1}^1_{<\ell} = \sum_{|\Delta_k(\omega)|<\ell} \mathbf{1}_{\Delta_k(\omega)}$.

Theorem

There exist $\rho_0 > 0$, $C > 0$ such that, for $\rho_\mu \in (0, \rho_0)$, ω-a.s., one has

$$\limsup_{\substack{L\to+\infty\\ n/L\to\rho}} \frac{1}{n} \left\| \left(\gamma^{(1)}_{\Psi^U_\omega(\Lambda,n)} - \gamma^{(1)}_{\Psi^{opt}_{\Lambda,n}} \right) \mathbf{1}^1_{<\ell_\rho + C} \right\|_{tr} \leq C\sqrt{\rho_\mu} \max\left(\frac{\sqrt{\rho_\mu}}{|\log \rho_\mu|}, \sqrt{Z(\ell_{\rho_\mu}/C)} \right),$$

$$\limsup_{\substack{L\to+\infty\\ n/L\to\rho}} \frac{1}{n} \left\| \left(\gamma^{(1)}_{\Psi^U_\omega(\Lambda,n)} - \gamma^{(1)}_{\Psi^{opt}_{\Lambda,n}} \right) \mathbf{1}^1_{\geq\ell_\rho + C} \right\|_{tr} \leq C\rho_\mu \max\left(\frac{1}{|\log \rho_\mu|}, \sqrt{Z(\ell_{\rho_\mu}/C)} \right).$$

When U decays slowly,

larger interaction between far away pieces

$\Longrightarrow$ interaction between short pieces more important (because of their larger number)

$\Longrightarrow$ optimal state in the short pieces quite different from ground state.

Does not change ground state energy to second order until $Z(x) \not\to 0$.

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Some open questions

1. When $x^3 \int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} 0$ not too fast, get a good control of the changes induced by the "long" range interactions.
   Get a good description of the ground state in the short pieces.
2. For U compactly supported, we actually have a better expansion for $\mathscr{E}^U(\rho)$. And we have a more precise description of the ground state.
   Does $\gamma^{(1)}_{\Psi^U_\omega(L,n)}$ converge as $L \to +\infty$?
3. What happens if $x^3 \int_x^{+\infty} U(t)dt \underset{x\to+\infty}{\longrightarrow} +\infty$? One may expect
   - if $\int_{\mathbb{R}} U(t)dt < +\infty$: interactions at a distance become more important than local interactions in the same piece.
   - if $\int_{\mathbb{R}} U(t)dt = +\infty$, interactions become more important than non interacting energy term.

In our model, no tunneling for a single particle

What happens for a more realistic model that includes tunneling?
In dimension 1, preliminary computations suggest same picture.

What happens in higher dimensions?