UNIQUENESS OF THE NON-EQUILIBRIUM STEADY STATE FOR A 1D BGK MODEL IN KINETIC THEORY

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Abstract. We continue our investigation of kinetic models of a one-dimensional gas in contact with homogeneous thermal reservoirs at different temperatures. Nonlinear collisional interactions between particles are modeled by a so-called BGK dynamics which conserves local energy and particle density. Weighting the nonlinear BGK term with a parameter $\alpha \in [0, 1]$, and the linear interaction with the reservoirs by $(1 - \alpha)$, we prove that for some α close enough to zero, the explicit spatially uniform non-equilibrium stable state (NESS) is *unique*, and there are no spatially non-uniform NESS with a spatial density ρ belonging to L^p for any $p > 1$. We also show that for all $\alpha \in [0, 1]$, the spatially uniform NESS is *dynamically stable*, with small perturbation converging to zero exponentially fast.

Contents

1. Introduction

This paper is a contribution to the theory of non-equilibrium steady states (NESS), of open systems in the particular context of kinetic theory. The understanding of NESS, their properties, uniqueness or lack thereof and stability or lack thereof, represents a challenge in mathematical physics due to the fact that the dynamics are nonlinear, non-Markovian and the absence of an entropy principle. Our main result is a uniqueness and stability theorem for the NESS in a simple nonlinear model.

1.1. **The model.** We briefly describe the sort of underlying particle model that would lead to the type of kinetic equation that we study here. It consists of a gas of particles on the one-dimensional torus $\mathbb{T}$, that interact only through binary energy conserving collisions, however we also suppose that there are two types of scatterers distributed on the torus according to some Poisson distribution, as in a Lorentz

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model, except that each scatterer has a temperature, T_1 or T_2 depending on its type, and a certain radius of interaction, so that when a gas particle travels across the interaction interval, a Poisson clock runs and if it goes off, the particle assumes a new velocity chosen at random according to the Maxwellian distribution for the temperature of the scatterer.

In an appropriate scaling limit, the net effect of the background scatterers is to produce two uniform thermal reservoirs. Whatever the speed of a gas particle, its rate of interaction with the reservoirs depends only on the density of the scatterers, again in an appropriate limit in which their intervals of interaction are unlikely to overlap. The kinetic equation that one would expect to arise from such a model in such a limit would be of the type (1.1) below, except that one might expect a Kac-Povzner type collision kernel, also known as a "soft-spheres" kernel [7, 13]. Our work concerns the kinetic equation itself, and not its rigorous derivation from an underlying particle system, although the brief description of such a system that we have given hopefully illuminates the physical context of our model.

We are concerned with the existence and uniqueness of NESS for our system. We make a further simplification, and model the gas particle collisions with a BGK collision kernel [4, 14, 15]. This will render the existence of NESS trivial, but the uniqueness is still a challenging problem, and we shall only prove part of what we conjecture to be true.

These considerations bring us to the following one dimensional kinetic model:

$$\partial_t f + v\partial_x f = \alpha \mathcal{M}_f + (1-\alpha)\rho_f \left(\frac{M_{T_1} + M_{T_2}}{2}\right) - f \tag{1.1}$$

where $\alpha \in [0,1]$, $f = f(t,x,v)$, $x \in \mathbb{T}$, $v \in \mathbb{R}$, and

$$M_{T_i}(v) := \frac{e^{-\frac{|v|^2}{2T_i}}}{\sqrt{2\pi T_i}}, \quad \mathcal{M}_f(t,x,v) := \rho_f(t,x)\frac{e^{-\frac{|v|^2}{2T_f(t,x)}}}{\sqrt{2\pi T_f(t,x)}}, \tag{1.2}$$

with

$$\begin{cases} \rho_f(t,x) := \displaystyle\int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v, \\ P_f(t,x) := \displaystyle\int_{\mathbb{R}} v^2 f(t,x,v)\,\mathrm{d}v = \rho_f(x) T_f(t,x), \end{cases} \tag{1.3}$$

being the *spatial density* and *pressure* corresponding to f. T_f is then the *temperature* corresponding to f, and T_1 and $T_2 \in (0,+\infty)$ being the two temperatures of the reservoirs.

The linear terms on the right in (1.1) that are multiplied by $(1-\alpha)$ model the interaction of particles with two reservoirs, both acting everywhere in space. Each time a particle interacts with one of the reservoirs, it velocity is replaced by a new velocity selected at random from the corresponding Maxwellian distribution. We have taken both of these Maxwellians to have zero mean velocity which is natural for static reservoirs.

The nonlinear term on the right in (1.1) that is multiplied by α represents the effect of collisions between particles. The collision term $\mathcal{M}_f$ is of the BGK type (see [4]), except that as usual in one dimensional kinetic models, it conserves only mass and energy, not momentum. Indeed, binary collisions that conserve both energy and momentum are trivial in one dimension: only an exchange of velocities is possible. For this reason, the mean velocity of $\mathcal{M}_f$ is zero.

The term $-f$ on the right in (1.1) is the *loss term* corresponding to both interactions with the reservoirs and other particles: after such interactions, particles vacate their pre-interaction state.

Without loss of generality, we choose units in which the torus has unit volume and there is unit total mass:

$$\mathbb{T} = [-1/2, 1/2] \quad \text{and} \quad \int_{-1/2}^{1/2}\int_{\mathbb{R}} f(x,v)\,\mathrm{d}x\,\mathrm{d}v = 1.$$

1.2. **Previous results.** In our previous papers [5, 6], we have studied related issues for related models. In [5] we proved the existence of spatially homogeneous non-equilibrium steady states and exponential convergence to them for related spatially homogeneous models, but with more realistic collision mechanisms, and also in higher dimensions. In [6] we studied the exponential rate of convergence to steady state for a non spatially homogeneous equation of the type (1.1) but with a modified collision mechanism that permitted the equation to be interpreted as the Kolmogorov forward equation for a non-stationary Markov process: we replaced the space-dependent local temperature $T_f(t,x)$ by the global temperature

$$\mathcal{T}_f(t) := \int_{\mathbb{T}\times\mathbb{R}} v^2 f\,\mathrm{d}x\,\mathrm{d}v$$

of f that depends only on time. We were then able to apply Doeblin's method [16] to prove the exponential convergence. The use of Doeblin's method to study linear dynamical models originates with [3, 11].

The rigorous study of NESS for nonlinear kinetic equations remains very challenging. One problem that has been studied by several authors is the Boltzmann equation in a slab with different temperatures on the two walls, with and without external forces. At this level of generality, one cannot always expect a unique NESS – there may be a symmetry breaking transition, such as the onset of Rayleigh-Bernard flow. Even without external forces, existence of NESS for the slab problem is a highly non-trivial, and existing results [2, 8] do not yield provide any information on uniqueness or non-uniqueness.

More recently, the Boltzmann equation in more general domains and with non-isothermal boundary conditions has been investigated [10] where it is proved that when the temperature on the boundary is sufficiently close to constant, then there is an NESS that is close to the uniform Maxwellian for the mean boundary temperature, and in a small neighborhood of this Maxwellian, there is no other NESS. However, it is not known that there are not other NESS further away, no matter how small the non-zero temperature difference may be.

1.3. **Question studied and conjecture.** Observe that the reservoirs will tend to damp out any mean velocity since M_{T_1} and M_{T_2} have zero mean velocity. Likewise, $\mathcal{M}_f$ has zero mean velocity at each x, so the collision gain term too will tend to damp out any mean velocity.

Therefore, if $f = f(v)$ is any spatially homogeneous steady state, $\int_{\mathbb{R}} vf \, \mathrm{d}v = 0$. Moreover, the time and space homogeneity yield $\partial_t f = 0$ and $v\partial_x f = 0$. Finally, multiplying both sides of (1.1) by v^2 and integrating over x and v shows

$$0 = (\alpha - 1)T_f + (1-\alpha)\frac{T_1 + T_2}{2}, \qquad T_f := \int_{\mathbb{R}} v^2 f(v) \, \mathrm{d}v.$$

Thus, the constant temperature in any spatially homogeneous steady state f must be $T_\infty := (T_1 + T_2)/2$ if $\alpha \neq 1$. Then for any spatially homogeneous steady state f, $\mathcal{M}_f = M_{(T_1+T_2)/2}$, and (1.1) reduces to

$$\alpha M_{\frac{T_1+T_2}{2}} + (1-\alpha)\rho_f \left(\frac{M_{T_1} + M_{T_2}}{2} \right) - f = 0.$$

Therefore the unique spatially homogeneous steady state is given by

$$f_\infty := \alpha M_{T_\infty} + (1-\alpha)\frac{M_{T_1} + M_{T_2}}{2}, \quad T_\infty = \frac{T_1 + T_2}{2}. \tag{1.4}$$

Observe that f_∞ is *not* Maxwellian as soon as $\alpha \neq 1$.

If $\alpha = 0$, the term $\mathcal{M}_f$ is not present, the only spatially homogeneous steady state is $f_\infty = \frac{1}{2}(M_{T_1} + M_{T_2})$, and the equation (1.1) is linear. It can be interpreted as the forward equation of a Markov process and in [6] we used probabilistic methods to prove that this steady state is unique and is approached exponentially fast. Hence, for $\alpha = 0$, there are no steady states that are spatially inhomogeneous.

Next, consider the case $\alpha = 1$: there are no thermal reservoirs and energy is conserved. There is a one-parameter infinite family of steady states, namely M_T for all $T > 0$. Moreover, if f_0 is such that

$$\int_{\mathbb{T}\times\mathbb{R}} v^2 f_0(x,v) \, \mathrm{d}x \, \mathrm{d}v = T, \qquad \int_{\mathbb{T}\times\mathbb{R}} f_0(x,v) \ln f_0(x,v) \, \mathrm{d}x \, \mathrm{d}v < +\infty,$$

and $f(t,x,v)$ is the solution of (1.1) with initial datum f_0, then

$$H\left(f(t,\cdot,\cdot)|M_T\right) = \int_{\mathbb{T}\times\mathbb{R}} f(t,x,v) \ln \frac{f(t,x,v)}{M_T(v)} \, \mathrm{d}x \, \mathrm{d}v$$

decreases monotonically to zero, and is stationary only when $f = M_T$. It follows that M_T is the unique steady state among solutions with second moment equal to T and finite entropy, and thus every steady state for $\alpha = 1$ with finite second moment and entropy is spatially homogeneous (and equal to M_T).

The question that motivates this paper is the study of the NESS in the intermediate region $\alpha \in (0,1)$. We conjecture the following:

Conjecture (Uniqueness of the NESS for the BGK model with reservoirs). *For all $\alpha \in [0,1]$, the non-equilibrium steady state of* (1.1) *is unique, regardless of the temperature difference, spatially homogeneous and stable under perturbations. We also expect this conjecture to hold in higher dimensions $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$.*

1.4. **Main results.** We give a partial answer to this conjecture, showing that it is satisfied when α is small enough. We first prove the uniqueness:

Theorem 1. *For all T_1, T_2, there is an explicitly computable $\alpha_0 > 0$ such that for all $\alpha \in [0, \alpha_0)$, every steady state solution f_∞ of (1.1) that belongs to $L^1(\mathbb{T} \times \mathbb{R})$, has finite second moment and is such that $\rho \in L^p(\mathbb{T})$ for some $p > 1$, is constant in x.*

Remark 1.1. Our method would also apply in higher dimensions *provided* there was non-trivial spatial dependence in only one direction on the torus, say the x_1 coordinate. The decomposition between odd and even parts in the next section should be then modified by splitting along v_1 only. The rest of the analysis would be similar.

We also prove the stability under perturbation for all $\alpha \in [0, 1]$. For this, we introduce the (real) Hilbert Space $\mathcal{H}^1_\alpha$ with inner product

$$\langle f, g\rangle_{\mathcal{H}^1_\alpha} = \int_{\mathbb{T}\times\mathbb{R}} (f(x,v)(1-\partial_x^2)g(x,v))\frac{1}{f_{\alpha,\infty}}\,\mathrm{d}x\,\mathrm{d}v\ . \tag{1.5}$$

Theorem 2. *For all $\alpha \in [0, 1]$, the spatially homogeneous steady state described above is asymptotically stable under perturbation in $\mathcal{H}^1_\alpha$. Small perturbations decay exponentially fast in time in this space.*

Theorem 2 shows that if for some $\alpha > \alpha_0$ there do exist non-uniform steady states, they do not arise as a branch bifurcating off the family of spatially homogeneous steady state solutions.

1.5. **Plan of the paper.** In Section 2, we establish some useful relations on the moments of any given NESS and introduce a decomposition between odd and even parts. In Section 3, we prove lower and upper pointwise bounds on local density and temperature of any given NESS. In Section 4, we explain the contraction mapping argument; it is in this section that we use that α is close to zero. Finally in Section 5, we prove a local stability result of the spatially homogeneous steady states for all $\alpha \in [0, 1]$.

2. Preliminaries: properties and decompositions of NESS

2.1. **Zero momentum and constant pressure.** A partial result supporting our conjecture is that the pressure is independent of x, as well as the momentum $m_f(x) := \rho_f(x)u_f(x)$, the latter being zero:

Lemma 3. *Let $f(x,v)$ be a probability density on $\mathbb{T} \times \mathbb{R}$ such that $v^2 f(x,v)$ is integrable, and suppose that $f(x,v)$ solves in a weak sense the equation*

$$v\partial_x f(x,v) = F(x,v) - f(x,v) \tag{2.1}$$

where $F(x,v)$ is a measurable function such that $(1+|v|)F(x,v)$ is integrable and

$$\forall\, x \in \mathbb{T}, \quad \int_{\mathbb{R}} F(x,v)\,\mathrm{d}v = \rho_f(x) \quad \text{and} \quad \int_{\mathbb{R}} vF(x,v)\,\mathrm{d}v = 0. \tag{2.2}$$

Then the pressure is constant and the momentum is zero:

$$\begin{cases} P_f(x) = \displaystyle\int_{\mathbb{R}} v^2 f(x,v)\,\mathrm{d}v = P_\infty \in \mathbb{R} \\ m_f(x) = \displaystyle\int_{\mathbb{R}} v f(x,v)\,\mathrm{d}v = \rho_f(x) u_f(x) = 0. \end{cases}$$

Remark 2.1. Evidently, Lemma 3 applies to any finite energy NESS of our equation.

Proof. Integrating both sides of (2.1) in v yields

$$\frac{\mathrm{d}}{\mathrm{d}x} m_f(x) = \rho_f(x) - \rho_f(x) = 0.$$

This proves that $m_f(x)$ is a constant $m_\infty \in \mathbb{R}$. Now multiplying both sides of (2.1) by v and integrating in v yields

$$\frac{\mathrm{d}}{\mathrm{d}x} P_f(x) = -m_f(x) = -m_\infty. \tag{2.3}$$

Integrating both sides of (2.3) in x shows that $m_\infty = 0$, and $P_f(x) = P_\infty \in \mathbb{R}$ is constant. ☐

Remark 2.2. The proof of Lemma 3 takes advantage of the dimension being one. In higher dimension, the argument above would only show that m_f is a divergence free vector field, but not necessarily constant. We shall make further use of the one dimensionality of the model when proving pointwise bounds on the NESS.

2.2. **Higher moments.** Multiplying the steady-state equation

$$v\partial_x f = \alpha \mathcal{M}_f + (1-\alpha)\rho_f \left(\frac{M_{T_1} + M_{T_2}}{2} \right) - f \tag{2.4}$$

by v^2, and integrating in v yields

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} v^3 f(x,v)\,\mathrm{d}v = (1-\alpha)\left(\frac{T_1+T_2}{2}\right)(\rho_f(x) - 1),$$

since by Lemma 3, $P_f(x) = (T_1 + T_2)/2$. Next, multiplying (2.4) by v^3 and integrating yields

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v = -\int_{\mathbb{R}} v^3 f(x,v)\,\mathrm{d}v.$$

Combining the last two equations yields

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v = (1-\alpha)\left(\frac{T_1+T_2}{2}\right)(\rho_f(x) - 1). \tag{2.5}$$

Since the right hand side integrates to zero, we have

$$\int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v - \int_{\mathbb{T}\times\mathbb{R}} v^4 f(x,v)\,\mathrm{d}x\,\mathrm{d}v = (1-\alpha)\left(\frac{T_1+T_2}{2}\right)\int_{\mathbb{T}} \psi(x-y)(\rho_f(y)-1)\mathrm{d}y$$

where

$$\psi(x) = \sum_{k\neq 0} \frac{e^{2\pi i k x}}{4\pi^2 k^2} \qquad \text{so that} \quad |\psi(x)| \leq \frac{1}{12}.$$

It follows that

$$\left|\int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v - \int_{\mathbb{T}\times\mathbb{R}} v^4 f(x,v)\,\mathrm{d}x\,\mathrm{d}v\right| \leq (1-\alpha)\left(\frac{T_1+T_2}{12}\right). \tag{2.6}$$

In particular, for α close to 1, $\int v^4 f(x,v)\,\mathrm{d}v$ is nearly constant; its average is

$$\int_{\mathbb{T}\times\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v\mathrm{d}x = 3\left[\alpha\left(\frac{T_1+T_2}{2}\right)^2 + (1-\alpha)\frac{T_1^2+T_2^2}{2}\right] \tag{2.7}$$

and thus the spatial fluctuations in $\int v^4 f(x,v)\,\mathrm{d}v$ are a small fraction of the mean for large temperatures.

Lemma 4. *Let f be a solution to* (2.4) *such that*

$$\int_{\mathbb{T}\times\mathbb{R}} (1+v^2) f(x,v)\,\mathrm{d}x\,\mathrm{d}v < \infty$$

and recall that $P_\infty := (T_1+T_2)/2$. Then (2.6) *is valid, and the spatial density ρ_f satisfies*

$$\rho_f(x) \geq \frac{1}{3(2-\alpha) + \frac{(1-\alpha)}{6P_\infty}}. \tag{2.8}$$

Proof. By the Cauchy-Schwarz inequality and Lemma 3,

$$\forall\, x \in \mathbb{T}, \quad P_\infty = P_f(x) = \int_{\mathbb{R}} v^2 f(x,v)\,\mathrm{d}v \leq \left(\int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v\right)^{1/2} \rho_f^{1/2}(x),$$

so that

$$\rho_f(x) \geq P_\infty^2 \left(\int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v\right)^{-1}.$$

From (2.7), we have the bounds

$$\begin{cases} \displaystyle\int_{\mathbb{T}\times\mathbb{R}} v^4 f(x,v)\,\mathrm{d}x\,\mathrm{d}v & \leq 3(2-\alpha)P_\infty^2 \\ \displaystyle\sup_{x\in\mathbb{T}} \int_{\mathbb{R}} v^4 f(x,v)\,\mathrm{d}v & \leq 3(2-\alpha)P_\infty^2 + \frac{(1-\alpha)P_\infty}{6}. \end{cases}$$

Combining bounds yields the result. □

2.3. **Splitting between odd and even parts and a wave-like system.** We split a given steady state f into even and odd parts $f = E + O$ with respect to the v variable. The steady state equation (1.1) can be rewritten as:

$$\begin{cases} v\partial_x E = -O \\ v\partial_x O = F_\alpha - E \end{cases} \tag{2.9}$$

where

$$F_\alpha(x,v) := \alpha \mathcal{M}_f(x,v) + (1-\alpha)\rho_f(x) G(v), \quad G := \left(\frac{M_{T_1}+M_{T_2}}{2}\right). \tag{2.10}$$

Combining the two equations in (2.9), we obtain

$$\left(1 - v^2\partial_x^2\right) E = F_\alpha. \tag{2.11}$$

Note that for each $v \neq 0$, the operator $(1 - v^2\partial_x^2)$ is invertible with a bounded kernel. The equation (2.11) conveniently and efficiently expresses the iterated effects of velocity averaging on the steady state, or rather on its even part.

Lemma 5. *For each density ρ on $\mathbb{T}$, there is at most one NESS f such that $\rho = \rho_f$.*

Proof. We have seen that for any NESS, $P_\infty = (T_1 + T_2)/2$, and then by Lemma 3, $\rho_f T_f = P_\infty$, so that $\mathcal{M}_f$ is determined by ρ. It follows that F_α is determined by ρ, and then since (2.11) is uniquely solvable, the uniqueness of f follows. □

The formal solution of (2.12) is

$$E = \left(1 - v^2\partial_x^2\right)^{-1} F_\alpha \tag{2.12}$$

and can be written in terms of an explicit Green's function. Integrating in v yields

$$\rho_f(x) = \int_{\mathbb{R}} \left[\left(1 - v^2\partial_x^2\right)^{-1} F_\alpha\right] \mathrm{d}v. \tag{2.13}$$

Lemma 6. *Let $T_1, T_2 > 0$ and $\alpha \in (0,1)$ and $P_\infty = (T_1 + T_2)/2$. For any probability density $\rho = \rho(x)$ on $\mathbb{T}$, define $T(x) = P_\infty/\rho(x)$ and*

$$\begin{cases} \mathcal{M}[\rho](x,v) & := \dfrac{\rho(x)}{\sqrt{2\pi T(x)}} e^{-\frac{v^2}{2T(x)}} = \dfrac{\rho^{3/2}(x)}{\sqrt{2\pi P_\infty}} e^{-\frac{v^2\rho(x)}{2P_\infty}} \\ F_\alpha[\rho](x,v) & := \alpha \mathcal{M}_\rho(x,v) + (1-\alpha)\rho(x)G(v) \\ \Psi_\alpha[\rho](x) & := \displaystyle\int_{\mathbb{R}} \left[\left(1 - v^2\partial_x^2\right)^{-1} F_\alpha[\rho]\right] \mathrm{d}v \end{cases} \tag{2.14}$$

with G defined as in (2.10)*. Then for all ρ, $\Psi_\alpha[\rho[$ is a probability density on $\mathbb{T}$, and ρ is the spatial density of some NESS f if and only if $\rho = \Psi_\alpha[\rho]$, and in this case the unique such NESS is given in terms of ρ by* (2.15) *and* (2.16) *below.*

Proof. Let ρ be any density. Then $F_\alpha[\rho]$ is a probability density on $\mathbb{T} \times \mathbb{R}$. Since $(1 + v^2\partial_x^2)^{-1}$ preserves both integrals and positivity, $(1 + v^2\partial_x^2)^{-1} F_\alpha[\rho]$ is also a a probability density on $\mathbb{T} \times \mathbb{R}$, and hence its marginal, $\Psi_\alpha[\rho]$, is a probability density on $\mathbb{T}$.

Next, suppose that $\rho(x) = \rho_f(x)$ for some NESS $f(x,v)$. By Lemma 3, $\mathcal{M}[\rho](x,v) = \mathcal{M}_f(x,v)$, and therefore $F_\alpha[\rho](x,v)$ is given in terms of f by (2.10). Then $E(x,v)$, given by (2.12), is a probability density on $\mathbb{T} \times \mathbb{R}$. $E(x,v)$ is the even part of $f(x,v)$ and finally $O(x,v)$, the odd part of $f(x,v)$, is given by the first equation in (2.9). Then by (2.13), $\rho = \Psi_\alpha[\rho]$.

Finally, suppose that $\rho(x)$ is a probability density on $\mathbb{T}$ such that $\rho = \Psi_\alpha[\rho]$. Define $F_\alpha[\rho]$ by (2.14), and then define $E_\alpha[\rho](x,v)$ by

$$E_\alpha[\rho] := \left(1 - v^2\partial_x^2\right)^{-1} F_\alpha[\rho], \tag{2.15}$$

and then define

$$O_\alpha[\rho] = -v\partial_x E_\alpha[\rho] \qquad \text{and} \qquad f_\rho(x,v) = E_\alpha[\rho](x,y) + O_\alpha[\rho](x,y). \tag{2.16}$$

Then

$$\begin{aligned} v\partial_x f_\rho(x,v) &= -O_\alpha[\rho] - v^2\partial_x^2 E_\alpha[\rho] \\ &= -O_\alpha[\rho] - E_\alpha[\rho] + (1 - v^2\partial_x^2)E_\alpha[\rho] \\ &= F_\alpha[\rho] - f_\rho. \end{aligned}$$

Lemma 3 applies to this equation, and we conclude that $\rho_{f_\rho} T_{f_\rho} = P_\infty$. The fixed point equation $\rho = \Psi_\alpha[\rho]$ implies $\rho_{f_\rho} = \rho$ and $\mathcal{M}[\rho] = \mathcal{M}_{f_\rho}$. This shows that f_ρ is an NESS for our equation, and concludes the proof that ρ is the spatial density of an NESS if and only if it is a fixed point of Ψ_α. □

It follows from Lemma 6 that our conjecture would be proved if we could show that the constant density is the only fixed point of Ψ_α for all $\alpha \in [0,1]$. We prove this for sufficiently small α in the next section.

3. Pointwise bounds on the moments of the NESS

3.1. Preliminaries.

We define the standard Fourier series of an integrable function $r = r(x)$ on the torus $\mathbb{T} = [-1/2, 1/2]$ by

$$\hat{r}(k) := \int_0^1 r(x) e^{-2i\pi kx}\,\mathrm{d}x, \quad k \in \mathbb{Z},$$

and we recall the inversion formula (when, say, the Fourier modes $(\hat{r}(k))_{k\in\mathbb{Z}}$ are absolutely summable)

$$r(x) = \sum_{k\in\mathbb{Z}} \hat{r}(k) e^{2i\pi kx}.$$

Define $\varphi_v(x)$ the fundamental solution to the Laplace equation

$$\left(1 - (v\partial_x)^2\right)^{-1} \varphi_v(x) = \delta_0(x)$$

on the circle $\mathbb{T} = [-1/2, 1/2]$. This fundamental solution is explicit:

$$\varphi_v(x) = \sum_{m\in\mathbb{Z}} \phi_v(x+m), \quad \phi_v(x) := \frac{1}{2|v|} e^{-\frac{|x|}{|v|}}$$

and its formula in Fourier is

$$\hat{\varphi}_v(k) = \int_{-1/2}^{1/2} e^{-i2\pi kx} \varphi_v(x)\,\mathrm{d}x = \frac{1}{1 + (2\pi)^2 v^2 k^2}, \quad k \in \mathbb{Z}.$$

The following bounds will be useful: $\varphi_v \in L^p(\mathbb{T})$ for all $p \in [1, +\infty]$, and

$$\|\varphi_v\|_{L^p(\mathbb{R})} = \left(\frac{1}{p}\right)^{\frac{1}{p}} \left(\frac{1}{2|v|}\right)^{\frac{p-1}{p}}, \ p \in [1, +\infty), \qquad \|\varphi_v\|_{L^\infty(\mathbb{R})} = \left(\frac{1}{2|v|}\right). \tag{3.1}$$

and it satisfies the lower bound

$$\forall\, x, y \in \mathbb{T}, \quad \varphi_v(x-y) \geq \frac{1}{2|v|} e^{-\frac{1}{|v|}}. \tag{3.2}$$

3.2. **Lower bound on the NESS.** We have already proved a uniform lower bound on the spatial density of any NESS in Lemma 4. We now prove a stronger result: a uniform lower bound holds for every density in the range of Ψ_α.

Lemma 7 (Pointwise lower bound)**.** *Let $\alpha \in [0,1]$ and let ρ be any probability density on $\mathbb{T}$ of the form $\rho = \Psi_\alpha[\rho_0]$ for a probability density ρ_0 on $\mathbb{T}$. Then*

$$\forall\, x \in \mathbb{T}, \quad \rho(x) \geq r_\infty := \frac{(1-\alpha)}{4\sqrt{e}} \int_{1\leq|v|\leq 2} \left(\frac{M_{T_1} + M_{T_2}}{2} \right) \mathrm{d}v. \tag{3.3}$$

Proof. Define $F_\alpha[\rho_0]$ in terms of ρ_0 using (2.14). Then

$$\rho(x) = \int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\, F_\alpha[\rho_0](y,v)\,\mathrm{d}y\,\mathrm{d}v. \tag{3.4}$$

The operator $(1-(v\partial_x)^2)^{-1}$ preserves positivity, and since $F_\alpha[\rho_0] \geq (1-\alpha)\rho_0 G$, we obtain

$$\begin{aligned} \rho(x) &\geq (1-\alpha) \int_{\mathbb{R}} G(v) \left(\int_{\mathbb{T}} \varphi_v(x-y)\,\rho_0(y)\,\mathrm{d}y \right) \mathrm{d}v \\ &\geq (1-\alpha) \int_{1\leq|v|\leq 2} G(v) \left(\int_{\mathbb{T}} \varphi_v(x-y)\,\rho_0(y)\,\mathrm{d}y \right) \mathrm{d}v. \end{aligned}$$

The kernel $\varphi_v(x-y)$ is bounded below by $e^{-1/|v|}/(2|v|)$. The function defined by $t>0 \mapsto t^{-1}e^{-t^{-1}}$ vanishes as t approaches zero or infinity, is maximum at $t=1$ and then decreases as t increases, and has the value $1/(2\sqrt{e})$ at $t=2$. This implies

$$\varphi_v(x-y) \geq \frac{1}{4\sqrt{e}}$$

for all x,y whenever $1 \leq |v| \leq 2$, which concludes the proof. □

3.3. **Upper bound on the NESS.**

Lemma 8 (Gain of integrability of Ψ_α)**.** *Let $\alpha \in [0,1]$ and $r \in [0,1)$, and let $\rho_0 \in L^{1+r/2}(\mathbb{T})$ a probability density. Then $\rho := \Psi_\alpha[\rho_0] \in L^{1+r}(\mathbb{T})$, and there are $A_r, B_r > 0$ depending only on r and degenerating as $r \to 1$ such that*

$$\int_{\mathbb{T}} \rho^{1+r}\,\mathrm{d}x \leq \alpha A_r \int_{\mathbb{T}} \rho_0^{1+r/2}\,\mathrm{d}x + B_r\ . \tag{3.5}$$

As a consequence, if $\rho = \rho_0 \in L^{1+r/2}(\mathbb{T})$ is a fixed point of Ψ_α, there is a constant $K_{r,\alpha} > 0$ depending only on r and α and monotone increasing in α such that

$$\int_{\mathbb{T}} \rho^{1+r}\,\mathrm{d}x \leq K_{r,\alpha}.$$

Remark 3.1. Note that the constant $K_{r,\alpha}$ is independent of $\|\rho\|_{L^{1+r/2}(\mathbb{T})}$. That is, knowing only that $\int_{\mathbb{T}} \rho^{1+r/2}\,\mathrm{d}x$ is finite, we obtain a universal bound on $\int_{\mathbb{T}} \rho^{1+r}\,\mathrm{d}x$.

Proof. Define again $F_\alpha[\rho_0]$ in terms of ρ_0 using (2.14) so that

$$\rho(x) = \int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\, F_\alpha[\rho_0](y,v)\,\mathrm{d}y\,\mathrm{d}v. \tag{3.6}$$

Multiply (3.6) by $\rho^r(x)$ and integrate in x:

$$\begin{aligned}\int_{\mathbb{T}} \rho^{1+r}(x)\,\mathrm{d}x &= \int_{\mathbb{T}} \rho^r(x)\left(\int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\, F_\alpha[\rho_0](y,v)\,\mathrm{d}y\,\mathrm{d}v\right)\mathrm{d}x\\ &= \int_{\mathbb{T}\times\mathbb{R}}\left(\int_{\mathbb{T}} \varphi_v(x-y)\,\rho^r(x)\,\mathrm{d}x\right) F_\alpha[\rho_0](y,v)\,\mathrm{d}y\,\mathrm{d}v.\end{aligned}$$

Equation (3.1) implies

$$\int_{\mathbb{T}} \varphi_v(x-y)\,\rho_0^r(x)\,\mathrm{d}x \ \le \|\rho_0\|^r_{L^1(\mathbb{T})}\|\varphi_v\|_{L^{\frac{1}{1-r}}(\mathbb{T})} \le \frac{1}{(2|v|)^r}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{T}} \rho^{1+r}(x)\,\mathrm{d}x \ &\le \ \int_{\mathbb{T}\times\mathbb{R}} |v|^{-r} F_\alpha[\rho_0](y,v)\,\mathrm{d}y\,\mathrm{d}v\\ &\le \ \int_{\mathbb{T}\times\mathbb{R}} |v|^{-r}\Big[\alpha\mathcal{M}_{\rho_0}(y,v) + (1-\alpha)\rho_0(y)G(v)\Big]\,\mathrm{d}y\,\mathrm{d}v\\ &\le \ \alpha\int_{\mathbb{T}\times\mathbb{R}} |v|^{-r}\mathcal{M}_{\rho_0}(y,v)\,\mathrm{d}y\,\mathrm{d}v + \int_{\mathbb{R}} |v|^{-r}G(v)\,\mathrm{d}v.\end{aligned}$$

Now using the definition of $\mathcal{M}_{\rho_0}$ and Lemma 3:

$$\int_{\mathbb{T}\times\mathbb{R}} |v|^{-r}\mathcal{M}_{\rho_0}\,\mathrm{d}y\,\mathrm{d}v \le \int_{\mathbb{T}}\left(\int_{\mathbb{R}} |v|^{-r}\frac{\rho_0^{3/2}(y)}{\sqrt{2\pi P_\infty}}e^{-\frac{v^2\rho_0(y)}{2P_\infty}}\,\mathrm{d}v\right)\mathrm{d}y$$

and making the change of variable $w = (\rho_0(y)/P_\infty)^{1/2}v$,

$$\int_{\mathbb{R}} |v|^{-r}\frac{\rho_0^{3/2}(y)}{\sqrt{2\pi P_\infty}}e^{-\frac{v^2\rho_0(y)}{2P_\infty}}\,\mathrm{d}v = \rho_0(y)^{1+r/2}\frac{1}{\sqrt{2\pi P_\infty^r}}\int_{\mathbb{R}} |w|^{-r}e^{-\frac{|w|^2}{2}}\,\mathrm{d}w.$$

This yields (3.5) with

$$A_r := \frac{1}{\sqrt{2\pi P_\infty^r}}\int_{\mathbb{R}} |w|^{-r}e^{-\frac{|w|^2}{2}}\,\mathrm{d}w \quad\text{and}\quad B_r := \int_{\mathbb{R}} |v|^{-r}G(v)\,\mathrm{d}v.$$

Now suppose that $\rho = \rho_0 \in L^{1+r/2}(\mathbb{T})$ is a fixed point of Ψ_α. We have from the inequality and Hölder's inequality that

$$\int_{\mathbb{T}} \rho^{1+r}\,\mathrm{d}x \le \alpha A_r\int_{\mathbb{T}} \rho^{1+r/2}\,\mathrm{d}x + B_r \le \alpha A_r\left(\int_{\mathbb{T}} \rho^{1+r}\,\mathrm{d}x\right)^{\frac{1+r/2}{1+r}} + B_r.$$

Since the exponent $(1+r/2)/(1+r)$ on the right is less than one, this proves that there is a constant $K_{r,\alpha} > 0$ depending only on r and α such that $\int_{\mathbb{T}} \rho^{1+r}\,\mathrm{d}x \le K_{r,\alpha}$. □

Lemma 9 (Pointwise upper bound)**.** *Let $\alpha \in [0,1]$ and let ρ be any probability density on $\mathbb{T}$ that is a fixed point of Ψ_α, and such that $\rho \in L^p$ for some $p > 1$. Then ρ satisfies the pointwise upper bound $\rho(x) \le R_\infty$ where $R_\infty < \infty$ only depends on the total energy, α and on p, and is monotone increasing in α.*

Proof. In the case where $p \in (1, 7/4)$, finitely many applications of the previous lemma will yield, for any $q \in (p, 2)$, a bound

$$\int_{\mathbb{T}} \rho^q(x)\,\mathrm{d}x \le C_{r,\alpha}$$

for some finite constant $C_{r,\alpha}$ depending only on r and α. We deduce that for all $p > 1$, the following control holds

$$\int_{\mathbb{T}} \rho^{7/4}(x)\,\mathrm{d}x \le C_{r,\alpha}.$$

We return to (3.6) and expand F_α to write

$$\rho(x) = \alpha \int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\mathcal{M}_\rho(x,v)\,\mathrm{d}y\,\mathrm{d}v + (1-\alpha)\int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\rho(y)G(v)\,\mathrm{d}y\,\mathrm{d}v. \tag{3.7}$$

Observe that Lemma 7 implies

$$\mathcal{M}_\rho(x,v) = \frac{\rho^{3/2}(x)}{\sqrt{2\pi P_\infty}} e^{-\frac{v^2\rho(x)}{2P_\infty}} \le \frac{\rho_0^{3/2}(x)}{\sqrt{2\pi P_\infty}} e^{-\frac{r_\infty v^2}{2P_\infty}}.$$

which yields

$$\int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\mathcal{M}_f(x,v)\,\mathrm{d}y\,\mathrm{d}v \le \frac{1}{\sqrt{2\pi P_\infty}} \int_{\mathbb{R}} \exp\left[-\frac{r_\infty v^2}{2P_\infty}\right]\left(\int_{\mathbb{T}} \varphi_v(x-y)\rho^{3/2}(y)\,\mathrm{d}y\right)\mathrm{d}v. \tag{3.8}$$

We apply Hölder's inequality with conjugate exponents $p = 7/6$ and $q = 7$ to obtain

$$\begin{aligned}
\left(\int_{\mathbb{T}} \varphi_v(x-y)\rho^{3/2}(y)\,\mathrm{d}y\right) &\le \|\varphi_v\|_{L^7(\mathbb{T})}\|\rho^{3/2}\|_{L^{7/6}(\mathbb{T})} \\
&\le \|\varphi_v\|_{L^7(\mathbb{T})}\|\rho\|^{3/2}_{L^{7/4}(\mathbb{T})} \\
&\le |v|^{-6/7}\|\rho\|^{3/2}_{L^{7/4}(\mathbb{T})} \\
&\le C_{r,\alpha}|v|^{-6/7}
\end{aligned}$$

where in the final step we have the used estimate (3.1) with $p = 7$. Using this in (3.8) and noting that

$$\int_{\mathbb{R}} \exp\left[-\frac{r_\infty v^2}{2P_\infty}\right]|v|^{-6/7}\,\mathrm{d}v < \infty,$$

we deduce a universal upper bound on $\int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\mathcal{M}_f(x,v)\,\mathrm{d}y\,\mathrm{d}v$.

The term $\int_{\mathbb{T}\times\mathbb{R}} \varphi_v(x-y)\rho(y)G(v)\,\mathrm{d}y\,\mathrm{d}v$ is bounded using Hölder inequality and (3.1):

$$\int_{\mathbb{T}} \varphi_v(x-y)\rho(y)\,\mathrm{d}y \le \|\varphi_v\|_{L^{7/3}(\mathbb{T})}\|\rho\|_{L^{7/4}(\mathbb{T})} \le C_{r,\alpha}|v|^{-4/7}.$$

The two last inequalities combined with (3.7) imply the pointwise bound on ρ. □

4. The contraction mapping argument

4.1. **Setting of the argument.** Recall the relation $\rho_f T_f = P_\infty$ and

$$\begin{cases} F_\alpha[\rho] = \alpha\rho M_{\frac{T_\infty}{\rho}} + (1-\alpha)\rho G, \\ G := \left(\dfrac{M_{T_1}+M_{T_2}}{2}\right) \\ \Psi_\alpha[\rho] := \displaystyle\int_{\mathbb{R}} (1-v^2\partial_x^2)^{-1} F_\alpha[\rho]\,\mathrm{d}v. \end{cases}$$

The local density of any steady state must be a fixed point of Ψ_α. When $\alpha = 0$, the map Ψ_0 is linear, and a consequence of the spectral analysis of the next Section 5 is that it is strictly contractive in H^1 or L^2 norms. To extend it to small positive α, we make use of the *a-priori* bounds proved in the previous Section 3.

4.2. **The contraction estimate.**

Lemma 10. *Given any $\varepsilon \in (0,1)$, define*

$$\mathcal{C}_\varepsilon := \left\{ \rho \in L^2(\mathbb{T}) \ : \ \int_{\mathbb{T}} \rho(x)\,\mathrm{d}x = 1 \quad \text{and} \quad 0 < \varepsilon < \rho < 1/\varepsilon \right\}.$$

Then $\Psi_\alpha(\mathcal{C}_\varepsilon) \subset L^2(\mathbb{T})$ for all $\alpha \in [0,1]$ and there are $\alpha_\varepsilon, \delta_\varepsilon \in (0,1)$ depending on ε such that for all $\alpha \in [0,\alpha_\varepsilon)$:

$$\forall\, \rho_1,\rho_2 \in \mathcal{C}_\varepsilon, \quad \|\Psi_\alpha(\rho_1) - \Psi_\alpha(\rho_2)\|_{L^2(\mathbb{T})} \le (1-\delta_\varepsilon)\|\rho_1-\rho_2\|_{L^2(\mathbb{T})}.$$

Proof. Recall that in our normalization the global conserved quantities satisfy

$$\rho_\infty := \int_{\mathbb{T}\times\mathbb{R}} f(x,v)\,\mathrm{d}x\,\mathrm{d}v = 1, \quad T_\infty := \frac{1}{\rho_\infty}\int_{\mathbb{T}\times\mathbb{R}} v^2 f(x,v)\,\mathrm{d}x\,\mathrm{d}v = \frac{T_1+T_2}{2}.$$

The fact that $\Psi_\alpha(\mathcal{C}_\varepsilon) \subset L^2(\mathbb{T})$ is straightforward and we only prove the contraction property. We linearize the map Ψ_α around a profile $\bar\rho \in \mathcal{C}_\varepsilon$ with global mass 1 and global temperature T_∞. The local temperature is $\bar T(x) = T_\infty/\bar\rho(x)$. We write the fluctuation

$$\rho = \bar\rho + \sigma \quad \text{with} \quad \sigma \in L^2(\mathbb{T}) \text{ and } \int_{\mathbb{T}} \sigma(x)\,\mathrm{d}x = 0.$$

The functional derivative of Ψ_α is:

$$D\Psi_\alpha[\bar\rho]\cdot\sigma = \alpha\int_{\mathbb{R}} \left[1-v^2\partial_x^2\right]^{-1}\left(\frac{3\sigma}{2}M_{\frac{T_\infty}{\bar\rho}}\sigma - \frac{v^2\bar\rho\sigma}{2T_\infty}M_{\frac{T_\infty}{\bar\rho}}\right)\mathrm{d}v$$
$$+ (1-\alpha)\int_{\mathbb{R}} \left[1-v^2\partial_x^2\right]^{-1}\sigma\, G(v)\,\mathrm{d}v.$$

We estimate by duality for $\sigma_1, \sigma_2 \in L^2(\mathbb{T})$:

$$\begin{aligned}\langle \sigma_2, D\Psi_\alpha[\bar\rho]\cdot\sigma_1\rangle &= \alpha \int_{\mathbb{T}\times\mathbb{R}} \left(\left[1 - v^2\partial_x^2\right]^{-1}\sigma_2\right)\left(\frac{3\sigma_1}{2} M_{\frac{T_\infty}{\bar\rho}} - \frac{v^2\bar\rho\sigma_1}{2T_\infty} M_{\frac{T_\infty}{\bar\rho}}\right) \mathrm{d}x\,\mathrm{d}v \\ &\quad + (1-\alpha)\int_{\mathbb{T}\times\mathbb{R}} \left(\left[1 - v^2\partial_x^2\right]^{-1}\sigma_2\right) G(v)\sigma_1(x)\,\mathrm{d}x\,\mathrm{d}v \\ &=: D_1 + D_2.\end{aligned}$$

Let us study the first term D_1. Using the controls $\bar\rho \in \mathcal{C}_\varepsilon$ we deduce

$$\left[\int_{\mathbb{T}} \left(\frac{3}{2} M_{\frac{T_\infty}{\bar\rho}} + \frac{v^2\bar\rho}{2T_\infty} M_{\frac{T_\infty}{\bar\rho}}\right)^2 \sigma(x)^2\,\mathrm{d}x\right]^{1/2} \le C_\varepsilon e^{-\nu_\varepsilon v^2}\|\sigma\|_{L^2(\mathbb{T})}$$

for some constant $C_\varepsilon, \nu_\varepsilon > 0$ depending on ε, and therefore

$$\begin{aligned}&\int_{\mathbb{T}\times\mathbb{R}} \left(\left[1 - v^2\partial_x^2\right]^{-1}\sigma_2\right)\left(\frac{3}{2} M_{\frac{T_\infty}{\bar\rho}} - \frac{v^2\bar\rho}{2T_\infty} M_{\frac{T_\infty}{\bar\rho}}\right)\sigma_1(x)\,\mathrm{d}x\,\mathrm{d}v \\ &\quad\le C_\varepsilon \|\sigma_1\|_{L^2(\mathbb{T})} \int_{\mathbb{R}} \left(\int_{\mathbb{T}} \left(\left[1 - v^2\partial_x^2\right]^{-1}\sigma_2\right)^2 \mathrm{d}x\right)^{1/2} e^{-\nu_\varepsilon v^2}\,\mathrm{d}v \\ &\quad\le C_\varepsilon \|\sigma_1\|_{L^2(\mathbb{T})} \int_{\mathbb{R}} \left(\int_{\mathbb{T}} \sigma_2^2\,\mathrm{d}x\right)^{1/2} e^{-\nu_\varepsilon v^2}\,\mathrm{d}v \\ &\quad\le C_\varepsilon \|\sigma_1\|_{L^2(\mathbb{T})}\|\sigma_2\|_{L^2(\mathbb{T})}\end{aligned}$$

where we have used that $\|\varphi_v\|_{L^1(\mathbb{T})} = 1$. We conclude that

$$D_1 \le \alpha C_\varepsilon \|\sigma_1\|_{L^2(\mathbb{T})}\|\sigma_2\|_{L^2(\mathbb{T})}.$$

Let us now study the second term D_2. We Fourier transform it in x:

$$\int_{\mathbb{T}\times\mathbb{R}} \left(\left[1 - v^2\partial_x^2\right]^{-1}\sigma_2\right) G(v)\sigma_1(x)\,\mathrm{d}x\,\mathrm{d}v = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\hat\sigma_2(k)}{[1+(2\pi)^2v^2k^2]} G(v)\hat\sigma_1(k)\,\mathrm{d}v.$$

The perturbation σ has zero mass, hence $\hat\sigma(0) = 0$ and

$$\begin{aligned}\left|\int_{\mathbb{T}\times\mathbb{R}} \left(\left[1 - v^2\partial_x^2\right]^{-1}\sigma_2\right) G(v)\sigma_1(x)\,\mathrm{d}x\,\mathrm{d}v\right| &\le \sum_{k\in\mathbb{Z},\ k\neq 0} \int_{\mathbb{R}} \frac{\hat\sigma_2(k)}{[1+(2\pi)^2v^2k^2]} G(v)\hat\sigma_1(k)\,\mathrm{d}v \\ &\le \left(\int_{\mathbb{R}} \frac{G(v)}{1+(2\pi)^2v^2}\,\mathrm{d}v\right) \sum_{k\in\mathbb{Z},\ k\neq 0} \hat\sigma_1(k)\hat\sigma_2(k) \\ &\le (1-\delta_G)\|\sigma_1\|_{L^2(\mathbb{T})}\|\sigma_2\|_{L^2(\mathbb{T})}\end{aligned}$$

with $\delta_G \in (0,1)$, where we have used that

$$\left(\int_{\mathbb{R}} \frac{G(v)}{1+(2\pi)^2v^2}\,\mathrm{d}v\right) < \left(\int_{\mathbb{R}} G(v)\,\mathrm{d}v\right) = 1.$$

Therefore the operator $D\Psi_\alpha[\bar\rho]$ is bounded from $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ with the operator norm bounded by

$$|||D\Psi_\alpha[\bar\rho]||| = \sup_{\|\sigma_1\|_{L^2(\mathbb{T})}\le 1,\ \|\sigma_2\|_{L^2(\mathbb{T})}\le 1} \langle \sigma_2, D\Psi_\alpha[\bar\rho]\cdot\sigma_1\rangle \le \alpha C_\varepsilon + (1-\alpha)(1-\delta_G).$$

For α small enough we deduce $|||D\Psi_\alpha[\rho]||| < (1-\delta_\varepsilon)$ with $\delta_\varepsilon \in (0,1)$. Finally, since the set $\mathcal{C}_\varepsilon$ is convex, the mean value theorem gives, for $\rho_1, \rho_2 \in \mathcal{C}_\varepsilon$:

$$\begin{aligned}\|\Psi_\alpha(\rho_1) - \Psi_\alpha(\rho_2)\|_{L^2(\mathbb{T})} &= \left\| \int_0^1 D\Psi_\alpha[(1-t)\rho_1 + t\rho_2]\cdot(\rho_2-\rho_1)\,\mathrm{d}t \right\|_{L^2(\mathbb{T})} \\ &\le (1-\delta_\varepsilon)\|\rho_1-\rho_2\|_{L^2(\mathbb{T})}\end{aligned}$$

which shows the contraction property for the nonlinear map. □

4.3. **Proof of the main Theorem 1.** We now prove Theorem 1. By Lemma 7 and Lemma 9, there is a $\varepsilon > 0$, depending only on T_1 and T_2 such that every steady state of (1.1) with $\alpha < 1/2$ belongs to the set $\mathcal{C}_\varepsilon$, defined in Lemma 10. Then by Lemma 10, there is an α_0 with $0 < \alpha_0 \le 1/2$ such that Ψ_α is a strictly contractive mapping on this convex set $\mathcal{C}_\varepsilon$, and hence there is a unique fixed point in $\mathcal{C}_\varepsilon$. Since there is always one spatially uniform steady state, it is the unique steady state.

5. Perturbative stability of the spatially uniform NESS

5.1. **Strategy.** In this section we investigate the perturbative stability of the spatially homogeneous NESS

$$f_{\infty,\alpha} := \alpha M_{T_\infty} + (1-\alpha)G(v), \quad G(v) := \frac{M_{T_1}+M_{T_2}}{2}, \quad T_\infty := \frac{T_1+T_2}{2}. \tag{5.1}$$

When the reservoirs have different temperatures and are coupled to the system, that is for $\alpha \in (0,1)$, there is transfer of heat through collisions from the hot reservoir to the cold reservoir, and there is no detailed balance; i.e., time reversal invariance is broken in the steady state. This is reflected in the fact that the linearized operator is a non-self-adjoint operator on $\mathcal{H}_\alpha = L^2(f_{\infty,\alpha}^{-1})$ for $\alpha \in (0,1)$, as we shall see.

Nonetheless, we shall prove that the linearized collision operator still satisfies a *microscopic coercivity* inequality (see (5.9)), expressing the dissipative nature of the linearized evolution on the orthogonal complement of the null space of the generator. This fact is striking since we do not derive it, through linearization, from a nonlinear entropy principle, which is the usual source of such inequalities. In our non-equilibrium setting, there is no analog of the H-Theorem, and therefore we must prove it by direct analysis of linearized collision operator.

Once the microscopic coercivity is proven, we can prove that our system is *hypocoercive* by a variety of methods, and we briefly describe two of these.

5.2. **Linearization around a spatially homogeneous NESS.** Consider densities f that are close to $f_{\infty,\alpha}$ with fluctuations denoted

$$\begin{cases} h(x,v) & := f(x,v) - f_{\infty,\alpha}(v) \\ \rho(x) & = 1 + \sigma(x) \qquad \text{with} \qquad \sigma(x) := \int_{\mathbb{R}} h(x,v)\,\mathrm{d}v, \\ P(x) & = T_\infty + \tau(x) \qquad \text{with} \qquad \tau(x) := \int_{\mathbb{R}} v^2 h(x,v)\,\mathrm{d}v, \end{cases} \tag{5.2}$$

The fluctuations of the local density and local pressure have zero mean:

$$\int_{\mathbb{T}} \sigma(x)\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_{\mathbb{T}} \tau(x)\,\mathrm{d}x = 0. \tag{5.3}$$

Consider the weighted L^2 Hilbert space of *real valued functions* defined by the norm

$$\|h\|^2_{\mathcal{H}_\alpha} = \int_{\mathbb{T}\times\mathbb{R}} |h(x,v)|^2 \frac{1}{f_{\infty,\alpha}(v)}\,\mathrm{d}x\,\mathrm{d}v. \tag{5.4}$$

We expand $\mathcal{M}_f - \mathcal{M}_{f_{\infty,\alpha}}$ to first order in terms of h, σ, τ:

$$\begin{aligned} \mathcal{M}_f(x,v) - \mathcal{M}_{f_{\infty,\alpha}}(v) &= \frac{(1+\sigma)^{3/2}(x)}{\sqrt{2\pi(T_\infty + \tau(x))}} e^{-\frac{v^2(1+\sigma(x))}{2(T_\infty+\tau(x))}} - \frac{1}{\sqrt{2\pi T_\infty}} e^{-\frac{v^2}{2T_\infty}} \\ &\approx \left(\frac{3}{2} - \frac{v^2}{2T_\infty}\right) M_{T_\infty}(v)\sigma(x) + \left(-\frac{1}{2T_\infty} + \frac{v^2}{2T_\infty^2}\right) M_{T_\infty}(v)\tau(x) \\ &= M_{T_\infty}(v)\sigma(x) + \frac{1}{2}\left(\frac{v^2}{T_\infty} - 1\right) M_{T_\infty}(v)\left(\frac{1}{T_\infty}\tau(x) - \sigma(x)\right). \end{aligned} \tag{5.5}$$

The fluctuation $h = f - f_{\infty,\alpha}$ satisfies the equation

$$\partial_t h + v\partial_x h = \alpha(\mathcal{M}_f - \mathcal{M}_{f_{\infty,\alpha}}) + (1-\alpha)\sigma G - h. \tag{5.6}$$

To first order we obtain the linearized equation

$$\partial_t h + \mathcal{S}h = \mathcal{L}_\alpha h \tag{5.7}$$

with the free streaming operator $\mathcal{S} := v\partial_x$ and the linearized collision operator

$$\mathcal{L}_\alpha h(x,v) := \sigma(x) f_{\infty,\alpha}(x,v) + \frac{\alpha}{2}\left(\frac{\tau(x)}{T_\infty} - \sigma(x)\right)\left(\frac{v^2}{T_\infty} - 1\right) M_{T_\infty}(v) - h(x,v). \tag{5.8}$$

Note that both $\mathcal{L}_\alpha$ and $\mathcal{H}_\alpha$ depend on α and that $\mathcal{L}_\alpha$ is bounded on $\mathcal{H}_\alpha$ for all $\alpha \in [0,1]$: observe that $f_{\infty,\alpha} \geq \alpha M_{T_\infty}$ and therefore

$$\begin{aligned} \|\mathcal{L}_\alpha h\|^2_{\mathcal{H}_\alpha} &\lesssim \|\sigma\|^2_{L^2(\mathbb{T})} + \alpha \left\|\frac{\tau}{T_\infty} - \sigma\right\|^2_{L^2(\mathbb{T})} \left(\int_{\mathbb{R}} \left(\frac{v^2}{T_\infty} - 1\right)^2 \frac{M_{T_\infty}(v)^2}{f_{\infty,\alpha}}\,\mathrm{d}v\right) \\ &\lesssim \|\sigma\|^2_{L^2(\mathbb{T})} + \left\|\frac{\tau}{T_\infty} - \sigma\right\|^2_{L^2(\mathbb{T})} \left(\int_{\mathbb{R}} \left(\frac{v^2}{T_\infty} - 1\right)^2 M_{T_\infty}(v)\,\mathrm{d}v\right) \\ &\lesssim \|\sigma\|^2_{L^2(\mathbb{T})} + \frac{1}{T_\infty}\|\tau\|^2_{L^2(\mathbb{T})} \lesssim \max\left\{1\,;\,\frac{1}{T_\infty}\right\} \|h\|^2_{\mathcal{H}_\alpha}. \end{aligned}$$

5.3. **Microscopic coercivity.** We shall now prove that the null space of $\mathcal{L}$ is the space of functions $\sigma(x)f_{\infty,\alpha}(v)$ for $\sigma \in L^2(\mathbb{T})$ and prove a spectral gap on the orthogonal of this null space.

Lemma 11. *Let $\alpha \in [0,1]$ and $\mathcal{L}_\alpha$ defined as in* (5.8)*. Then for all $h \in \mathcal{H}_\alpha$,*

$$\langle h, \mathcal{L}_\alpha h\rangle_{\mathcal{H}} \leq -\frac{1-\alpha}{2}\int_{\mathbb{T}\times\mathbb{R}} \left| h(x,v) - f_{\infty,\alpha}(v)\int_{\mathbb{R}} h(x,w)\,\mathrm{d}w\right|^2 \frac{1}{f_{\infty,\alpha}(v)}\,\mathrm{d}x\,\mathrm{d}v\ . \tag{5.9}$$

Remark 5.1. If we had taken $\mathcal{H}_\alpha$ to consist if complex valued functions, we would need a real part on the left side of the inequality since $\mathcal{L}_\alpha$ is not self-adjoint. Note that for $\alpha \in [0,1)$, the constant λ_α is strictly positive but $\lambda_\alpha \to 0$ as $\alpha \to 1$. This reflects the fact, see the proof below, that the dissipativity in the energy mode is lost because there is energy conservation in this limit. In fact in this limit case the microscopic coercivity nevertheless holds once accounting for the larger null space of $\mathcal{L}_1$. We are not concerned in this case for which the NESS is already known, and we refer to [1] for a study of the microscopic coercivity and hypocoercivity for the equation of this limit case; This could also be deduced from the abstract results in [12].

Proof. Let us define the following orthonormal family in $L^2(f_{\infty,\alpha}^{-1}\mathrm{d}v)$:

$$H_0(v) := f_{\infty,\alpha}, \quad H_1(v) := \frac{1}{\sqrt{T_\infty}} v f_{\infty,\alpha}, \quad H_2(v) := c_\alpha\left(\frac{v^2}{T_\infty}-1\right)f_{\infty,\alpha}, \tag{5.10}$$

where $c_\alpha > 0$ is the normalizing constant so that $\|H_2\|_{\mathcal{H}_\alpha} = 1$ (one can check that $c_\alpha^{-2} = 3\left(\alpha + (1-\alpha)\left(2 - \frac{T_1T_2}{T_\infty^2}\right)\right) - 1$). We also define the corresponding orthogonal projections Π_0, Π_1, Π_2 in $L^2(f_{\infty,\alpha}^{-1}\mathrm{d}v)$ (note that they all depend on α):

$$\begin{cases} \Pi_0(h)(v) := \left(\int_{\mathbb{R}} h(w)\,\mathrm{d}w\right) H_0(v), \\ \Pi_1(h)(v) := \left(\int_{\mathbb{R}} h(w)\frac{w}{\sqrt{T_\infty}}\,\mathrm{d}w\right) H_1(v), \\ \Pi_2(h)(v) := \left(\int_{\mathbb{R}} h(w)c_\alpha\left(\frac{w^2}{T_\infty}-1\right)\,\mathrm{d}w\right) H_2(v) = \left(\frac{\tau}{T_\infty}-\sigma\right)c_\alpha H_2(v). \end{cases} \tag{5.11}$$

Finally we denote $\Pi^\perp$ the orthogonal projection on $\{H_0, H_2\}^\perp$ (note that this projection includes Π_1 in its range).

The linearized collision operator $\mathcal{L}_\alpha$ can be written using this notation a

$$\mathcal{L}_\alpha h = \Pi_0(h) + \frac{\alpha}{2c_\alpha^2}\Pi_2(h)\frac{M_{T_\infty}}{f_{\infty,\alpha}} - h. \tag{5.12}$$

We then compute the Dirichlet form

$$\begin{aligned} \langle \mathcal{L}_\alpha h, h\rangle_{\mathcal{H}_\alpha} =& \|\Pi_0(h)\|_{\mathcal{H}_\alpha}^2 + \frac{\alpha}{2c_\alpha^2}\left\langle \Pi_2(h)\frac{M_{T_\infty}}{f_{\infty,\alpha}}, \Pi_2(h)\right\rangle_{\mathcal{H}_\alpha} \\ &+ \frac{\alpha}{2c_\alpha^2}\left\langle \Pi_2(h)\frac{M_{T_\infty}}{f_{\infty,\alpha}}, \Pi^\perp(h)\right\rangle_{\mathcal{H}_\alpha} - \|h\|_{\mathcal{H}_\alpha}^2 \end{aligned}$$

where we have used that $H_2 M_{T_\infty} f_{\infty,\alpha}^{-1}$ is orthogonal to H_0 and H_1 in $L^2(f_{\infty,\alpha}^{-1}\mathrm{d}v)$.

Let us define

$$\mathcal{U}_\alpha(v) := H_2(v)\frac{M_{T_\infty}(v)}{f_{\infty,\alpha}(v)}. \tag{5.13}$$

The projection of this function on H_0 and H_1 in $L^2(f_{\infty,\alpha}^{-1}\mathrm{d}v)$ is zero and its projection on H_2 has coefficient

$$\int_{\mathbb{R}} \mathcal{U}_\alpha(v) H_2(v) \frac{1}{f_{\infty,\alpha}(v)}\,\mathrm{d}v = c_\alpha^2 \int_{\mathbb{R}} \left(\frac{v^2}{T_\infty} - 1\right)^2 M_{T_\infty}(v)\,\mathrm{d}v = 2c_\alpha^2.$$

Its norm satisfies

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{U}_\alpha(v)^2 \frac{1}{f_{\infty,\alpha}(v)}\,\mathrm{d}v &= c_\alpha^2 \int_{\mathbb{R}} \left(\frac{v^2}{T_\infty} - 1\right)^2 \frac{M_{T_\infty}(v)^2}{f_{\infty,\alpha}(v)}\,\mathrm{d}v \\ &\le \frac{c_\alpha^2}{\alpha} \int_{\mathbb{R}} \left(\frac{v^2}{T_\infty} - 1\right)^2 M_{T_\infty}(v)\,\mathrm{d}v \le \frac{2c_\alpha^2}{\alpha}\end{aligned}$$

where we have used in the last line $f_{\infty,\alpha} \ge \alpha M_{T_\infty}$. We then decompose orthogonally $\mathcal{U}_\alpha = \Pi_2(\mathcal{U}_\alpha) + \Pi^\perp(\mathcal{U}_\alpha)$ and deduce by Pythagoras' theorem that

$$\left\|\Pi^\perp(\mathcal{U}_\alpha)\right\|^2_{L^2(f_{\infty,\alpha}^{-1})} \le 4c_\alpha^4 \frac{1-\alpha^2}{\alpha^2}. \tag{5.14}$$

We deduce on the one hand that

$$\begin{aligned}\frac{\alpha}{2c_\alpha^2}\left\langle \Pi_2(h)\frac{M_{T_\infty}}{f_{\infty,\alpha}}, \Pi_2(h)\right\rangle_{\mathcal{H}_\alpha} &= \frac{\alpha}{2}\int_{\mathbb{T}\times\mathbb{R}} \left(\frac{\tau}{T_\infty} - \sigma\right)^2 \mathcal{U}_\alpha(v) H_2(v)\,\mathrm{d}x\,\mathrm{d}v \\ &= \alpha c_\alpha^2 \int_{\mathbb{T}} \left(\frac{\tau}{T_\infty} - \sigma\right)^2 \mathrm{d}x = \alpha \|\Pi_2(h)\|^2_{\mathcal{H}_\alpha}.\end{aligned}$$

We deduce on the other hand

$$\begin{aligned}&\frac{\alpha}{2c_\alpha^2}\left\langle \Pi_2(h)\frac{M_{T_\infty}}{f_{\infty,\alpha}}, \Pi^\perp(h)\right\rangle_{\mathcal{H}_\alpha} \\ &= \frac{\alpha}{2c_\alpha}\int_{\mathbb{T}\times\mathbb{R}} \left(\frac{\tau}{T_\infty} - \sigma\right) \Pi^\perp(\mathcal{U}_\alpha)\Pi^\perp(h)\,\mathrm{d}x\,\mathrm{d}v \\ &\le \frac{\alpha}{2c_\alpha}\left\|\Pi^\perp(\mathcal{U}_\alpha)\right\|_{L^2(f_{\infty,\alpha}^{-1})} \left(\int_{\mathbb{T}} \left(\frac{\tau}{T_\infty} - \sigma\right)^2 \mathrm{d}x\right)^{1/2} \|\Pi^\perp(h)\|_{\mathcal{H}_\alpha} \\ &\le \frac{\alpha}{2c_\alpha^2}\left\|\Pi^\perp(\mathcal{U}_\alpha)\right\|_{L^2(f_{\infty,\alpha}^{-1})} \|\Pi_2(h)\|_{\mathcal{H}_\alpha}\|\Pi^\perp(h)\|_{\mathcal{H}_\alpha} \\ &\le (1-\alpha^2)^{1/2}\|\Pi_2(h)\|_{\mathcal{H}_\alpha}\|\Pi^\perp(h)\|_{\mathcal{H}_\alpha}\end{aligned}$$

where we have used (5.14) in the last line.

We therefore obtain

$$\langle \mathcal{L}_\alpha h, h\rangle_{\mathcal{H}_\alpha} \le (\alpha - 1)\|\Pi_2(h)\|^2_{\mathcal{H}_\alpha} + (1-\alpha^2)^{1/2}\|\Pi_2(h)\|_{\mathcal{H}_\alpha}\|\Pi^\perp(h)\|_{\mathcal{H}_\alpha} - \|\Pi^\perp h\|^2_{\mathcal{H}_\alpha}.$$

The quadratic form on the right hand side is negative for $\alpha \in [0,1)$ since then $(1-\alpha^2) < 4(1-\alpha)^2$. It degenerates at $\alpha = 1$. The matrix of the quadratic form

$$(V_1, V_2) \in \mathbb{R}^2 \mapsto (\alpha-1)V_1 - V_2 + (1-\alpha^2)^{1/2} V_1 V_2$$

has characteristic polynomial

$$P(X) = X^2 + (2-\alpha)X + \frac{1}{4}(1-\alpha)(3-\alpha)$$

whose roots are $\alpha/2 + 1 \pm 1/2$. The greatest eigenvalue is therefore $(\alpha-1)/2$ which concludes the proof. □

5.4. **Hypocoercivity.** With the microcoercivity at hand we can now readily prove prove hypocoercivity: That is, we shall prove that for some constant $C < \infty$ and some $\lambda > 0$, and solution h_t in $\mathcal{H}_\alpha$ of our linearized evolution equation satisfies

$$\|h_t\|_{\mathcal{H}_\alpha} \leq Ce^{-\lambda t} \|h_0\|_{\mathcal{H}_\alpha} \ . \tag{5.15}$$

With this in hand, it is a simple matter to prove the nonlinear stability. We discuss two approaches to proving (5.15) for our model. One approach applies when the steady state is symmetric in v, as in our case. As noted in [1], whenever this is the case, there is a natural orthonormal basis such that in this basis the streaming operator is represented by uncoupled tridiagonal blocks, while the gain term in $\mathcal{L}_\alpha$ is represented by uncoupled lower triangular blocks. This structure permits the extraction of simple, explicit bounds on μ. Another approach, developed in [9] is more abstract and not requiring symmetry of the steady state, provides an efficient route to (5.15). In this section we prove:

Theorem 12. *The decay estimate* (5.15) *is valid with the following explicit values of λ and C:*

(1) If $c_\alpha^2 \frac{1-\alpha}{2\sqrt{T_\infty}} < \frac{1}{2}$, we may take $C = 4$ and $\lambda = \frac{1-\alpha}{8}$.

(2) $c_\alpha^2 \frac{1-\alpha}{2\sqrt{T_\infty}} \geq \frac{1}{2}$, we may take $C = 4$ and $\lambda = \frac{\sqrt{T_\infty}}{8}$.

This is also true, with the same constants, if we replace $\mathcal{H}_\alpha$ by $\mathcal{H}^1_\alpha$, the latter Hilbert space being defined in (1.5).

Remark 5.2. This result is stronger than Theorem 2 in that it provides explicit bounds on the exponential rate of convergence. The reason for the $\sqrt{T_\infty}$ dependence of λ for small T_∞ is that hypocoercivity depends on the effects of the streaming operator $v\partial_x$ to "mix" the conserved mass mode into the dissipated modes, and the tridiagonal representation of the streaming operator given in (5.17) shows that its mixing effects are proportional to $\sqrt{T_\infty}$. When T_∞ is large, there is rapid mixing, but this can do only so much good: The dissipativity of the non-conserved modes as estimated in Lemma (11) is bounded independent of T_∞, but the mixing only shares this dissipativity around, it cannot improve the dissipativity no matter how fast it runs.

It is also worth noting that we have simple bounds on c_α: By the arithmetic-geometric mean inequality, $0 \leq \frac{T_1 T_2}{T^2} \leq 1$, and hence $2 \leq c_{\alpha,T}^{-2} \leq 3(2-\alpha) - 1$.

The proof of Theorem 12 is quite short once one has computed matrix representations of $v\partial_x$ and $\mathcal{L}_\alpha$ with respect to a basis that we now introduce: The basis is $\{e^{ikx}g_m(v)\}_{m\geq 0,k\in\mathbb{Z}}$ where $\{g_m(v)\}_{m\geq 0}$ is the sequence one gets by applying this Gram-Schmidt orthonormalization procedure to the sequence of functions $v^m f_{\infty,\alpha}(v)$ for $m\geq 0$. We write these in the form $g_m(v) = H_m(v)f_{\infty,\alpha}(v)$ where H_m is a polynomial of degree m. For $h\in\mathcal{H}_\alpha$, we write

$$h = \sum_{m\geq 0,k\in\mathbb{Z}} ik\widehat{h}_m(k)e^{ikx}g_m(v)\ . \tag{5.16}$$

The action of the free streaming operator $\mathcal{S} := v\partial_x$ then is

$$\mathcal{S}h(x,v) = \sum_{m\geq 0,k\in\mathbb{Z}} ik\widehat{h}_m(k)e^{ikx}vg_m(v)\ .$$

It is a simple consequence of the fact that $f_{\infty,\alpha}$ is even in v that for each $m\geq 1$, vg_m is a linear combination of g_{m-1} and g_{m+1}; see [1]. Since the operation of multiplication by v is self-adjoint, it follows that that there exist numbers $\{a_n\}_{m\geq 1}$ such that

$$vg_m(v) = \sqrt{T_\infty}a_{m+1}g_{m+1}(v) + \sqrt{T_\infty}a_m g_{m-1}(v) \ \ \text{with the convention } g_{-1} := 0.$$

For $m = 0$, by (5.10), $vg_0(v) = vf_{\infty,\alpha} = \sqrt{T_\infty}g_1$, and hence $a_1 = 1$. Likewise,

$$vg_1(v) = \frac{1}{\sqrt{T_\infty}}v^2 f_{\infty,\alpha} = \sqrt{T_\infty}\left(\frac{v^2}{T_\infty} - 1\right) f_{\infty,\alpha} + \sqrt{T_\infty}g_0(v)$$

and hence $a_2 = c_\alpha^{-1}$. One can work out a_m for or higher values of m, but these are not needed here.

Let $\widehat{\mathbf{h}}(k)$ denote the element of ℓ_2 whose mth component is $\widehat{h}_m(k)$. Then the corresponding vector of coefficients for $\mathcal{S}h$ is given by $ik\mathbf{S}\widehat{\mathbf{h}}(k)$ where $\mathbf{S}$ is the tri-diagonal matrix

$$\mathbf{S} = \sqrt{T_\infty}\begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots \\ a_1 & 0 & a_2 & 0 & \cdots \\ 0 & a_2 & 0 & a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \sqrt{T_\infty}\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & c_\alpha^{-1} & 0 & \cdots \\ 0 & c_\alpha^{-1} & 0 & a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.17}$$

We now turn to the gain term in the linearized collision operator. The linearized collision operator does not act on the spatial variable. By projecting (5.12) in our basis we get

$$\widehat{\mathcal{L}_\alpha h}(k,v) = \widehat{h}_0(k)g_0(v) + \frac{\alpha}{2c_\alpha^2}\widehat{h}_2(k)H_2(v)M_{T_\infty}(v) - \widehat{h}(k,v). \tag{5.18}$$

For each k, the action of $\mathcal{L}_\alpha$ in the $\{g_n\}_{m\geq 0}$ basis is given by

$$\widehat{\mathcal{L}_\alpha h}_m(k) = (\mathbf{L}_\alpha\widehat{\mathbf{h}}(k))_m$$

where $\mathbf{L}_\alpha + \mathbf{Id}$ is the matrix whose first column is the unit vector $(1, 0, 0, \dots)$, whose third column is the vector $(0, 0, \alpha, b_3, b_4, \dots)$, with all other columns being zero, and with

$$b_m := \frac{\alpha}{2c_\alpha^2}\widehat{h}_2(k)\int H_2(v)H_m(v)M_{T_\infty}(v)\mathrm{d}v\ ,$$

so that, in particular, $b_0 = b_1 = 0$ and $b_2 = \alpha$.

Therefore, we may rewrite our linearized equation as the decoupled system of equations

$$\partial_t \widehat{\mathbf{h}}(k) = (\mathbf{L}_\alpha - ik\mathbf{S})\widehat{\mathbf{h}}(k)\ , \tag{5.19}$$

for each $k \in \mathbb{Z}$. For $k = 0$ we simply have from (5.18) $\partial_t \widehat{\mathbf{h}}(0) = -\widehat{\mathbf{h}}(0)$ since $\widehat{h}(0)_0 = \widehat{h}(0)_2 = 0$.

For each $k \neq 0$, define $\mathbf{C}_k = -(\mathbf{L}_\alpha - ik\mathbf{S})$. We seek a positive definite matrix $\mathbf{P}_k$ such that for some fixed $\lambda > 0$,

$$\forall\, k \in \mathbb{Z}, \quad \mathbf{C}_k^*\mathbf{P}_k + \mathbf{P}_k\mathbf{C}_k \geq 2\lambda\mathbf{P}_k. \tag{5.20}$$

Then if we define the Lyapunov function $e(h)$ by $e(h) = \sum_{k\in\mathbb{Z}} \omega_k \left\langle \widehat{\mathbf{h}}(k), \mathbf{P}_k\widehat{\mathbf{h}}(k)\right\rangle_{\ell_2}$, for *any* sequence $\{\omega_k\}_{k\in\mathbb{Z}}$ of positive numbers we have that for any solution of our linearized equation with initial data h_0 with $e(h) < \infty$,

$$\frac{\mathrm{d}}{\mathrm{d}t}e(h(t)) \leq -2\lambda e(h(t)) \quad \Longrightarrow \quad e(h(t)) \leq e^{-2t\lambda}e(h_0). \tag{5.21}$$

We will construct the matrices $\mathbf{P}_k$ so that for some $C > 0$

$$\forall\, k \in \mathbb{Z}\backslash\{0\}, \quad K\mathbf{I} \leq \mathbf{P}_k < \frac{1}{K}\mathbf{I}. \tag{5.22}$$

This implies that the function $e(h)$ is equivalent to the norm on $\mathcal{H}_\alpha$ if we take each $\omega_k = 1$. We then conclude that (5.15) is valid with $C = \frac{1}{K}$ and the value of λ appearing in (5.20). By making other choices for ω_k, we obtain decay in various Sobolev type norms. For instance, taking $\omega_k = 1 + k^2$, we would obtain decay in the norm defined in (1.5). With the matrix representation computed, and Lemma 11 at our disposal, we are ready to prove Theorem 12

Proof of Theorem 12. Because $\mathcal{L}_\alpha$ is lower triangular with positive diagonal entries that are uniformly bounded away from zero except of course for the zero in the upper left, and because $\mathbf{S}$ is tridiagonal, a simple prescription from [1] provides $\mathbf{P}_k$. For a parameter $c \in (0, 1)$ to be chosen later, define $\mathbf{P}_k(a)$ by entering

$$\begin{pmatrix} 1 & -ic/k \\ ic/k & 1 \end{pmatrix} \tag{5.23}$$

as its upper-left 2×2 block, with all other entries being those of the identity. The eigenvalues of $\mathbf{P}_k(a)$ are 1, $1 + c/k$ and $1 - c/k$, and hence (5.22) is satisfied with $K = 1 - c$.

Then simple computations show that

$$\mathbf{C}_k^*\mathbf{P}_k(a) + \mathbf{P}_k(a)\mathbf{C}_k = (2\mathbf{I} - \mathbf{L} - \mathbf{L}^T) + \mathbf{R}$$

where $\mathbf{R}$ is the matrix whose upper 3×3 block is

$$\begin{pmatrix} 2c\sqrt{T_\infty} & -ic/k & c\sqrt{T_\infty}c_\alpha^{-1} \\ ic/k & 0 & 0 \\ c\sqrt{T_\infty}c_\alpha^{-1} & 0 & 0 \end{pmatrix} \tag{5.24}$$

and whose remaining entries are all zero. Hence for any (real) $\mathbf{h} = (h_0, h_1, h_2 \dots) \in \ell^2$,

$$\langle \mathbf{h}, \mathbf{R}\mathbf{h}\rangle_{\ell^2} = 2c\sqrt{T_\infty}h_0^2 + 2c\sqrt{T_\infty}c_\alpha^{-1}h_0h_2 \geq c\sqrt{T_\infty}(h_0^2 - c_\alpha^{-2}h_2^2) \ .$$

By Lemma (11),

$$\langle \mathbf{h}, (2\mathbf{I} - \mathbf{L} - \mathbf{L}^T)\mathbf{h}\rangle_{\ell_2} \geq (1-\alpha)\sum_{m=1}^{\infty} h_m^2 \ .$$

Combining these estimates with $\mathbf{I} \geq (1-c)\mathbf{P}_k$, (5.20) then holds with

$$2\lambda = \sqrt{T_\infty}\min\left\{c \ , \ \frac{1-\alpha}{\sqrt{T_\infty}} - cc_\alpha^{-2}\right\}(1-c) \ .$$

We now choose c so that $c = \min\{\frac{1}{2}, c_\alpha^2\frac{1-\alpha}{2\sqrt{T_\infty}}\}$. If $c_\alpha^2\frac{1-\alpha}{2\sqrt{T_\infty}} < \frac{1}{2}$, then $\lambda \geq \frac{1-\alpha}{8}$. If $c_\alpha^2\frac{1-\alpha}{2\sqrt{T_\infty}} \geq \frac{1}{2}$, then $\lambda \geq \frac{\sqrt{T_\infty}}{8}$. □

We now explain another route to (5.15) which is relies on general abstract results obtained in [9].:

Theorem 13 (Abstract result of hypocoercivity from [9])**.** *Consider a Hilbert space $\mathcal{H}$ and two closed unbounded operators $\mathcal{S}$ and $\mathcal{L}$ such that:*

- **(H1)** ***Microscopic coercivity****: There exists an orthogonal projection Π_0 such that $\mathcal{L}\Pi_0 = \Pi_0\mathcal{L} = 0$ and there is $\lambda_m > 0$ such that*

$$\forall\, h \in \mathrm{Domain}(\mathcal{L}), \quad -\langle \mathcal{L}h, h\rangle \geq \lambda_m \|(I - \Pi_0)\, h\|^2.$$

- **(H2)** ***Macroscopic coercivity****: The operator $\mathcal{S}$ satisfies $\mathcal{S}^* = -\mathcal{S}$ (skew symmetry) and there exists $\lambda_M > 0$ such that*

$$\|\mathcal{S}\Pi_0 h\|^2 \geq \lambda_M \|\Pi_0 h\|^2 \quad \text{for all } h \in \mathcal{H} \text{ such that } \Pi_0(h) \in \mathrm{Domain}(\mathcal{S}).$$

- **(H3)** ***Consistency****:* $\Pi_0\mathcal{S}\Pi_0 = 0$.
- **(H4)** ***Auxiliary operator****: Define $\mathcal{A} := (1 + (\mathcal{S}\Pi_0)^*(\mathcal{S}\Pi_0))^{-1}(\mathcal{S}\Pi_0)^*$ and assume that $\mathcal{A}\mathcal{S}(1-\Pi_0)$ and $\mathcal{A}\mathcal{L}$ are bounded with a constant $C_M > 0$ such that*

$$\forall\, h \in \mathcal{H}, \quad \|\mathcal{A}\mathcal{S}(1-\Pi_0)h\| + \|\mathcal{A}\mathcal{L}h\| \leq C_M \|(1-\Pi_0)h\|.$$

Then there exist positive constants $\lambda > 0$ and $C > 0$, which are explicitly computable in terms of λ_m, λ_M, and C_M, such that

$$\forall\, h \in \mathcal{H}, \ \forall\, t \geq 0, \quad \left\|e^{t(\mathcal{L}-\mathcal{S})}h\right\| \leq Ce^{-\lambda t}\|h\|.$$

We give a short proof of here for the convenience of the reader.

Proof of Theorem 13. Define the modified norm

$$\mathcal{H}[h] := \frac{1}{2}\,\|h\|^2 + \varepsilon\,\langle \mathcal{A}h, h\rangle$$

where $\varepsilon > 0$ will be chosen small enough below. Given $h_t := e^{t(\mathcal{L}-\mathcal{S})}h$, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{H}[h_t] =& \langle \mathcal{L}h_t, h_t\rangle - \varepsilon\langle \mathcal{A}\mathcal{S}\Pi_0 h_t, h_t\rangle - \varepsilon\langle \mathcal{A}\mathcal{S}(1-\Pi_0)h_t, h_t\rangle \\ &+ \varepsilon\langle \mathcal{S}\mathcal{A}h_t, h_t\rangle + \varepsilon\langle \mathcal{A}\mathcal{L}h_t, h_t\rangle \\ =:& -\mathcal{D}[h_t].\end{aligned}$$

We have used here that $\mathcal{L}^*\mathcal{A} = 0$ which follows from $\mathcal{A} = \Pi_0\mathcal{A}$ and $\Pi_0\mathcal{L} = 0$ in **(H1)**. By **(H1)**, **(H2)**, and by $\mathcal{A}\mathcal{S}\Pi_0 = (1 + (\mathcal{S}\Pi_0)^*(\mathcal{S}\Pi_0))^{-1}(\mathcal{S}\Pi_0)^*(\mathcal{S}\Pi_0)$, the sum of the first two terms in $\mathcal{D}[h_t]$ is coercive:

$$-\langle \mathcal{L}h_t, h_t\rangle + \varepsilon\langle \mathcal{A}\mathcal{S}\Pi_0 h_t, h_t\rangle \geq \min\left\{\lambda_m, \frac{\varepsilon\lambda_M}{1+\lambda_M}\right\}\|h_t\|^2.$$

Let us prove that the operators $\mathcal{A}$ and $\mathcal{S}\mathcal{A}$ are bounded:

$$\forall\, h \in \mathcal{H}, \quad \|\mathcal{A}h\| \leq \frac{1}{2}\|(1-\Pi_0)h\| \quad \text{and} \quad \|\mathcal{S}\mathcal{A}h\| \leq \|(1-\Pi_0)h\|. \tag{5.25}$$

The equation $\mathcal{A}h = g = \Pi_0 g$ (remember that $\mathcal{A} = \Pi_0\mathcal{A}$) is equivalent to

$$(\mathcal{S}\Pi_0)^*h = g + (\mathcal{S}\Pi_0)^*(\mathcal{S}\Pi_0)g.$$

Taking the scalar product of the above equality with g and using **(H3)**, we get

$$\begin{aligned}\|g\|^2 + \|\mathcal{S}\Pi_0 g\|^2 &= \langle h, \mathcal{S}\Pi_0 g\rangle = \langle (1-\Pi_0)h, \mathcal{S}\Pi_0 g\rangle \\ &\leq \|(1-\Pi_0)h\|\|\mathcal{S}\Pi_0 g\| \leq \frac{1}{4}\|(1-\Pi_0)h\|^2 + \|\mathcal{S}\Pi_0 g\|^2,\end{aligned}$$

which completes the proof of (5.25).

The first inequality in (5.25) implies that $\mathcal{H}[h]$ is equivalent to $\|h\|^2$:

$$\frac{1}{2}(1-\varepsilon)\|h\|^2 \leq \mathcal{H}[h] \leq \frac{1}{2}(1+\varepsilon)\|h\|^2. \tag{5.26}$$

The second inequality in (5.25) and **(H1)-(H2)-(H3)-(H4)** imply

$$\begin{aligned}\mathcal{D}[f] &\geq \lambda_m\|(1-\Pi_0)h_t\|^2 + \frac{\varepsilon\lambda_M}{1+\lambda_M}\|\Pi_0 h_t\|^2 - \varepsilon(1+C_M)\|(1-\Pi_0)h_t\|\|h_t\| \\ &\geq \left[\lambda_m - \varepsilon(1+C_M)\frac{1}{2\delta}\right]\|(1-\Pi_0)h_t\|^2 + \varepsilon\left[\frac{\lambda_M}{1+\lambda_M} - (1+C_M)\frac{\delta}{2}\right]\|\Pi_0 h_t\|^2\end{aligned}$$

for an arbitrary $\delta > 0$. By choosing first δ and then ε small enough, a positive constant κ can be found, such that $\mathcal{D}[h_t] \geq \kappa\|h_t\|^2$. Using (5.26), this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{H}[h_t] \leq -\frac{2\kappa}{1+\varepsilon}\mathcal{H}[h_t],$$

completing the proof with $\lambda = \kappa/(1+\varepsilon)$ and $C = \sqrt{1+\varepsilon}/\sqrt{1-\varepsilon}$. □

Second proof of Theorem 2. We consider $\alpha \in [0,1)$ (for $\alpha = 1$ the result is already known from [1]). We apply the previous Theorem 13 with $\mathcal{H}$ being the subspace of $\mathcal{H}_\alpha = L^2(f_{\infty,\alpha}^{-1} \mathrm{d}x\mathrm{d}v)$ consisting of functions that satisfy the zero global mass condition $\int_{\mathbb{T}\times\mathbb{R}} h(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0$. We take $\mathcal{S} = v\partial_x$ and $\mathcal{L} = \mathcal{L}_\alpha$, the linearized operator defined in (5.12), and $\Pi_0(h) = (\int_{\mathbb{R}} h\,\mathrm{d}v) f_{\infty,\alpha}$ defined in (5.11). Then **(H1)** is proved in Lemma 11 and **(H2)** follows from

$$\int_{\mathbb{T}\times\mathbb{R}} (v\partial_x \sigma f_{\infty,\alpha})^2 \frac{\mathrm{d}x\,\mathrm{d}v}{f_{\infty,\alpha}} = T_\infty \|\partial_x \sigma\|^2_{L^2(\mathbb{T})} \geq T_\infty \|\sigma\|_{L^2(\mathbb{T})}$$

where we have used the Poincaré inequality in the unit torus, and the zero global mass condition. **(H3)** follows from $\int_{\mathbb{R}} v f_{\infty,\alpha}(v)\,\mathrm{d}v = 0$. Finally, to prove **(H4)**, we first figure out some explicit formula for $\mathcal{A}$:

$$\begin{cases} \mathcal{A}h = -\left[\left(1 - T_\infty \partial_x^2\right)^{-1} \partial_x j\right] f_{\infty,\alpha}, & j(x) := \displaystyle\int_{\mathbb{R}} v h(x,v)\,\mathrm{d}v \\ \mathcal{A}\mathcal{S}h = -\left[\left(1 - T_\infty \partial_x^2\right)^{-1} \partial_x^2 \tau\right] f_{\infty,\alpha}, & \tau(x) := \displaystyle\int_{\mathbb{R}} v^2 h(x,v)\,\mathrm{d}v. \end{cases}$$

Since $\left(1 - T_\infty\partial_x^2\right)^{-1}\partial_x$ and $\left(1 - T_\infty\partial_x^2\right)^{-1}\partial_x^2$ are bounded operators on $L^2(\mathbb{T})$, we deduce that $\mathcal{A}$ and $\mathcal{A}\mathcal{S}$ are bounded, and **(H4)** follows since $\mathcal{L}_\alpha$ is bounded and $\mathcal{L}_\alpha \Pi_0 = 0$. This concludes the proof of hypocoercivity for our linearized operator $\mathcal{L}_\alpha - \mathcal{S}$.

5.5. **Nonlinear stability.** We close this section by proving nonlinear stability. Let $\alpha \in [0,1)$ and consider a probability density $f_{in} \in \mathcal{H}_\alpha$ and define $h_{in} := f_{in} - f_{\infty,\alpha} \in \mathcal{H}_\alpha$. This fluctuation has zero global mass by definition. We define the solution through

$$h_t = e^{t(\mathcal{L}_\alpha - \mathcal{S})} h_{in} + \int_0^t e^{(t-s)(\mathcal{L}_\alpha - \mathcal{S})} \mathcal{R}[h_s]\,\mathrm{d}s$$

with the nonlinear remainder term defined by

$$\mathcal{R}[h] := \mathcal{M}_f - \mathcal{M}_{f_{\infty,\alpha}} - \left[M_{T_\infty}\sigma + \frac{1}{2}\left(\frac{v^2}{T_\infty} - 1\right) M_{T_\infty}(v) \left(\frac{1}{T_\infty}\tau - \sigma\right)\right]$$

with σ and τ defined in terms of h as before. Taylor expansions and straightforward calculations, using the multiplicative property of the $H^1(\mathbb{T})$ Sobolev norm, show that

$$\|\mathcal{R}[h]\|_{\mathcal{H}^1_\alpha} \lesssim \|\sigma\|^2_{H^1(\mathbb{T})} + \|\tau\|^2_{H^1(\mathbb{T})} \lesssim \|h\|^2_{\mathcal{H}^1_\alpha}.$$

(For more detail in a closely related argument, see [1].) We deduce the a priori estimate

$$\|h_t\|_{\mathcal{H}^1_\alpha} \leq C e^{-\lambda t} \|h_{in}\|_{\mathcal{H}^1_\alpha} + C' \int_0^t e^{-\lambda(t-s)} \|h_s\|^2_{\mathcal{H}^1_\alpha}\,\mathrm{d}s$$

for some constants $C, C', \lambda > 0$, and one can use the values of C and λ provided by Theorem 12. By a standard argument, this shows global existence and exponential decay at rate λ when $\|h_{in}\|_{\mathcal{H}^1_\alpha}$ is small enough. $\square$

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