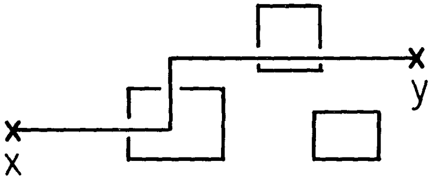
Spin systems and random walks



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with Tyler Helmuth and Andrew Swan

119th Statistical Mechanics Conference

Interacting random walks

Vertices Λ. Edge weights $\beta = (\beta_{ij})_{i,j\in\Lambda}$. E.g. nearest neighbors on $\Lambda \subset \mathbb{Z}^2$.

Simple random walk (SRW)

$$\mathbb{P}(X_{t+\delta t} = j | X_j = i) = \beta_{ij}\delta t + o(\delta t).$$

The process (X_t) is a Markov process on Λ with generator

$$\mathcal{L}g(i) = \sum_{j\in\Lambda} \beta_{ij}(g(j) - g(i)).$$

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Vertex reinforced jump process (VRJP)

$$\mathbb{P}(X_{t+\delta t} = j | X_t = i, L_t) = \beta_{ij}(1 + L_t^j)\delta t + o(\delta t), \qquad L_t^j = \int_0^t 1_{X_s=j}\, ds.$$

The joint process (X_t, L_t) is a Markov process on $\Lambda \times \mathbb{R}^\Lambda$ with generator

$$\mathcal{L}g(i,\ell) = \sum_{j\in\Lambda} \beta_{ij}(1 + \ell_j)(g(j,\ell) - g(i,\ell)) + \frac{d}{d\ell_i} g(i,\ell).$$

Expectation with initial condition $(X_0, L_0) = (i, \ell)$ denoted by $\mathbb{E}_{i,\ell}$.

Sigma models

Vertices Λ. Edge weights $\beta = (\beta_{ij})_{i,j\in\Lambda}$.

Free fields: target space is Euclidean space $\mathbb{R}^n$.

$$\langle F\rangle \propto \int_{(\mathbb{R}^n)^\Lambda} \exp\left(-\frac{1}{4}\sum_{i,j\in\Lambda}\beta_{ij}|\varphi_i-\varphi_j|^2 - \frac{m^2}{2}\sum_{i\in\Lambda}|\varphi_i|^2\right) F(\varphi)\,d\varphi$$

where $\varphi_i = (x_i^1,\dots,x_i^n)\in\mathbb{R}^n$ and $|\varphi|^2 = \varphi\cdot\varphi$ is the Euclidean inner product.

Hyperbolic sigma models: target space is Hyperbolic space $\mathbb{H}^n$.

$$\langle F\rangle \propto \int_{(\mathbb{H}^n)^\Lambda} \exp\left(-\frac{1}{4}\sum_{i,j\in\Lambda}\beta_{ij}|u_i-u_j|^2 - h\sum_{i\in\Lambda}(z_i-1)\right) F(u)\,du$$

where $u_i = (x_i^1,\dots,x_i^n,z_i)\in\mathbb{R}^{n,1}$ with Minkowski inner product

$$|u|^2 = u\cdot u = (x^1)^2+\cdots+(x^n)^2 - z^2$$

and

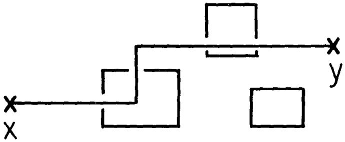
$$\mathbb{H}^n = \{u\in\mathbb{R}^{n,1} : u\cdot u = -1, z>0\}.$$

Isomorphism theorems

Let $g : \mathbb{R}^\Lambda \to \mathbb{R}$.

Free fields $\leftrightarrow$ SRW (Symanzik, Brydges–Fröhlich–Spencer, Dynkin)

$$\langle g(\tfrac{1}{2}|\varphi|^2)x_a^1 x_b^1\rangle_{\mathbb{R}^n} = \left\langle \int_0^\infty \mathbb{E}_a^{\mathrm{SRW}}\left(1_{X_t=b}\, g(\tfrac{1}{2}|\varphi|^2 + L_t)\right) e^{-m^2 t}\, dt \right\rangle_{\mathbb{R}^n}$$



Related: Ising model $\leftrightarrow$ random currents (Griffiths–Hurst–Sherman, Aizenman)

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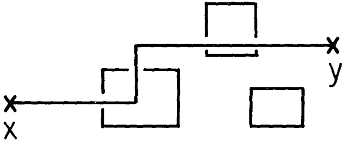

Related: Ising model $\leftrightarrow$ random currents (Griffiths–Hurst–Sherman, Aizenman)

Hyperbolic sigma models $\leftrightarrow$ VRJP (B–Helmuth–Swan)

$$\langle g(z-1) x_a^1 x_b^1 \rangle_{\mathbb{H}^n} = \left\langle \int_0^\infty \mathbb{E}_{a,z-1}^{\mathrm{VRJP}} \left(1_{X_t = b}\, g(L_t) \right) e^{-ht}\, dt \right\rangle_{\mathbb{H}^n}$$

x^1 is any direction orthogonal to the z-direction in which the symmetry is broken.

Supersymmetric extension

Introduce internal supersymmetry.

- Free field $\mathbb{R}^{2|2}$: replace $\varphi = (x, y) \in \mathbb{R}^2$ by $\varphi = (x, y, \eta, \xi) \in \mathbb{R}^{2|2}$

  Then $|\varphi|^2 = x^2 + y^2 + \eta\xi - \xi\eta$ is supersymmetric.

- Hyperbolic sigma model $\mathbb{H}^{2|2}$: replace $(x, y) \in \mathbb{R}^2$ by $(x, y, \eta, \xi) \in \mathbb{R}^{2|2}$

  Then $z = \sqrt{1 + x^2 + y^2 + \eta\xi - \xi\eta}$ is supersymmetric.

Isomorphism theorems remain true with exactly the same statement.

Isomorphism theorems + Supersymmetric localization

$\mathbb{R}^{2|2}$ and SRW.

$$\langle g(\frac{1}{2}|\varphi|^2)x_a^1 x_b^1\rangle_{\mathbb{R}^{2|2}} = \left\langle \int_0^\infty \mathbb{E}_a^{\text{SRW}} \left(1_{X_t=b}\, g(\frac{1}{2}|\varphi|^2 + L_t)\right) e^{-m^2 t}\, dt \right\rangle_{\mathbb{R}^{2|2}}$$

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In particular. Let $g = 1$. Then

$$\langle x_a^1 x_b^1\rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}_{a,0}^{\mathsf{VRJP}}\left(1_{X_t=b}\right) e^{-ht}\, dt.$$

The two-point function of the $\mathbb{H}^{2|2}$ model equals that of the VRJP.

Coordinates and symmetry

Horospherical coordinates for $\mathbb{H}^{2|2}$ (Disertori–Spencer–Zirnbauer):

$$\begin{aligned} x &= \sinh(t) - (\frac{1}{2}s^2 + \psi\bar{\psi})e^t \\ y &= se^t \\ z &= \cosh(t) + (\frac{1}{2}s^2 + \psi\bar{\psi})e^t \end{aligned}$$

This choice of coordinates is not essential, but symmetry plays a role.

$$|u_i - u_j|^2 = \cosh(t_i - t_j) - 1 + \left(\frac{1}{2}(s_i - s_j)^2 + (\psi_i - \psi_j)(\bar{\psi}_i - \bar{\psi}_j)\right) e^{t_i + t_j}$$

The hyperbolic energy is invariant under global translations of s field.

Local translations lead to local Ward identities (see, e.g., Aizenman–Simon).

Proof of hyperbolic isomorphism theorem

- Integration by parts ('local Ward identity'): Let $g : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$.

$$-\sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H} y_a y_b \mathcal{L} g(b, z-1) = \int_{(\mathbb{H}^n)^\Lambda} e^{-H} z_a g(a, z-1)$$

  where recall

$$\mathcal{L} g(b, z-1) = \sum_{c \in \Lambda} \beta_{bc} z_c (g(c, z-1) - g(b, z-1)) + \frac{\partial}{\partial z_b} g(b, z-1)$$

  and

$$y_b \frac{\partial}{\partial z_b} g(b, z-1) = \frac{\partial}{\partial s_b} g(b, z-1).$$

- Markov property: $g_t(a, \ell) = \mathbb{E}_{a,\ell} g_0(X_t, L_t)$ satisfies $\partial_t g = \mathcal{L} g$.
- Supersymmetric localization.

VRJP and $\mathbb{H}^{2|2}$

Sabot–Tarres (2013), Merkl–Rolles–Tarres (2016):

- Random time change for VRJP such that $\tilde{L}_t^i = \log L_t^i$.
- [ST] In the limit $t \to \infty$ the $\tilde{L}_t^i - \tilde{L}_t^j$ converge to the measure with density

$$\exp\left(-\frac{1}{4}\sum_{i,j}\beta_{ij}\cosh(t_i - t_j) - \frac{1}{2}\log\det D(t)\right)$$

  This is t-marginal of the $\mathbb{H}^{2|2}$ model in horospherical coordinates [DSZ].
- [MRT] The s-marginal can also be obtained in the limit from a rescaled VRJP.

Our relation:

- Non-asymptotic relation between original local time and $\mathbb{H}^{2|2}$ model.
- Only symmetry: Lorentz group (local Ward identity) + SUSY (localization).
- VRJP describes correlations of Goldstone mode in non-supersymmetric case.

Mermin–Wagner Theorem

Let Λ be discrete torus, β finite range, translation invariant.

$$\hat{G}(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_{a \in \Lambda} \langle x_0^1 x_a^1 \rangle e^{-ip \cdot a}.$$

Theorem. For $\mathbb{H}^n$ or $\mathbb{H}^{2|2}$ (with $n = 0$),

$$\hat{G}(p) \geqslant \frac{1}{C_\beta |p|^2 (n+1) G(0) + h}.$$

Proof: Cauchy–Schwarz exactly as in Mermin–Wagner, but integrate by parts in the horospherical s-coordinate rather than polar coordinates.

Corollary. In $d = 2$, for any β as above,

$$\langle (x_0^1)^2 \rangle \to \infty \quad \text{as } h \downarrow 0.$$

In particular, the expected time spent by VRJP at 0 is always infinite in $d \leqslant 2$.

ERRW. Recurrence for VRJP at $\beta \ll 1$: Sabot–Tarres, Angel–Kozma–Crawford, Merkl–Rolles, Disertori–Spencer, Collevecchio–Xeng by **fractional moments**.

Thank you!