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# Dimerization in a class of SU(n) invariant quantum spin chains

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**Happy Birthday, Jennifer, Michael, and Paul!**

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# Outline

- The $-P^{(0)}$ spin chains
- A random loop representation
- Prior results
- New result
- Comments and outlook

# The $-P^{(0)}$ quantum spin chains

For each $n \geq 2$, we consider finite quantum spin chains indexed by $x \in [a, b] \subset \mathbb{Z}$, with Hilbert space $\mathcal{H}_{[a,b]} = (\mathbb{C}^n)^{\otimes b-a+1}$, $a \leq b \in \mathbb{Z}$, and Hamiltonians

$$H_{[a,b]} = -\sum_{x=a}^{b-1} P^{(0)}_{x,x+1},$$

where $P^{(0)} = |\phi\rangle\langle\phi|$ is the orthogonal projection on the $SU(2)$ spin singlet state $(n = 2S + 1)$ $\phi \in \mathbb{C}^n \otimes \mathbb{C}^n$, given by

$$\phi = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} (-1)^k |n-k, k\rangle.$$

In fact, $H_{[a,b]}$ has an $SU(n)$ symmetry: $[P^{(0)}, v \otimes \overline{v}] = 0$, where $v$ and $\overline{v}$ denote the fundamental and antifundamental representations of $SU(n)$.

# Dimerized ground states

In this talk, a ground state of a finite chain is given by a unit eigenvector, $\psi_{[a,b]}$, belonging to the smallest eigenvalue of $H_{[a,b]}$.

A ground state of the infinite chain is any linear functional $\omega$ on the algebra observables obtained as a limit of the form

$$\omega(A) = \lim_{\substack{a_n \to -\infty \\ b_n \to +\infty}} \langle \psi_{[a_n,b_n]}, A\psi_{[a_n,b_n]} \rangle, \quad A \in \mathcal{A}_{\mathrm{loc}},$$

where each $\psi_{[a,b]}$ is a ground state of $H_{[a,b]}$. We say that

dimerization occurs in the ground states if there is a ground state $\omega$ for which

$$\omega(|\phi\rangle\langle\phi|_{x-1,x}) \neq \omega(|\phi\rangle\langle\phi|_{x,x+1}).$$

It has been conjectured that dimerization occurs iff $n \geq 3$.

# Remarks

- The $-P^{(0)}$-model with $n = 2$ is equivalent to the standard Heisenberg antiferromagnetic chain:

$$H^{AF}_{[a,b]} = \sum_{x=a}^{b-1} \sigma^1_x \sigma^1_{x+1} + \sigma^2_x \sigma^2_{x+1} + \sigma^3_x \sigma^3_{x+1}.$$

  The Bethe ansatz predicts a unique, translation-invariant ground state for the infinite chain.
- Barber and Batchelor (1989), for the case $n = 3$, and Affleck (1990), for all $n \geq 2$ argued that $-P^{(0)}$-model can be understood in terms of the transfer matrix of the $n^2$-state Potts model on the square lattice. Klümper (1990) used properties of the Temperley-Lieb algebra to describe an exact solution of the $n = 3$ model, demonstrating dimerization and a spectral gap.

# A random loop representation

Consider intervals of the form $[-\ell+1,\ell]$ ($2\ell$ spins), and denote the Hamiltonian by $H_\ell$, and let $\psi_\ell$ be a normalized eigenvector of its smallest eigenvalue, which turns out to be simple. Then

$$|\psi_\ell\rangle\langle\psi_\ell| = \lim_{\beta\to\infty} \frac{e^{-2\beta H_\ell}}{\mathrm{Tr} e^{-2\beta H_\ell}},$$

and therefore

$$\langle\psi_\ell, P^{(0)}_{x,x+1}\psi_\ell\rangle = \mathrm{Tr}|\psi_\ell\rangle\langle\psi_\ell| P^{(0)}_{x,x+1} = \lim_{\beta\to\infty} \frac{\mathrm{Tr} e^{-\beta H_\ell} P^{(0)}_{x,x+1} e^{-\beta H_\ell}}{\mathrm{Tr} e^{-2\beta H_\ell}}.$$

Both the numerator and the denominator can be given a nice representation by writing (for integer $\beta$)

$$e^{-\beta H_\ell} = \lim_{N\to\infty}(\mathbb{1} - \frac{1}{N}H_\ell)^{\beta N} = \lim_{N\to\infty}\left(\mathbb{1} + \frac{1}{N}\sum_{x=-\ell+1}^{\ell-1} P^{(0)}_{x,x+1}\right)^{\beta N} .$$

The $(2\ell)^{\beta N}$ terms in the RHS resulting from expanding the product are labeled by a set $\Omega_{\ell,N}$ of diagrams we call random loop configurations, which are helpful to calculate the trace of each term using the matrix representation of each factor $P^{(0)}_{x,x+1}$:

$$P^{(0)} = \frac{1}{n}\sum_{k,l=1}^{n}(-1)^{k-l}|k, n+1-k\rangle\langle l, n+1-l|$$

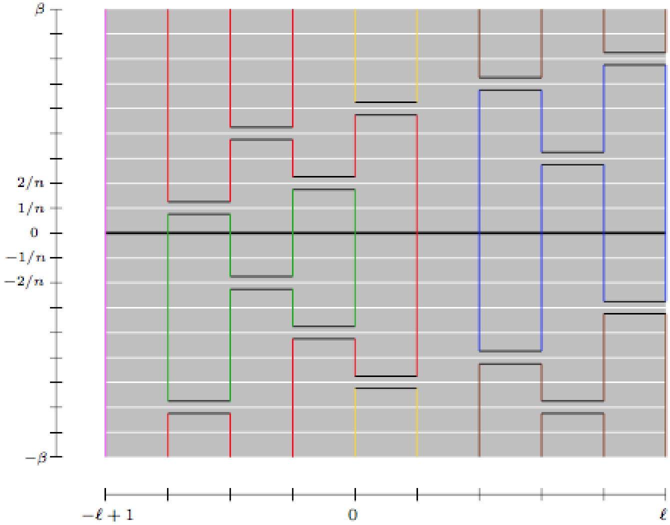



FIGURE 2. Loop representation of the $SU(2S+1)$-invariant quantum spin chains.

The trace of each term, labeled by $\omega \in \Omega_{\ell,N}$, is positive and depends only on the number of factors $P^{(0)}$, denoted by $|\omega|$, and the number of loops in $\omega$, denoted by $\mathcal{L}(\omega)$. This allows us to define a probability measure on $\Omega_{\ell,N}$. It is given by

$$\mu_{\beta,\ell,N}(\omega) = \frac{1}{Z_N(\beta,\ell)} \left(\tfrac{1}{N}\right)^{|\omega|} n^{\mathcal{L}(\omega)-|\omega|}, \tag{1}$$

with

$$Z_N(\beta,\ell) = \sum_{\omega\in\Omega_{\ell,N}} \left(\tfrac{1}{N}\right)^{|\omega|} n^{\mathcal{L}(\omega)-|\omega|}. \tag{2}$$

In the following, we use the notation $\mathbb{P}_{\beta,\ell,N}$ for the probability with respect to the measure $\mu_{\beta,\ell,N}$. The event $x \leftrightarrow y$ (resp. $x \not\leftrightarrow y$) represents the set of all $\omega \in \Omega_{\ell,N}$ where $(x,0)$ and $(y,0)$ belong to the same loop (resp. belong to distinct loops).

## Theorem (Aizenman-N, 1994)

*(a)* $\mathrm{Tr} e^{-2\beta H_\ell} = \lim_{N\to\infty} Z_N(\beta,\ell)$;
*(b)* $\langle P^{(0)}_{x,x+1}\rangle_{2\beta,\ell} = C(n) + D(n)\lim_{N\to\infty}\mathbb{P}_{\beta,\ell,N}(x \leftrightarrow x+1)$.
*(c)* $\langle \psi_\ell, S^3_x S^3_y \psi_\ell\rangle = (-1)^{x-y}\frac{n^2}{6}\lim_{\beta\to\infty}\lim_{N\to\infty}\mathbb{P}_{\beta,\ell,N}(x \leftrightarrow y)$.
*(d) If*

$$\sum_{r=1}^{\infty} r \lim_{\ell\to\infty} |\langle \psi_\ell, S^3_0 S^3_r \psi_\ell\rangle| < \infty,$$

*translation invariance is broken and there are at least two 2-periodic ground states of the infinite chain.*
*(e) In the situation of (d), broken translation invariance, there is a dimerized ground state, i.e., a 2-periodic ground state* $\omega$ *for which*

$$\omega(P^{(0)}_{x,x+1}) \neq \omega(P^{(0)}_{x-1,x}).$$

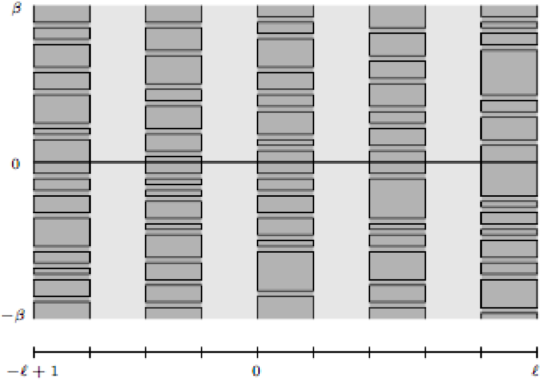
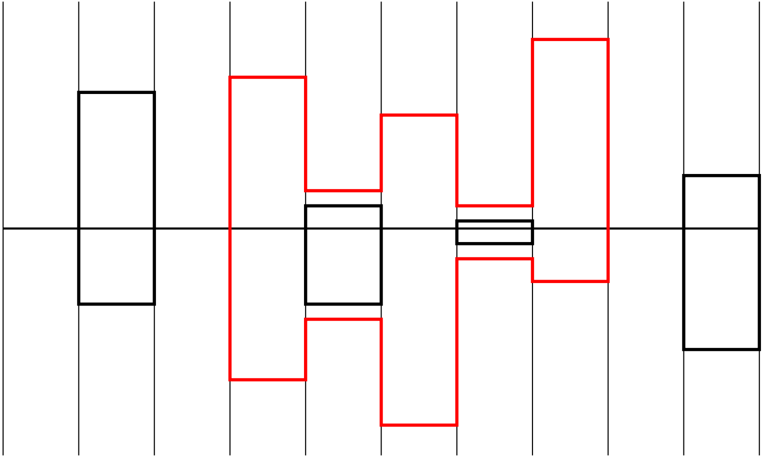



FIGURE 3. Illustration for 'optimal' configurations that allow for large numbers of loops.

# Proof of dimerization by Peierls argument

The simplest direct proof of discrete symmetry breaking is by a Peierls argument.



Large loop encloses a region where the state is a dimerized state shifted by one lattice spacing with respect to the exterior.

# Main result

## Theorem (N-Ueltschi, Lett. Math. Phys. 2017)

*For $n \geq 17$, the $-P^{(0)}$-model has at least two ground states of the infinite chain that are dimerized, i.e., there is a 2-periodic ground state $\omega$ and $\delta_n > 0$ such that*

$$\omega(P^{(0)}_{x,x+1} - P^{(0)}_{x-1,x}) = (-1)^x \delta_n.$$

*For $n \geq 81$,*

$$\delta_n > (1 - n^{-2})(1 - 2c_n),$$

*with*

$$c_n = \frac{64}{n^{3/2}} + \frac{128}{n^2} + \frac{1}{12} \sum_{k=7}^{\infty} \frac{(k+1)4^k}{n^{\frac{1}{2}k-1}}.$$

# Comments and Outlook

- The 'direct proof' proof gives an estimate for the dimerization order parameter for $n \geq 81$. In combination with FKG magic, we can conclude from the same arguments dimerization occurs for $n \geq 17$.
- The exponential decay of correlations is indicative of a spectral gap above the ground state of the infinite chain.
- A topic of great current interest is the stability of 'gapped phases'. It should be possible to use the random loop representation to prove that any spin chain with an interaction $-P^{(0)} + \Delta$ , with $\Delta$ also finite-range, translation-invariant, and not too large, also has dimerized ground states.
- It is believed, by some, that there are several inequivalent types of dimerization. Are there interactions with random loop models for all of them?