Heat kernel and graph distance for Liouville Brownian motion

Ofer Zeitouni

May, 2018

Short time heat kernels estimates for diffusions

• Brownian motion W_t in $\mathbb{R}^d$, generator $\Delta/2$.
$p_t(x,y) = \frac{1}{\sqrt{2\pi t}^d} e^{-|x-y|^2/2t}$.
Thus...

$$t \log p_t(x,y) \to_{t\to 0} -\frac{|x-y|^2}{2}.$$

• For general diffusions:

Theorem (Varadhan '67)

With uniformly eliptic generator $\frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$, *heat kernel satisfies*

$$t \log p_t(x,y) \to_{t\to 0} -\frac{d(x,y)^2}{2},$$

where d is geodesic distance determined by a.

Many generalizations (hypoelliptic, on diagonal), using large deviation theory and refinements: Azencott, Stroock, Kusuoka, Ben Arous, Leandre...

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Time change

- A particularly simple case: $a(x) = \sigma^2(x)$ scalar, strictly positive.

$$dX_t = \sigma(X_t)dB_t$$

. From time change, $X_t = B_{F^{-1}(t)}$, where

$$F(t) = \int_0^t \frac{1}{a(B_s)} ds \quad \text{is strictly increasing}(PCAF).$$

Still,

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- Take $a(x) = e^{-V(x) + \frac{1}{2}\mathbb{E}V^2(x)}$ where V is a centered Gaussian field, then

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Equilibrium measure $\mu_V(dx) = \frac{1}{a(x)} dx = e^{V(x) - \frac{1}{2}\mathbb{E}V^2(x)} dx$

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Log correlated fields

- Focus on 2D torus.

$$\mu_V(dx) = e^{V(x) - \frac{1}{2}\mathbb{E}V^2(x)}dx$$

Interest in cases where $V(\cdot)$ is not smooth, not even pointwise defined:

$$\mathbb{E}V(x)V(y) = \log \frac{1}{|x-y|} + g(x,y)$$

where g is bounded, continuous off the diagonal.
Defined as distribution. Particular case - the Gaussian free field.
Formal limit of $V_\epsilon(x) = \int \phi_\epsilon(x-y)V(y)dy$.

- The measures $\mu_\epsilon^\gamma(dx) = e^{\gamma V_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}V_\epsilon^2(x)}dx$ converge, if $\gamma < 2$, to a limit formally given by

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GMC-properties

$$\mu_V = e^{\gamma V(x)dx - \frac{\gamma^2}{2}\mathbb{E}V^2(x)}dx, \, I = \mu_V(\mathbb{T}^2)$$

and approximations

$$\mu_V^\epsilon = e^{\gamma V_\epsilon(x)dx - \frac{\gamma^2}{2}\mathbb{E}V_\epsilon^2(x)}dx, \, I_\epsilon = \mu_V^\epsilon(\mathbb{T}^2)$$

- For $\gamma < 2$, I_ϵ is uniformly integrable.
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- μ_V does not depend on particular mollifiers.
- For $\gamma < \sqrt{2}$, I_ϵ is in L^2, analysis simpler.
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Liouville BM

Given μ_V, can inquire on "geometry" associated with μ_V, taken as "Riemannian volume" (Sheffield, Miller).

One possible approach: use Brownian motion.

$$F_t = \int_0^t e^{\gamma V(B_s) - \frac{1}{2}\gamma^2 \mathbb{E} V^2(B_s)} ds, \quad X_t = B_{F_t^{-1}}. \tag{1}$$

Of course, not defined as written. But using approximations, one has:

Theorem (Garban,Rhodes,Vargas '13; N. Berestycki '13)

There exists a diffusion process X_t with continuous paths corresponding to (1).

The corresponding Dirichlet form is the usual one, but with domain $L^2(\mu_V) \cap H^1_{loc}$, not $L^2(dx) \cap H^1_{loc}$.

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Heat kernel and geometry

Can we use LBM to study geometry?

A general paradigm (verified for BM on many fractals, etc.) is that heat kernel *should* behave, for short time, as

$$p_t(x,y) \sim \frac{1}{t^{d_H/b}} e^{-\left(\frac{d(x,y)}{t^{1/b}}\right)^{b/(b-1)}}$$

d_H- Hausdorff dimension, $2d_H/b$ - spectral dimension.
For BM, $2d_H/b = d$, $b = 2$.

Theorem (Rhodes-Vargas '14)

$d_H/b = 1$ *in the sense that* $\log p_t(x,x)/\log t \to_{t\to 0} -1$.

(Logarithmic corrections, improved by Andres-Kajino '14).
To have any hope of identifying distances, we thus need to find d_H!

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Some speculations

Watabiki ('93):

$$d_H = 1 + \frac{\gamma^2}{4} + \sqrt{\left(1 + \frac{\gamma^2}{4}\right)^2 + \gamma^2}$$

• For γ small, $d_H(Watabiki) \sim 2 + \gamma^2$.
• Remarkable: for $\gamma = \sqrt{8/3}$, $d_H(Watabiki) = 4$, consistent with convergence to Brownian map results of Miller-Sheffield.
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GFF results

Take covariance corresponding to GFF on torus:

$$G(x,y) = \sum_{n \geq 1} \frac{1}{\lambda_n} e_n(x) e_n(y)$$

where (λ_n, e_n) are eigenvalues and eigenfunctions of the (minus) standard Laplacian.

Theorem (Maillard,Rhodes,Vargas, Z. '14)

Upper bound: $\exists$ *(explicit,deterministic)* β_{UB} *so that*

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Lower bound: for all $\eta > 0$ *there exists* $C_V = C_V(\eta)$ *so that for* $t < 1$*,*

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$\beta_{UB} \sim 2 + 2\gamma$. Improved by Andres-Kajino.

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Relation to geometry

Recall the GMC

$$\mu_V = e^{\gamma V(x)dx - \frac{\gamma^2}{2}\mathbb{E}V^2(x)}dx$$

with V the GFF in 2D.

Can define the Liouville graph distance $D_{\gamma,\delta}(u,v)$ as the minimal number of balls of γ-Liouville mass at most δ needed to create a path connecting u, v.

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• Upper bound (both in MRVZ and AK) is based on *uniform* bound on exit times from balls, i.e. on $F(\tau_{B(x,r)})$. As such, can't hope it is tight.
• Off diagonal lower bound (MRVZ): use *bridge* formula

$$\int_0^\infty \psi(t) p_t^\gamma(x,y)dt = \int_0^\infty p_t^0(x,y) E_B^{x \xrightarrow{t} y}(\psi(F(t)))dt.$$

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$$r_\lambda(x,y) = \int_0^\infty E_B^{x \xrightarrow{t} y}(e^{-\lambda F(t)}) p_t(x,y)dt,$$

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Key estimate:

$$\int_{w \in B_{t/2}(x)} \inf_{z \in B_{t/2}(y)} E_B^{w \xrightarrow{t} z}\left(e^{-\lambda F(t)}\right) \geq e^{-c\left(\frac{1}{t} + \lambda t^{1+\gamma^2/4-\eta}\right)}.$$

Strategy: force BM to stay in tube of width t connecting w to z.

- Probability of this event is $\sim e^{-c/t}$.
- Fix $\delta < \sqrt{2}$. Call a box S_k δ-thick if $\mu_V(S_k) \sim t^{2+\gamma^2/2-\gamma\delta}$. Then

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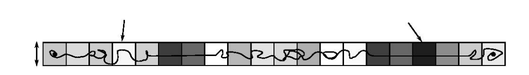

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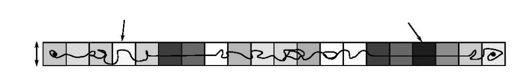

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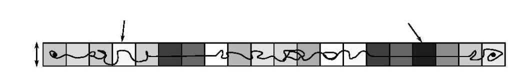

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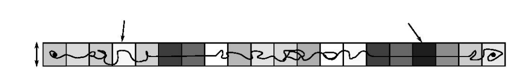

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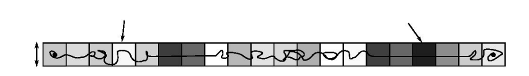
$$\int_{w\in B_{t/2}(x)} \inf_{z\in B_{t/2}(y)} E_B^{w\overset{t}{\to}z}(e^{-\lambda F(t)}) \geq e^{-c(\frac{1}{t}+\lambda t^{1+\gamma^2/4-\eta})}.$$

Strategy: force BM to stay in tube of width t connecting w to z.



- Probability of this event is $\sim e^{-c/t}$.
- Fix $\delta < \sqrt{2}$. Call a box S_k δ-thick if $\mu_V(S_k) \sim t^{2+\gamma^2/2-\gamma\delta}$. Then

$$\#\delta\text{-thick boxes} \sim t^{\delta^2/2-1}.$$

- Contribution of δ-thick boxes to $F(t)$ is roughly $T_\delta \cdot t^{\gamma^2/2-\delta\gamma}$, where T_δ is crossing time.
- now accelerate: force BM to have velocity v_δ (instead of $1/t$) in δ-thick boxes. Cost is $e^{-v_\delta t^{\delta^2/2}}$.
- optimize: $v_\delta = t^{-(1+\delta^2/2)}$, $\delta = \gamma/2$, get main estimate.

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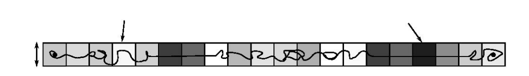

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Improvements

Improving on LB requires (at least) to optimize over paths (instead of straight line). This is a hard percolation problem.

Can somewhat simplify percolation problem by modifying the Gaussian field. Introduce the *k-coarse MBRW*

$$G(x,y) = k \log 2 \sum_{j=0}^{\infty} A(x,y,2^{-kj})$$

where

$$A(x,y,R) = \frac{|B(x,R) \cap B(y,R)|}{|B(x,R)|}$$

(For $k = 1$, it is continuous version of *modified branching random walk* introduced with Bramson to study tightness of max of discrete GFF.)

$$|G(x,y) - \log(1/d_{\mathbb{T}^2}(x,y))| \leq g_k(x,y)$$

and g_k is bounded by $6k$ and continuous off diagonal.

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Improvements II

$$G(x,y) = k \log 2 \sum_{j=0}^{\infty} A(x,y,2^{-kj})$$

Can write the field as $V(x) = \sum_{j=0}^{\infty} h_j(x)$, with h_j fields independent. Given t, define r as $t \sim 2^{-kr(1+\gamma^2/2)}$, $s = 2^{-kr}$, and define the coarse and fine fields

$$\varphi_r = \sum_{j=0}^{r-1} h_j, \quad \psi_r = \sum_{j=r}^{\infty} h_j.$$

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Improvements III

Lemma (Ding-Zhang-Z. '17, based on Ding-Zhang '16)

For k large enough, there exists a path of neighboring $s = 2^{-kr}$-boxes connecting x and y, of total number $2^{kr(1+\delta)}$, so that:
a) Coarse field φ_r for each box is small ($\leq \delta kr \log 2$).
b) LBM associated with fine field Ψ crosses each box within time $s^{2-\delta}$, with probability at least s^{δ}.

Forcing LBM through sequence, can check that total time is $\sim t$ while probability is at least $e^{-1/(t^{1+\gamma^2/2+\epsilon})}$.
Upper bound uses a complementary percolation estimate: can't find a better path.

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Convergence

Main idea: exploit sub-additivity.

- Write GFF as integral of white noise againts Brownian heat kernel.
- Localize GFF.
- Truncate GFF at appropriate scales by controlling variability of field.
- Move to diadic grid.
- Apply sub-additivity (requires a percolation argument due to variability of end-points)
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