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Approach to the steady state in kinetic models with thermal reservoirs at different temperatures

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Abstract

We investigate kinetic models of a system in contact with several spatially homogeneous thermal reservoirs at different temperatures. We explicitly find all spatially homogeneous non-equilibrium stationary states (NESS).

We then consider the question of whether there are also non spatially uniform NESS. This remains partly open in general, except for a class of models to which we can apply probabilistic methods. For these models we show not only that all NESS are spatially homogeneous, but that solutions of these kinetic equation relax exponentially fast, at an explictely computable rate, to the spatially homogeneous NESS also for general spatially inhomogeneous initial data.

1 Introduction

In this note we investigate the time evolution and nonequilibium steady states (NESS) of a gas, described on the mesoscopic scale by a one particle phase space probability distribution $f(x, v, t)$ in contact with heat reservoirs at different temperatures.. Here x is in some domain $V \subset \mathbb{R}^d$ and $v \in \mathbb{R}^d$. Starting with some initial state $f(x, v, 0)$, f changes in time according to an autonomous equation of the general form

$$\frac{\partial}{\partial t} f(x, v, t) + \mathrm{div}(vf(x, v, t) = Q[f](x, v, t) + \sum_{j=1}^{k} L_j f(x, v, t) \ . \qquad (1.1)$$

Here $Q[f]$ represents the effects of interactions between the particles, and thus specifies the changes in f caused by the internal dynamics, e.g. through a Boltzmann collision kernel. Each L_j is a linear operator which represents the effect on f of stochastic interactions with the jth thermal reservoir at temperature T_j. We require that each L_j conserve particle density, but not total momentum and energy.

In the models that are the main focus of the paper, equation (1.1) can be interpreted as the Kolmogorov forward equation of a time dependent Markovian stochastic process, and this opens the way to the use of probabilistic methods to study both the stationary states and the rate of approach to them. This point of view has been fruitfully developed in a setting without thermal reservoirs by Fournier and Méléard [4, 5] for the actual Boltzmann collision without cut-off. Our collision models will be simpler, but we include thermal reservoirs.

The L_j have the property that $L_j M_j = 0$, for $M_j = M_{T_j}$ the uniform centered Maxwellian distribution at temperature T_j. If Q is a Boltzmann collision operator, $Q[M] = 0$ for global Maxwellian densities M of any temperature. In fact, when Q is the Boltzmann collision operator, $Q[M] = 0$ for any local Maxwellian density M. While this feature is not shared the models we consider our Q will always conserve particle density and total energy.

Moreover, in the absence of reservoirs, the isolated system would have for its stationary solutions the spatially uniform Maxwellian density M_T with temperature T determined by the initial value of the energy. (We assume that the total momentum is zero.) On the other hand, when the system is in contact with only one reservoir, or $T_j = T_1$ for all j, the Maxwellian M_{T_1} will then be a stationary solution. In fact, we expect in both cases that these stationary states will be approached by $f(x, v, t)$ as $t \to \infty$. This can be proven in some cases; see below.

The question then arises: What happens when the T_j are different? In physical situations the different reservoirs would occupy different spatial regions, a common case being that in which Maxwell boundary conditions with different temperatures are imposed on different parts of the boundary of the domain V. Recently Esposito et. al. [3] proved that when the T_j are all close to T_1 and when $f(x, v, 0)$ is close to $M_1(v)$, and $Q[f]$ is the Boltzmann collision term, then $f(x, v, t)$ will indeed approach a stationary state close to M_1. Their methods do not extend to the case where the initial state is not close to M_1.

In [2] the spatially uniform case was considered, and it was proved there, for a certain type of Q and L_j, that when the initial f is independent of x, $f(v,t)$ has an exponential approach to a unique stationary state $F(v)$. But, what happens when the initial state is not spatially homogeneous: Do there then exist spatially inhomogeneous stationary states, different from $F(v)$? Supposing there are no stationary states other than $F(v)$, will it be approached as $t \to \infty$ from an arbitrary initial f_0? These questions seem difficult to answer for the case where Q is a Boltzmann collision operator. Therefore here we consider some different collision operators Q which share the above listed properties with the Boltzmann Q and prove for them that F is indeed the unique stationary state, and that it is approached exponentially fast from arbitrary initial data.

1.1 Thermostatted kinetic equations

Let Λ be a torus in $\mathbb{R}^d$ with volume $|\Lambda| = L^d$. For any time dependent one particle distribution $f(x,v,t)$ on the phase space $\Lambda \times \mathbb{R}^d$, define the *hydrodynamic moments*

$$\begin{aligned} \rho(x,t) &= \int_{\mathbb{R}^d} f(x,v,t)\mathrm{d}v \\ u(x,t) &= \frac{1}{\rho(x,t)} \int_{\mathbb{R}^d} v f(x,v,t)\mathrm{d}v \\ T(x,t) &= \frac{1}{d\rho(x,t)} \int_{\mathbb{R}^d} |v-u(x,t)|^2 f(x,v,t)\mathrm{d}v \ . \end{aligned} \tag{1.2}$$

(Note that $u(x,t)$ and $T(x,t)$ are only defined when $\rho(x,t) \neq 0$.) Define the instantaneous (at time t) *local Maxwellian corresponding to* f, denoted by $M_f(x,v,t)$, as

$$M_f(x,v,t) = \rho(x,t)(2\pi T(x,t))^{-d/2} \exp(-|v-u(x,t)|^2/2T(x,t)) \ , \tag{1.3}$$

with the natural convention that $M_f(x,v,t) = 0$ when $\rho(x,t) = 0$.

We consider two forms of Q. The first is a kinetic (self consistent) Fokker-Planck form, and the second is of BGK form.

The *kinetic Fokker-Planck equation* (KFP) is the self-consistent equation

$$\frac{\partial}{\partial t} f(x,v,t) + \mathrm{div}_x(v f(x,v,t)) = T(x,t)\mathrm{div}_v \left(M_f(x,v,t))\nabla_v \frac{f(x,v,t)}{M_f(x,v,t)} \right) \ , \tag{1.4}$$

where a diffusion constant has been absorbed into the time scale for convenience. The right hand side of (1.4) can be written as

$$\mathcal{G}_f f(x,v,t) := T(x,t)\Delta_v f(x,v,t) + \mathrm{div}_v\left((v-u(x,t))f(x,v,t)\right) \ . \tag{1.5}$$

The BGK form of the equation is

$$\frac{\partial}{\partial t}f(x,v,t) + \mathrm{div}_x(vf(x,v,t)) = \alpha[M_f(x,v,t) - f(x,v,t)] \tag{1.6}$$

for a constant $\alpha > 0$.

Both of these equations conserve total energy, momentum and mass. Without loss of generality, we restrict our attention to initial data (for these equations and others to be considered) f_0 such that

$$\int_{\Lambda\times\mathbb{R}^d} vf(x,v,t)\mathrm{d}x\mathrm{d}v = 0\ . \tag{1.7}$$

Under this condition, the only spatially homogeneous steady states of (1.4) and (1.6) are the *global Maxwellian* phase space probability densities of the form

$$M_T(x,v) = \frac{1}{|\Lambda|}(2\pi T)^{-d/2}e^{-|v|^2/2T}\ . \tag{1.8}$$

The situation is more interesting for the thermostatted version of the equation with thermal reservoirs (thermostats) at more than one temperature.

We couple our system to thermal reservoirs defined as follows. For each $j = 1,\ldots,k$, fix a temperature $T_j > 0$, and a coupling constant $\eta_j > 0$, and define

$$L_jf(x,v,t) = \eta_j[\rho(x,t)M_{T_j}(v) - f(x,v,t)]\ .$$

Note that the thermostats act globally in space, not only at walls (which we do not have for the torus Λ in any case). The *thermostatted Fokker-Planck equation* is

$$\frac{\partial}{\partial t}f(x,v,t) + \mathrm{div}_x(vf(x,v,t) = \mathcal{G}_f f(x,v,t)) + \sum_{j=1}^k L_jf(x,v,t) \tag{1.9}$$

where $\mathcal{G}_f f(x,v,t)$ is given by (1.5). The thermostatted BGK equation is

$$\frac{\partial}{\partial t}f(x,v,t) + \mathrm{div}_x(vf(x,v,t) = \alpha[M_f(x,v,t) - f(x,v,t)] + \sum_{j=1}^k L_jf(x,v,t) \tag{1.10}$$

We are interested in the steady states of these equations. It is natural to conjecture that because our thermal reservoirs act in a spatially uniform manner, *there are no steady states that are not spatially uniform.* This is not so easy to verify, and we do not have a proof of this in full generality.

In fact, let $f_*(x,v)$ be any steady state solution of (1.9). Multiplying by a smooth function $\phi(x)$ and integrating over phase space, we obtain

$$\int_\Lambda \nabla\phi(x)\cdot u(x)\rho(x)\mathrm{d}x = 0\ . \tag{1.11}$$

In two or three dimensions, this says that the divergence of ρu is zero, and in one dimension it says that ρu is constant, and then under the condition (1.7), $u(x) = 0$ for all x.

Next, in one dimension, multiplying by $\phi(x)v$ and integrating over phase space

$$\int_\Lambda \nabla\phi(x)\rho(x)T(x)\mathrm{d}x = 0\ , \tag{1.12}$$

so that $\rho(x)T(x)$, the pressure, is constant. But even in one dimension, this does not imply that $\rho(x)$ and $T(x)$ are individually constant. The same remarks apply in the BGK case as well.

There is one simple result that can be proved about convergence to a steady state for solutions of (1.9) and (1.10): More precisely, for a phase space density f define

$$T_f = \frac{1}{d}\int_V\int_{\mathbb{R}^d}|v|^2 f(x,v)\mathrm{d}v\mathrm{d}x\ . \tag{1.13}$$

and let M_{T_f} denote the global Maxwellian with temperature T_f; i.e., the probability density given by (1.8) with $T = T_f$.

1.1 LEMMA. *For any solution $f(x,v,t)$ of (1.9) and(1.10)*

$$\frac{\mathrm{d}}{\mathrm{d}t}T_f(t) = \sum_{j=1}^{d}\eta_j\left(T_j - T_f(t)\right)\ . \tag{1.14}$$

Therefore, if we define the quantities η and T_∞ by

$$\eta := \sum_{j=1}^{k}\eta_j \qquad \text{and} \qquad T_\infty = \frac{1}{\eta}\sum_{j=1}^{k}\eta_j T_j\ , \tag{1.15}$$

we have that

$$T_f(t) = T_\infty + e^{-t\eta}(T_f(0) - T_\infty)\ . \tag{1.16}$$

Proof. Note that $\int_V \int_{\mathbb{R}^d} |v|^2 \mathrm{div}_x(vf(x,v,t))\mathrm{d}v\mathrm{d}x = 0$ so that the term representing the effects of spatial inhomogeneity drops out of (1.9). Also, since energy is conserved globally by the Fokker-Planck term.

$$\int_V \int_{\mathbb{R}^d} |v|^2 G_{T_f(t)}[f](x,v,t)\mathrm{d}v\mathrm{d}x = 0 \ .$$

The rest follows from the definition of the reservoir terms. The analysis for (1.10) is essentially the same. Solving the equation (1.14) yields (1.15) and (1.16). □

In particular, Lemma 1.1 says that for any solution $f(x,v,t)$ of (1.9) or (1.10) $\lim_{t\to\infty} T_f(t) = T_\infty$ and the convergence is exponentially fast.

In Section 2, we use Lemma 1.1, among other devices, to determine the explicit forms of all *spatially homogeneous* steady states for both thermostatted equations.

To go beyond the determination of the spatially uniform steady states, and to prove exponential convergence to them from general initial data, or even that there are no steady states that are not spatially uniform, we simplify our model. We modify Q so that energy is conserved globally but not locally: We replace $M_f(x,v,t)$ in (1.4) by

$$\rho(x,t)M_{T_f(t)}(v) \ .$$

Defining the operator G_T by

$$G_T[g](v) = T\mathrm{div}_v\left(M_T\nabla\frac{f}{M_T}\right) = T\Delta g(v) + \mathrm{div}[vg(v)] \ , \tag{1.17}$$

we may write the resulting equation as

$$\frac{\partial}{\partial t}f(x,v,t) + \mathrm{div}_x(vf(x,v,t) = G_{T_f(t)}[f](x,v,t) + \sum_{j=1}^k L_j f(x,v,t) \ , \tag{1.18}$$

Likewise, in the BGK case, we replace $M_f(x,v,t)$ in the gain term of Q by $\rho(x,t)M_{T_f(t)}(v)$. The resulting equation is

$$\frac{\partial}{\partial t}f(x,v,t) + \mathrm{div}_x(vf(x,v,t) = \\ \alpha[\rho_f(x,t)Mf_{T_f(t)}v,t) - f(x,v,t)] + \sum_{j=1}^k L_j f(x,v,t) \tag{1.19}$$

Equations (1.18) and (1.19) are still non-linear, but only in a superficial way.

This means that we may regard $T_f(t)$ as given in equations (1.18) or (1.19), and then they become the (*linear*) Kolmogorov forward equations of Markov processes with time-dependent generators. This allows us to apply probabilisitic methods to the problem of uniqueness of steady states, and to the problem of proving exponentially fast relaxation to the steady states.

Our main results are the following:

1.2 THEOREM. *Let $f_\infty(x,v)$ be an NESS for (1.18) or (1.19). Then $f_\infty(x,v)$ does not depend on x. That is, every NESS for (1.18) and (1.19) is spatially uniform.*

1.3 THEOREM. *Let f be a solution of either (1.18) or (1.19) with $T_f(0) < \infty$. Let f_∞ be the unique stationary state, which for solutions of (1.18) is given by Theorem 2.2, and for solutions of (1.19) is given by Lemma 2.1. Then there are finite positive and explicitly computable constants C and c such that*

$$\int_{\Lambda\times\mathbb{R}^d} |f(x,v,t) - f_\infty(v)|\mathrm{d}x\mathrm{d}v \le Ce^{-ct} \ .$$

Theorem 1.3 is proved in Section 3. We emphasize that the method provides an explicitly computable bound on the rate of convergence that is likely to be useful elsewhere.

The proof of Theorem 1.2 is much simpler than that of Theorem 1.3. By an ergodicity argument, all we need to do is to find *one* spatially homogeneous steady state. Ergodicity then implies that there are no others. In Section 2 we find the spatially homogeneous steady states for (1.18) or (1.19). Granted that these exist, we may prove Theorem 1.2 as follows:

Proof of Theorem 1.2. By the convergence of $T_f(t)$ to T_∞, any steady state solution of (1.18) must also be a steady state solution of

$$\frac{\partial}{\partial t}f(x,v,t) + \mathrm{div}_x(vf(x,v,t) = G_{T_\infty}[f](x,v,t) + \sum_{j=1}^{k} L_j f(x,v,t) \ . \tag{1.20}$$

In Theorem 2.2 below we find the explicit form of the unique *spatially homogeneous* stationary state of this equation. It remains to show that there are no other stationary states of any kind. However, this is an immediate consequence of the fact that (1.20) is the forward equation of an ergodic process. This applies equally well to the BGK model. □

2 Explicit form of the spatially homogeneous NESS

If the initial data f_0 for either (1.9) or (1.10) is spatially homogenous; i.e., translation invariant on Λ, then the solution $f(x,v,t)$ of (1.4) or (1.10) will be spatially homogeneous for all t, so that $f(x,v,t) = |\Lambda|^{-1}g(v,t)$ for a time dependent probability density $g(v,t)$. In this case,

$$M_f(x,v,t) = M_{T_f(t)} \quad \text{with} \quad T_g(t) = d^{-1}\int_{\mathbb{R}^d} |v|^2 g(v,t)\mathrm{d}v \ . \tag{2.1}$$

It is simplest to write down the stationary states for (1.10). Under the assumptions of stationarity and spatial homogeneity, (1.10) reduces to

$$[M_{T_\infty} - g] + \sum_{j-1}^{k} \eta_j [M_{t_j} - g] = 0$$

which immediately yields

2.1 LEMMA. *The unique normalized spatially homogeneous steady state solution of (1.10), $g_\infty(v)$, is given by*

$$g_\infty(v) = |\Lambda|^{-1}\left(1 + \sum_{j=1}^{k}\eta_j\right)^{-1}\left(M_{T_\infty} + \sum_{j=1}^{k}\eta_j M_{T_j}\right) \tag{2.2}$$

where T_∞ is given by (1.16).

An explicit formula can also be given in the kinetic Fokker-Planck case, but its derivation is not so immediate. Under the assumptions of stationarity and spatial homogeneity, (1.9) reduces to

$$\frac{\partial}{\partial t} g(v,t) = G_{T_g(t)} g(v,t)) + \sum_{j=1}^{k} L_j g(v,t) \tag{2.3}$$

where for $T > 0$, the operator G_T is given by (1.17).

By Lemma 1.1, any steady state solution g_∞ of (2.3) must satisfy

$$G_{T_\infty} g_\infty(v)) + \sum_{j=1}^{k} L_j g_\infty(v) = 0 \tag{2.4}$$

where T_∞ is given by (1.15).

The equation

$$\frac{\partial}{\partial t} g(v,t) = G_{T_\infty}[g](v,t) + \sum_{j=1}^{k} L_j g(v,t)\ , \tag{2.5}$$

is the forward Kolmogorov equation of a stochastic process v_t in $\mathbb{R}^d$ that has the following description: At random times t, arriving in a Poisson stream with rate η, there are interactions with the k reservoirs. When an interaction occurs, an index $j \in \{1,\dots,k\}$ is chosen with probability η_j/η, and then v_t jumps to a new point chosen from the distribution M_{T_j}. Between interactions with the reservoir, the particle diffuses, governed by the SDE

$$\mathrm{d}v_t = -v_t \mathrm{d}t + 2\sqrt{T_\infty}\mathrm{d}w_t \tag{2.6}$$

where w_t is a standard Brownian motion.

Consider a large time t_1, so that with very high probability, there has been at least one collision. Almost surely, there are at most finitely many collisions. Let $\hat{t}_1$ be the *last* collision before time t_1. Then at time $(\hat{t}_1)_+$, supposing the j-th reservoir is selected for the interaction, the conditional distribution is given by M_{T_j}. If one starts a solution of the SDE (2.6) with initial distribution M_{T_j}, at each later time the distribution is M_T for some T in between T_j and T_∞. This heuristic suggest that the invariant measure of the process governed by (2.5) is a convex combination of Gaussians M_T with $T \in [T_1, T_k]$, where we have assumed, without loss of generality that $T_1 < T_2 < \cdots < T_k$. More precisely, we expect an invariant density f_∞ (2.5) of the form

$$f_\infty(v) = |\Lambda|^{-1} \int_{[T_1,T_k]} M_T(v)\mathrm{d}\nu(T) \tag{2.7}$$

Of course, if $k > 2$ and $T_j = T_\infty$ for some $T_j \in (T_1, T_2)$, one can expect ν to have a point mass at $T_j = T_\infty$, since if at the last interaction with the reservoirs the particle has jumped to a point distributed according to M_{T_∞}, the diffusion does not change this distribution. Otherwise, we expect ν to be absolutely continuous, so that for some probability density $w(T)$ on $[T_1, T_k]$,

$$\mathrm{d}\nu(T) = w(T)\mathrm{d}T\ . \tag{2.8}$$

One has to be careful about conditioning a Markov process on a random time that depend on future events (such as the time of the *last* interaction with the reservoirs before time t_1). The heuristic argument put forward can be made rigorous in several ways, using the fact that the Poisson stream of interaction times is independent of the diffusion process. However, since we need the explicit form of the probability measure

ν in what follows, it is simplest to treat (2.7) as an ansatz, and to derive the form of ν. The next theorem gives the explicit form of the unique steady state g_∞ of (2.5) for $k = 2$.

2.2 THEOREM (Steady state formula)**.** *Suppose $k = 2$ with $T_1 < T_2$, and $\eta_1, \eta_2 > 0$. Then there is a unique steady state solution g_∞ of (2.5) which is given by (2.7) and (2.8) where*

$$w(T) = \begin{cases} \dfrac{\eta_1}{2(T_\infty - T_1)^{\eta/2}}(T_\infty - T)^{\eta/2-1} & T \in [T_1, T_\infty) \\ \dfrac{\eta_2}{2(T_2 - T_\infty)^{\eta/2}}(T - T_\infty)^{\eta/2-1} & T \in (T_\infty, T_2] \end{cases} . \tag{2.9}$$

Proof. Let

$$g(v) := \int_{T_1}^{T_2} w(T) M_T \mathrm{d}T \tag{2.10}$$

where $w(T)$ is a probability density on $[T_1, T_2]$. If g is to be a steady state solution of (2.5), then we must have

$$G_{T_\infty} g + \eta_1 M_{T_1} + \eta_2 M_{T_2} - \eta g = 0 \ . \tag{2.11}$$

Note that

$$G_{T_\infty} M_T = G_T M_T + (T_\infty - T)\Delta M_T = (T_\infty - T)\Delta M_T \ .$$

Then since

$$\frac{\partial}{\partial T} M_T(v) = 2\Delta M_T(v) \ ,$$

$$\begin{aligned} G_{T_\infty} g &= 2\int_{T_1}^{T_2} (T_\infty - T) w(T) \frac{\partial}{\partial T} M_T \mathrm{d}T \\ &= 2\int_{T_1}^{T_\infty} (T_\infty - T) w(T) \frac{\partial}{\partial T} M_T \mathrm{d}T \\ &+ 2\int_{T_\infty}^{T_2} (T_\infty - T) w(T) \frac{\partial}{\partial T} M_T \mathrm{d}T \ . \end{aligned} \tag{2.12}$$

Integrating by parts,

$$\begin{aligned} &\int_{T_1}^{T_\infty} (T_\infty - T) w(T) \frac{\partial}{\partial T} M_T \mathrm{d}T = \\ &\qquad - (T_\infty - T_1) w(T_1) M_{T_1} - \int_{T_1}^{T_\infty} \left(\frac{\partial}{\partial T}[(T_\infty - T) w(T)] \right) M_T \mathrm{d}T \ . \end{aligned} \tag{2.13}$$

Making a similar integration by parts in the last integral in (2.12), we obtain

$$G_{T_\infty} g = -2(T_\infty - T_1)w(T_1)M_{T_1} - 2(T_2 - T_\infty)w(T_2)M_{T_2} + 2\int_{T_1}^{T_2} \left(\frac{\partial}{\partial T}[(T_\infty - T)w(T)]\right) M_T \mathrm{d}T \ . \quad (2.14)$$

Then (2.11) implies that

$$(\eta_1 - 2(T_\infty - T_1)w(T_1))\, M_{T_1} + (\eta_2 - 2(T_2 - T_\infty)w(T_2))\, M_{T_2} + \int_{T_1}^{T_2} \left(2\frac{\partial}{\partial T}[(T_\infty - T)w(T)] - \eta w(T)\right) M_T \mathrm{d}T = 0 \ . \quad (2.15)$$

For $c_1, c_2 > 0$ to be determined, suppose

$$w(T) = \begin{cases} c_1(T_\infty - T)^{\eta/2-1} & T \in [T_1, T_\infty) \\ c_2(T - T_\infty)^{\eta/2-1} & T \in (T_\infty, T_2] \end{cases} \ . \quad (2.16)$$

Then

$$2\frac{\partial}{\partial T}[(T_\infty - T)w(T)] - \eta w(T) = 0$$

everywhere in $[T_1, T_\infty) \cup (T_\infty, T_2]$, and so defining $w(T_\infty) = 0$, for example, (2.15) reduces to

$$\left(\eta_1 - 2c_1(T_\infty - T_1)^{\eta/2}\right) M_{T_1} + \left(\eta_2 - 2c_2(T_2 - T_\infty)^{\eta/2}\right) M_{T_2} = 0 \ .$$

This is satisfied for all v if and only if

$$c_1 = \frac{\eta_1}{2(T_\infty - T_1)^{\eta/2}} \qquad \text{and} \qquad c_2 = \frac{\eta_2}{2(T_2 - T_\infty)^{\eta/2}} \ ,$$

which yields (2.9). The uniqueness is an immediate consequence of the fact that our equations is the forwards equation of an ergodic process; details are given in the proof of Theorem 1.2 below which gives an even stronger uniqueness result. □

Notice that when $w(T)$ is given by (2.9), then indeed

$$\int_{T_1}^{T_2} w(T)\mathrm{d}T = 1$$

for all $\eta > 0$. If we vary $\eta > 0$ but keep

$$p_1 = \frac{\eta_1}{\eta} \qquad \text{and} \qquad p_2 = \frac{\eta_2}{\eta}$$

fixed, then it is easy to see that

$$\lim_{\eta\to 0} w(T)\mathrm{d}T = \delta_{T_\infty} \qquad \text{and} \qquad \lim_{\eta\to\infty} w(T)\mathrm{d}T = p_1\delta_{T_1} + p_2\delta_{T_2}\ .$$

An entirely analogous analysis applies in the case $k > 2$, except that one must then take into account the possibility that $T_j = T_\infty$ for some j. This simply introduces a point mass at T_∞ into ν. We now prove Theorem 1.2.

3 Approach to the NESS

We continue our study of (1.18) and (1.19), and prove Theorem 1.3.

The starting point for the proof of Theorem 1.3 is based on the elementary fact, proved in Lemma 1.1, that

$$T_f(t) = T_\infty + e^{-t\eta}(T_f(0) - T_\infty)\ , \tag{3.1}$$

and that equation (1.18) is the Kolmogorov forward equation of a stochastic process with a time dependent generator, but one in which the time dependence damps out exponentially fast due to (3.1). Exploiting this and making use of a variant Doeblin's Theorem to control the rate at which memory of the initial data is lost is the basis of the proof that follows. The basic stategy is the one introduced in [1], though here we use quantitative estimates in place of compactness arguments. We only give the details in the case of kinetic Fokker-Planck equation (1.18); the other case is similar but simpler.

We preface the proof itself with a few further remarks on the strategy. On account of (3.1), (1.18) can be rewritten as

$$\frac{\partial}{\partial t} f(x,v,t) + \mathrm{div}_x(vf(x,v,t) = G_{T_\infty + e^{-t\eta}(T_f(0)-T_\infty)}[f](x,v,t) + \sum_{j=1}^{k} L_j f(x,v,t)\ . \tag{3.2}$$

For $t > s$, let $\widetilde{f}(x,v,t)$ be the solution of this equation started at time s with the data $\widetilde{f}(x,v,s) = f_\infty(v)$. Since $T_\infty + e^{-t\eta}(T_f(0) - T_\infty) \neq T_\infty$ (except in the trivial case $T_f(0) = T_\infty$), $\widetilde{f}(x,v,t)$ is not independent of t. However, an argument using Duhamel's formula, (3.1) and the regularity of f_∞ that is provided by Theorem 2.2 shows that the $L^1(\Lambda\times\mathbb{R}^d)$ distance between $\widetilde{f}(x,v,t)$ and $f_\infty(v)$ is bounded by a fixed multiple of e^{-s} for all $t > s$. Since a variant of Doeblin's Theorem may be applied to show that memory of initial data is lost at an exponential rate, for t much larger than s, there will only be a small difference between $f(x,v,t)$ and $\widetilde{f}(x,v,t)$, and hence only a small difference between $f(x,v,t)$ and $f_\infty(v)$. Choosing $s = t/2$ for large t then gives us the bound we seek.

We break the proof into several lemmas, after fixing notation. First, pick some large value t_0, to be specified later, but for now, take large to mean that $T_f(t_0)$ is very close to T_∞. Consider the stochastic process governed by (3.2) and started at time $s \geq t_0$ at the phase-space point $(x_0, v_0) \in \Lambda \times \mathbb{R}^d$ with probability 1. (A more detailed description of the process is provided below.) Let $\mathbb{P}_{s,(x_0,v_0)}$ be the law; i.e., the path-space measure of the stochastic process. For $t > s$, and measurable $A \subset \Lambda \times \mathbb{R}^d$, we are interested in the transition probabilities

$$P_{s,t}((x_0, v_0), A) = \mathbb{P}_{s,(x_0,v_0)}(\{(x_t, v_t) \in A\}) \tag{3.3}$$

as a function of (x_0, v_0), and aim to prove that the memory of (x_0, v_0) is lost at an exponential rate. The stochastic process consists of a Poisson stream of interactions with the thermal reservoirs, and an independent degenerate diffusion process between these interactions. We make use of a conditioning argument on the event described in the next lemma:

3.1 LEMMA. *Consider the event E that in this stochastic process there is exactly one interaction with the reservoirs in the time interval $(t_0, t_0 + 1/\eta]$, and none in the interval $(t_0 + 1/\eta, t_0 + 2/\eta]$. Then, independent of the initial data $(x_0, v_0) \in \Lambda \times \mathbb{R}^d$ at $s \in [t_0, t_0 + 1/\eta]$, the time of the interaction with the thermal reservoir,*

$$\mathbb{P}_{s,(x_0,v_0)}(E) = e^{-2} \ .$$

Proof. Since the interaction with the reservoirs occur in a Poisson stream with rate η, the probability of the event E is $\displaystyle\int_{t_0}^{t_0+1/\eta} \eta e^{-s\eta} e^{s\eta-2} \mathrm{d}s = e^{-2}$. □

When any interaction with the reservoirs takes place, the velocity before the interaction is replaced with a new velocity chosen according the the distribution $\eta^{-1}[\eta_1 M_1(v) + \eta_2 M_2(v)]$, independent of what the velocity was before.

Between interactions with the reservoirs, the motion of the particle is governed by the stochastic differential equation

$$\begin{aligned} \mathrm{d}v_t &= -v\mathrm{d}t + \sqrt{2T(t)}\mathrm{d}w_t \\ \mathrm{d}x_t &= \quad v_t \mathrm{d}t \end{aligned} \tag{3.4}$$

where $w(t)$ is a standard Wiener process, and $T(t) = T_\infty + e^{-t\eta}(T_f(0) - T_\infty)$.

We compute the distribution of (x, v) at time $t_0 + 2/\eta$ conditional on the even E defined above. Let b denote the new velocity after the interaction with the reservoir at time s, and let a denote the position x_s. Then the distribution we seek is the distribution of $(x_{t_0+2/\eta}, v_{t_0+2/\eta})$ for the solution of (3.4) started at $(x_s, v_s) = (a, b)$ with probability 1, averaged in a over the distribution of x_s and averaged in b over the distribution $\eta^{-1}[\eta_1 M_1(v) + \eta_2 M_2(v)]$.

We wish to determine the dependence of

$$\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}|E) \tag{3.5}$$

on (x_0, v_0). This has the following structure. Let $\mu_{(x_0,v_0),t_0,s}$, be the probability measure on Λ given by

$$\mu_{(x_0,v_0),t_0,s}(B) = \mathbb{P}_{t_0,(x_0,v_0)}(\{x_s \in B\}|E) \ ,$$

be the law of x_s at the time of the single collision in the interval $(t_0, t_0 + 1/\eta]$. Define $u = t_0 + 2/\eta - s$, and let $p_{b,u}(v)$ be the density for v_{s+u} where v_t is given by (3.4) with $v_s = b$ with probability 1. An explicit expression will be obtained below, but all that matters for us at present is that this is independent of a, x_0 and v_0.

For fixed a and b let $p_{a,b,s,s+u}(x, v)$ be the probability density of (x_{s+u}, v_{s+u}) for the solution of (3.4) started at $(x_s, v_s) = (a, b)$ with probability 1. We shall derive an explicit formula for this below. Our focus will be on the conditional probability density for x_{x+u} given v_{s+u}.

$$p_{a,b,s,s+u}(x|v) = \frac{p_{a,b,s,s+u}(x, v)}{p_{b,u}(v)} \ .$$

Now define the probability measure ν on $\mathbb{R}^d$ by

$$\mathrm{d}\nu(b) = \frac{1}{\eta}[\eta_1 M_1(b) + \eta_2 M_2(b)] \ ,$$

which is the law of the velocity immediately following an interaction with the reservoirs. Putting the components together, the probability in (3.5) is obtained by integrating

$$h_{(x_0,v_0)}(x, v) := e^2\eta \int_{t_0}^{t_0+1/\eta} \int_{\mathbb{R}^d} \left[\int_\Lambda [p_{a,b,s,s+u}(x|v)] \mu_{(x_0,v_0),t_0,s}(\mathrm{d}a) \right] p_{b,u}(v) \mathrm{d}\nu(b) \mathrm{d}s \tag{3.6}$$

over the set A. We shall obtain control over the (x_0, v_0) dependence in (3.5) by quantifying the rate of coupling as follows: Consider two copies of the process, started from (x_0, v_0) and (x_1, v_1) respectively. After the first interaction with the reservoir, the velocity variable jumps to a new velocity chosen independent of the starting point, and thus we obtain perfect coupling of the velocities at this time. It remains to consider the spatial coupling. To do this, we get bounds on the conditional spatial density, conditioning on the new velocity after the collision and the time of the collision. We aim to show that these conditional spatial densities all dominate a fixed multiple of one another. From this we get a fixed minimum amount of cancelation (the effect of coupling) when subtracting transition probabilities for any two phase space starting points. The following lemma allows us to combine velocity coupling and spatial coupling to get phase space coupling.

3.2 THEOREM. *Let $(X\times Y\times Z, \mathcal{F}_X\otimes\mathcal{F}_Y\otimes\mathcal{F}_Z)$ be a product of three measure spaces. Let ρ_1 and ρ_2 be two probability measures on this measure space. Suppose that their marginal distributions on $Y\times Z$ have the following form: There is a fixed probability measure ν on $(Z,\mathcal{F}_Z)$ and two probability measures μ_1,μ_2 on $(Y,\mathcal{F}_Y)$ such that for all $B\in\mathcal{F}_Y\otimes\mathcal{F}_Z$,*

$$\rho_j(X\times B) = \mu_j\otimes\nu(B)\ .$$

Suppose also that ρ_1 and ρ_2 possess proper conditional probabilities for Y,Z, so that there is a representation

$$\rho_j(C) = \int_Z \rho_j(C|y,z)\mathrm{d}\mu_j(y)\otimes\mathrm{d}\nu(z)$$

valid for all $C\in\mathcal{F}_X\otimes\mathcal{F}_Y\otimes\mathcal{F}_Z$. Suppose finally that there exists a constant $0<c<\infty$ such that for all $y,y'\in Y$, all $z\in Z$, and all $A\in\mathcal{F}_X$

$$c\rho_1(A|y,z)\le\rho_2(A|y',z)\le\frac{1}{c}\rho_1(A|y,z)\ . \tag{3.7}$$

Then

$$\sup\{|\rho_1(C)-\rho_2(C)|\ :\ C\in\mathcal{F}_X\otimes\mathcal{F}_Y\otimes\mathcal{F}_Z\ \}\le 1-c\ . \tag{3.8}$$

We shall apply this in (3.6) for any two different (x_0,v_0) and (x_1,v_1) in $\Lambda\times\mathbb{R}^d$ as follows: We take $Z=\mathbb{R}^d\times\mathbb{R}^d$, and let the z variable be (b,v).

Proof. Pick any $C \in \mathcal{F}_X \otimes \mathcal{F}_Y \otimes \mathcal{F}_Z$. Then

$$\begin{aligned}
\rho_1(C) &= \int_{Y\times Z} \left[\int_X 1_C(x,y,z)\rho_1(\mathrm{d}x|y,z)\right] \mathrm{d}\mu_1(y)\mathrm{d}\nu(z) \\
&= \int_Y \left(\int_{Y\times Z} \left[\int_X 1_C(x,y,z)\rho_1(\mathrm{d}x|y,z)\right] \mathrm{d}\mu_1(y)\mathrm{d}\nu(z)\right) \mathrm{d}\mu_2(y') \\
&= \int_Y \left(\int_{Y\times Z} \left[\int_X 1_C(x,y,z)\rho_1(\mathrm{d}x|y,z)\right] \mathrm{d}\mu_2(y')\mathrm{d}\nu(z)\right) \mathrm{d}\mu_1(y) \\
&\geq c\int_Y \left(\int_{Y\times Z} \left[\int_X 1_C(x,y,z)\rho_1(\mathrm{d}x|y',z)\right] \mathrm{d}\mu_2(y')\mathrm{d}\nu(z)\right) \mathrm{d}\mu_1(y) \\
&= c\rho_2(C)\ ,
\end{aligned}$$

where the first equality is trivial, the second is valid by the Fubini-Tonelli Theorem, and the inequality is (3.7). Then (3.8) follows directly. □

In the context of (3.6), we apply Theorem 3.2 for *fixed* $s \in [t_0, t_0 + 1/\eta]$ as follows: We take $Z = \mathbb{R}^d \times \mathbb{R}^d$ with variables b and v. We take $Y = \Lambda$ with variable a, and we take $X = \Lambda$ with variable x. We choose $C \in \mathcal{F}_X \otimes \mathcal{F}_Z$, and in fact, independent of the b component of Z. In the next subsection we shall derive an explicit formula for the conditional probability $p_{a,b,s,s+u}(x|v)$, and along the way, $p_{b,u}(v)$. Using this formula, we shall show that (3.7) is satisfied. The constant c in (3.7) will be shown to depend only on T_∞ and η. Then, integrating in s, we shall have proved:

3.3 LEMMA. *For fixed a and b, and for fixed $u > 0$, let $p_{a,b,s,s+u}(x,v)$ be the probability density of (x_{s+u}, v_{s+u}) for the solution of (3.4) started at $(x_s, v_s) = (a,b)$ with probability 1. Let $p_{a,b,s,s+u}(x|v)$ be the conditional density of x_{s+u} given $v_{s+u} = v$. Then there is an explicitly computable constant $C_{\eta,T_\infty,|\Lambda|} > 0$ depending only on η, T_∞ and $|\Lambda|$ and an explicitly computable $s_0 < \infty$ such that for all $s \geq s_0$ and all $u \in [1/\eta, 2/\eta]$, the following bound holds uniformly in $x \in \Lambda$:*

$$C_{\eta,T_\infty,|\Lambda|} \leq p_{a,b,s,s+u}(x|v) \leq \frac{1}{C_{\eta,T_\infty,|\Lambda|}}\ . \tag{3.9}$$

The proof is given at the end of the next subsection.

3.1 Estimates for the degenerate diffusion

We now derive an explicit formula for this probability, which will be an analog of a well-known formula obtained by Kolmogorov [7] in 1934. The variant we need, taking into account our time-varying temperature, is easily derived using stochastic calculus as done by McKean in [8]. We will write down the formula in the case $d = 1$ in order to keep the notation simple. It will be clear from these formulas and their derivation that the conclusions we draw from them are valid in all dimensions.

If $x(s) = a$ and $v(s) = b$, we have that for $t > s$

$$v_t = e^{s-t}b + \int_s^t e^{r-t}\sqrt{2T(r)}\mathrm{d}w_r \ . \tag{3.10}$$

In our application, Λ is a circle of circumference L. However, we may solve (3.4) for $x_t \in \mathbb{R}$, and then later "wrap" this into the circle. So for the moment, let us take $x_t \in \mathbb{R}$. We may then integrate to find x_t. To do so, note that

$$\int_s^t \int_s^u e^{r-u}\sqrt{2T(r)}\mathrm{d}w_r\mathrm{d}u = \int_s^t (1 - e^{r-t})\sqrt{2T(r)}\mathrm{d}w_r \ .$$

Therefore,

$$x_t = a + (1 - e^{s-t})b + \int_s^t (1 - e^{r-t})\sqrt{2T(r)}\mathrm{d}w_r \ . \tag{3.11}$$

The random variables x_t and v_t are evidently Gaussian, and their joint distribution is determined by their means, variances, and correlation. Let μ_x and μ_v denote the means of x_t and v_t respectively. Let σ_x and σ_v denote the standard deviations of x_t and v_t respectively. Finally, let ρ denote their correlation, which means that

$$\rho\sigma_x\sigma_v = \mathbb{E}(x_t - \mu_x)(v_t - \mu_v) \ .$$

The joint density of (x_t, v_y), again taking x_t to be $\mathbb{R}$ valued, is the probability density given by

$$f(x,v) = \frac{1}{2\pi\sigma_x\sigma_v\sqrt{1-\rho^2}} \times \exp\left(\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(v-\mu_v)^2}{\sigma_v^2} - \frac{2\rho(x-\mu_x)(v-\mu_v)}{\sigma_x\sigma_v}\right]\right) \ . \tag{3.12}$$

Completing the square, we obtain the alternate form

$$f(x,v) = \frac{1}{2\pi\sigma_x\sigma_v\sqrt{1-\rho^2}} \times \exp\left(\frac{-1}{2(1-\rho^2)}\left[\left(\frac{(x-\mu_x)}{\sigma_x} - \rho\frac{(v-\mu_v)}{\sigma_v}\right)^2 + (1-\rho^2)\frac{(v-\mu_v)^2}{\sigma_v^2}\right]\right) \ . \tag{3.13}$$

Evidently, the conditional density of x_t given $v_t = v$ is

$$f(x|v) := \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left(\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)}{\sigma_x} - \rho\frac{(v-\mu_v)}{\sigma_v}\right)^2\right) \ , \tag{3.14}$$

and the density of v_t is

$$f(v) = \frac{1}{\sigma_v\sqrt{2\pi}} \exp\left(-\frac{(v-\mu_v)^2}{2\sigma_v^2}\right) \ . \tag{3.15}$$

Note that $f(x,v) = f(x|v)f(v)$. Now we wrap this density onto Λ which we identify with the interval $(-L/2, L/2] \subset \mathbb{R}$. The wrapped density is, for $x \in (-L/2, L/2]$,

$$\widetilde{f}(x,v) = \sum_{k\in\mathbb{Z}} f(x+kL|v)f(v) \ . \tag{3.16}$$

Recall that for fixed a and b, and for fixed $u > 0$, we have defined $p_{a,b,s,s+u}(x,v)$ to be the probability density of (x_{s+u}, v_{s+u}) for the solution of (3.4) started at $(x_s, v_s) = (a,b)$ with probability 1. By the computations above, $p_{a,b,s,s+u}(x,v)$ is obtained by substituting $\sigma_x(s+u)$, $\sigma_v(s+u)$ and $\rho(s+u)$ into (3.16). We now compute these quantities using the Ito isometry for the variance and covariance.

From (3.10) and (3.11) we readily compute

$$\mu_x = a + (1-e^{s-t}b) \quad \text{and} \quad \mu_v = e^{s-t}b \ . \tag{3.17}$$

Next, we compute:

$$\begin{aligned} \sigma_v^2(s+u) &= \int_s^{s+u} e^{2(r-(s+u))} 2T(r)\mathrm{d}r \\ &= (1-e^{-2u})T_\infty + \frac{2(T_{f(0)} - T_\infty)}{2-\eta} e^{-\eta s}(e^{-\eta u} - e^{-2u}) \ . \end{aligned} \tag{3.18}$$

(Note that a limit must be taken at $\eta = 2$.)

$$\sigma_x^2(s+u) = \int_s^{s+u} (1-e^{r-(s+u)})^2 2T(r)\mathrm{d}r \tag{3.19}$$

$$\begin{aligned} &= 4(e^{-u} - 1 + u)T_\infty - (e^{-2u} - 1 + 2u)T_\infty \\ &+ \frac{2(T_{f(0)} - T_\infty)}{\eta(\eta-1)(\eta-2)} \times \\ &\quad e^{-\eta s}(\eta^2(1-e^{-u})^2 - \eta(1-e^{-u})(3-e^{-u}) + 2(1-e^{-\eta u})) \ . \end{aligned} \tag{3.20}$$

(Note that a limit must be taken at $\eta = 1$ or $\eta = 2$. Note also that $\sigma_x^2(s+u)$ is of order u^3 for small u, as one would expect from Kolmogorov's formula, and of order u for large u.) Next, we compute the covariance. For $t = s + u$,

$$\begin{aligned}\mathbb{E}(x_t - \mu_x)(v_t - \mu_v) &= \int_s^t e^{r-t}(1 - e^{r-t})2T(r)\mathrm{d}r \\ &= T_\infty(1 - e^{-u})^2 \\ &+ \frac{2(T_{f(0)} - T_\infty)}{(\eta - 1)(\eta - 2)} e^{-\eta s}((\eta - 2)(1 - e^{-u}) - (\eta - 1)e^{-2u} + e^{-\eta u}) \ .\end{aligned} \tag{3.21}$$

In our application, we are concerned with $s \in (t_0 + 1/\eta]$ and $t = t_0 + 2/\eta$, and hence with $u \in [1/\eta, 2/\eta]$. For such u, and large t_0 (and hence large s, the following approximations are accurate up to exponentially small (in t_0) percentage-wise corrections:

$$\sigma_v^2(s+u) \approx (1 - e^{-2u})T_\infty \tag{3.22}$$

$$\sigma_x^2(s+u) \approx [4(e^{-u} - 1 + u) - (e^{-2u} - 1 + 2u)]T_\infty \tag{3.23}$$

$$\rho(s+u) \approx \frac{(1 - e^{-u})^2}{\sqrt{(1 - e^{-2u})(4(e^{-u} - 1 + u) - (e^{-2u} - 1 + 2u))}} \ . \tag{3.24}$$

Define $\hat{\rho}(s+u)$ to be the quantity on the right side of (3.24). One readily checks that

$$\hat{\rho}(s+u) = \frac{\sqrt{3}}{2} + \mathcal{O}(u)$$

for small u, that $\hat{\rho}(s+u)\sqrt{1 + u/5}$ is monotone decreasing, and that $\lim_{u\to\infty} \hat{\rho}(s + u)\sqrt{1 + u/5} = 1/\sqrt{10}$. Altogether,

$$\frac{\sqrt{3}}{2}\frac{1}{\sqrt{5+u}} \geq \hat{\rho}(s+u) \geq \frac{1}{\sqrt{2}}\frac{1}{\sqrt{5+u}} \ . \tag{3.25}$$

It now follows that for all s sufficiently large, and all $u \in [1/\eta, 2/\eta]$,

$$\frac{\sqrt{3}}{2}\frac{1}{\sqrt{4 + 1/\eta}} \geq \rho(s+u) \geq \frac{1}{\sqrt{2}}\frac{1}{\sqrt{6 + 2\eta}} \ . \tag{3.26}$$

Likewise, define $\hat{\sigma}_x^2(s+u)$ to be the right side of (3.23). Simple calculation show that $\hat{\sigma}_x^2(s+u)(1+2u)^2/u^3$ is monotone for $u > 0$ increasing with

$$8T_\infty \geq \hat{\sigma}_x^2(s+u)(1+2u)^2/u^3 \geq \frac{2}{3}T_\infty \ .$$

It now follows that for all s sufficiently large, and all $u \in [1/\eta, 2/\eta]$,

$$64\frac{\eta^{-3}}{(1+2/\eta)^2}T_\infty \geq \sigma_x^2(s+u) \geq \frac{2}{\sqrt{3}}\frac{\eta^{-3}}{(1+5/\eta)^2}T_\infty \ . \tag{3.27}$$

Proof of Lemma 3.3. By the computations above, for any $x \in [-L/2, L/2]$ by substituting into

$$\sum_{k\in\mathbb{Z}} \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left(\frac{-1}{2(1-\rho^2)}\left(\frac{(x+kL-\mu_x)}{\sigma_x} - \rho\frac{(v-\mu_v)}{\sigma_v}\right)^2\right)$$

the appropriate values of μ_x, μ_v, σ_x, σ_v and ρ, given by (3.17), (3.18), (3.19) and (3.21).

Now, whatever, the value of v, there is some $k \in Z$ so that

$$|x + kL - \mu_x - (\sigma_x/\sigma_v)\rho(v-\mu_v)| \leq L/2 \ .$$

Retaining only this term in the sum,

$$p_{a,b,s,s+u}(x|v) \geq \frac{1}{\sigma_x\sqrt{2\pi(1-\rho^2)}} \exp\left(\frac{-L^2}{8(1-\rho^2)\sigma_x^2}\right) \ .$$

Now using the upper and lower bounds for ρ and σ_x that are given in (3.26) and (3.27), which are valid for all $s \geq s_0$ that may be explicitly computed keeping track of constants leading up to (3.26) and (3.27). The corresponding uniform upper bound is readily derived by estimating the sum in k, which converges extremely rapidly. □

It is evident from the proof that the analogous lemma for the d dimensional version of our process is also valid.

3.2 Bounds on transition functions

Recall that for any $t > s \geq t_0 > 0$, and any measurable $A \subset \Lambda \times \mathbb{R}^d$, and any $(x_0, v_0) \in \Lambda \times \mathbb{R}^d$, $P_{t_0,t}((x_0,v_0), A)$ is the probability that our original stochastic process (with interactions with the reservoirs), started at (x_0, v_0) at time s satisfies $(x_t, v_t) \in A$. Let $\mathbb{P}_{s,(x_0,v_0)}$ be the path-space measure of the stochastic process. Then, with E being the event considered in Lemma 3.1,

$$\begin{aligned} P_{t_0,t}((x_0,v_0), A) &= \mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t,v_t) \in A\}) \\ &= \mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t,v_t) \in A\} \cap E) + \mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t,v_t) \in A\} \cap E^c) \end{aligned} \tag{3.28}$$

By Lemma 3.1, we can express $\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t,v_t) \in A\} \cap E)$ in terms of the conditional probabilities we have estimated in the previous subsections:

$$\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t,v_t) \in A\} \cap E) = e^{-2}\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t,v_t) \in A \mid E) \ .$$

3.4 THEOREM. *There is an explicitly computable constant $C_{\eta,T_\infty,|\Lambda|} > 0$ depending only on η and T_∞ and an explicitly computable $t_0 < \infty$ such that for $t = t_0 + 2/\eta$, and all (x_0, v_0) and (x_1, v_1) in $\Lambda \times \mathbb{R}^d$*

$$\sup\{|\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}) - \mathbb{P}_{t_0,(x_1,v_1)}(\{(x_t, v_t) \in A\})|\} \leq 1 - e^{-2}C_{\eta,T_\infty,|\Lambda|}\ ,$$

where the supremum is taken over all measurable subset of $\Lambda \times \mathbb{R}^d$,

Proof. By Lemma 3.1,

$$\begin{aligned}&\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\} = \\ &\qquad e^{-2}\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}|E) + (1 - e^{-2})\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}|E^c)\ . \end{aligned} \tag{3.29}$$

By Lemma 3.3, (3.7) is satisfied with $c = C_{\eta,T_\infty,|\Lambda|}$ when we apply Theorem 3.2 to the probabilities $\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\})$ given by (3.5) and (3.6). Therefore,

$$\begin{aligned}&|\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}) - \mathbb{P}_{t_0,(x_1,v_1)}(\{(x_t, v_t) \in A\})|\\ &\leq e^{-2}|\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}|E) - \mathbb{P}_{t_0,(x_1,v_1)}(\{(x_t, v_t) \in A\}|E)|\\ &+ (1 - e^{-2})|\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}|E^c) - \mathbb{P}_{t_0,(x_1,v_1)}(\{(x_t, v_t) \in A\}|E^c)|\\ &\leq e^{-2}\Big|\mathbb{P}_{t_0,(x_0,v_0)}(\{(x_t, v_t) \in A\}|E) - \mathbb{P}_{t_0,(x_1,v_1)}(\{(x_t, v_t) \in A\}|E)\Big| + (1 - e^{-2})\\ &\leq e^{-2}(1 - C^2_{\eta,T_\infty,|\Lambda|}) + (1 - e^2) = 1 - e^{-2}C^2_{\eta,T_\infty,|\Lambda|})\ .\end{aligned}$$

Theorem 3.4 now follows from what has been said in the paragraph preceding it. □

We next recall a form of Doeblin's Theorem in the context of temporally non-homogeneous processes.

Let $P_{s,t}(z, A)$, $t > s$, be a family of Markov kernels on a measure space $(Z, \mathcal{F})$ such that for $r < s < t$

$$P_{r,t}(z, A) = \int_Z P_{r,s}(z, \mathrm{d}y)P_{s,t}(y, A)\ , \tag{3.30}$$

and we suppose also that for fixed A, s and t, $P_{s,t}(z, A)$ is a continuous function of z. Define the quantity

$$\rho_{s,t} = \sup_{z,y\in Z}\sup_{A\in\mathcal{F}}\{|P_{s,t}(z, A) - P_{s,t}(y, A)|\}\ .$$

The following lemma is an adaptation of a proof in Varadhan's text [9]:

3.5 LEMMA. *For all $r < s < t$, $\rho_{r,t} \leq \rho_{r,s}\rho_{s,t}$.*

Proof. By (3.30),

$$|P_{r,t}(z,A) - P_{r,t}(y,A)| = \left| \int_Z P_{r,s}(z,\mathrm{d}w) P_{s,t}(w,A) - \int_Z P_{r,s}(y,\mathrm{d}w) P_{s,t}(w,A) \right| \quad (3.31)$$

To write this more compactly, introduce the continuous function $f(w) = P_{s,t}(w,A)$, and let ν denote the signed measure

$$\mathrm{d}\nu(w) = P_{r,s}(z,\mathrm{d}w) - P_{r,s}(y,\mathrm{d}w) \ .$$

Then (3.31) becomes

$$|P_{r,t}(z,A) - P_{r,t}(y,A)| = \left| \int_Z f(w)\mathrm{d}\nu(w) \right| \ . \quad (3.32)$$

Define $\|\nu\| = \sup_{A\in\mathcal{F}} |\nu(A)|$. Then if $\nu = \nu_+ - \nu_-$ is the Hahn decomposition of ν into its positive and negative parts, $\nu_+(Z) = \nu_-(Z)$ (since $\nu(Z) = 0$), and

$$\|\nu\| = \nu_+(Z) \ .$$

Let $\mathrm{d}|\nu|$ be the measure defined by $\mathrm{d}|\nu| = \mathrm{d}\nu_+ + \mathrm{d}\nu_-$. Then

$$|\nu|(X) = 2\|\nu\| \le 2\rho_{t,s} \ . \quad (3.33)$$

If φ is any continuous function on Z, we have

$$\left| \int_Z \varphi \mathrm{d}\nu \right| \le \int_Z |\varphi| \mathrm{d}|\nu| \le \|\varphi\|_\infty |\nu|(Z) = 2\|\varphi\|_\infty \|\nu\| \ . \quad (3.34)$$

By hypothesis

$$|f(z_1) - f(z_2)| \le \rho_{s,r} \quad \text{for all} \quad z_1, z_2 \in Z \ .$$

It follows that the range of f is contained in an interval of width at most $\rho_{s,r}$, and hence

$$\inf_{c\in\mathbb{R}} \|f - c\|_\infty \le \frac{1}{2}\rho_{s,r} \ . \quad (3.35)$$

Since $P_{s,t}(z,\mathrm{d}w)$ and $P_{s,t}(y,\mathrm{d}w)$ are both probability measures, for any constant $c \in \mathbb{R}$,

$$\int_Z f(w)\mathrm{d}\nu(w) = \int_Z (f(w) - c)\mathrm{d}\nu(w) \ , \quad (3.36)$$

and hence (3.32) becomes

$$\begin{aligned} |P_{r,t}(z,A) - P_{r,t}(y,A)| &= \inf_{c\in\mathbb{R}} \left| \int_X (f(w) - c)\mathrm{d}\nu(w) \right| \\ &\le \inf_{c\in\mathbb{R}} \|f - c\|_\infty |\nu(Z)|. \end{aligned}$$

Now using the estimate (3.33) and (3.35) we obtain the result. □

The following lemma is now a direct consequence of Theorem 3.4 and the definitions made just above.

3.6 LEMMA. *Let $P_{s,t}((x,v),A)$ be the transition function for the process associated to (3.2). There exist explicitly computable $\delta > 0$, $t_0 > 0$ and $t_1 > 0$ such that for all A and all $t > s \geq t_0$ with $t - s \geq t_1$,*

$$\rho_{s,t} \leq 1 - \delta \ . \tag{3.37}$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 for the kinetic Fokker-Planck equation. Combining Lemma 3.5 (For $Z = \Lambda \times \mathbb{R}^d$) and Lemma 3.6, whenever $t - t_0 \geq nt_1$,

$$\rho_{t_0,t} \leq (1-\delta)^n \ .$$

Let $\mu^{(1)}$ and $\mu^{(2)}$ are any two probability measures on $\Lambda \times \mathbb{R}^d$, and use them to initialize the the Markov process associated to our family of transition kernels at time t_0. For $j = 1, 2$ define

$$\mu_t^{(j)}(A) = \int_{\Lambda\times\mathbb{R}^d} \mathrm{d}\mu_j(z) P_{t_0,t}(z, A) \ .$$

Then for all $t > t_0$

$$\|\mu_t^{(1)} - \mu_t^{(2)}\| \leq \rho_{t_0,t} \ .$$

Thus, when (3.37) is valid, the memory of the initial condition is washed out exponentially fast. Of course this by itself does not imply convergence to a stationary state, and if the time inhomogeneity of our process had a strong oscillatory character, for example, we would not expect convergence to a stationary state. However, our process has an asymptotic temporal homogeneity property; namely the generator converges exponentially fast to the generator of a stationary process with an invariant measure $\mu_\star$. We use this to show that for any μ_0, μ_t converges exponentially fast to $\mu_\star$.

Let $R_{s,t}(z, A)$ be a homogenous family of Markov kernels on $(Z, \mathcal{F})$, where in our case $Z = \Lambda \times \mathbb{R}^d$. That is, for $s < t < u$

$$R_{s,u}(z, A) = \int_Z R_{s,t}(z, \mathrm{d}z') R_{t,u}(z', A) \ .$$

In our application, $R_{s,t}(z, A)$ will be the transition kernel for the Markov process corresponding to (1.20), however, it is convenient to continue the discussion in a general context for now.

Let L be the generator for the process governed by $R_{s,t}(z,A)$. Then for $s<t$, the Kolmogorov backward equation

$$-\frac{\partial}{\partial s}R_{s,t}(z,A) = LR_{s,t}(z,A)\ , \tag{3.38}$$

with the final condition $R_{t,t}(z,A) = 1_A(z)$, where 1_A is the indicator function of A. For $t>s$, $R_{s,t}(z,\cdot)$, considered as a time dependent measure, satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial t}R_{s,t}(z,\cdot) = L^*R_{s,t}(z,\cdot)\ , \tag{3.39}$$

where L^* is the adjoint of L, meaning that for any bounded continuous function φ,

$$\int L^*R_{s,t}(z,\mathrm{d}z')\varphi(z') = \int R_{s,t}(z,\mathrm{d}z')L\varphi(z')\ . \tag{3.40}$$

Let K_t be the generator of the inhomogeneous process described by $P_{s,t}$. Note that for any fixed measurable set A in phase space, $P_{s,t}((x,v),A)$ with $s<t$, satisfies the Kolmogorov backward equation

$$-\frac{\partial}{\partial s}P_{s,t}((x,v),A) = K_sP_{s,t}((x,v),A)\ , \tag{3.41}$$

Then by the Fundamental Theorem of Calculus, (3.39), (3.40) and (3.41)

$$\begin{aligned}P_{s,t}(z,A) - R_{s,t}(z,A) &= -\int_s^t\left(\frac{\mathrm{d}}{\mathrm{d}u}\int R_{s,u}(z,z')P_{u,t}(z,,A)\mathrm{d}z\right)\mathrm{d}u\\ &= -\int_s^t\left(\int R_{s,u}(z,z')(L-K_u)P_{u,t}(z,A)\mathrm{d}z\right)\mathrm{d}u\end{aligned}$$

Let $B_t = K_t - L$, the difference of the generators. Then we can rewrite the result of this computation as

$$P_{s,t}(z,A) - R_{s,t}(z,A) = \int_s^t\left(\int R_{s,u}(z,z')B_uP_{u,t}(z,A)\mathrm{d}z\right)\mathrm{d}u\ . \tag{3.42}$$

For any measure μ, let $\mu P_{s,t}$ denote the measure $\mu P_{s,t}(A) = \int \mathrm{d}\mu(z)P_{s,t}(z,A)$, and likewise for $\mu R_{s,t}$. Then if $\mu_\star$ is an invariant measure for R, meaning that $\mu_\star = \mu_\star R_{s,t}$, we have from (3.42) that

$$\mu_\star P_{s,t}(A) - \mu_\star(A) = \int_s^t\left(\int \mu_\star(z)B_uP_{u,t}(z,A)\mathrm{d}z\right)\mathrm{d}u\ . \tag{3.43}$$

We now apply the general formula (3.43) in our specific context, in which $B_u = (T(u) - T_\infty)\Delta_v$ where $(T(s) - T_\infty) = Ce^{-cu}$ for some C and $c > 0$ and $\mu_\star = f_\infty(v)\mathrm{d}v$. Then

$$\|B_u^*\mu_\star\| \le Ce^{-cu}\|\Delta_v f_\infty\|_{L^1}\ . \tag{3.44}$$

Crucially, $\|\Delta_v f_\infty\|_{L^1} < \infty$ by the explicit formula we have found for f_∞. Integrating, for all $t < t_2$,

$$\left\|\int_t^{t_2} (B_s^*\mu_\star)P_{s,t_2}\mathrm{d}s\right\| \le \int_t^{t_1} Ce^{-cu}\mathrm{d}u \le \frac{C}{c}e^{-ct}\ .$$

Consequently,

$$\|\mu_\star P_{t,t_2} - \mu_\star\| \le \frac{C}{c}e^{-ct}\ .$$

Now for any probability measure μ_0 on phase space, and any $s > t_0$ and any $t > t_0 + n(t_1 - t_0) + s$,

$$\|\mu_0 P_{0,t} - \mu_\star\| = \|(\mu_0 P_{0,s})P_{s,t} - \mu_\star P_{s,t}\| + \|\mu_\star P_{s,t} - \mu_\star\|$$

By Lemma 3.6, $\|(\mu_0 P_{0,s})P_{s,t} - \mu_\star P_{s,t}\| \le (1-\delta)^n$. By the computations above, $\|\mu_\star P_{s,t} - \mu_\star\| \le (C/c)e^{-cs}$.

Now for $t > t_0$, we have that

$$\|\mu_0 P_{0,t} - \mu_\star\| \le (C/c)e^{(t-t_0)/2} + \frac{1}{1-\delta}e^{\ln(1-\delta)(t-t_0)/2}\ .$$

This gives us the exponential relaxation. □

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