Dissipative stationary plasmas: Kinetic modeling, Bennett’s pinch and generalizations

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The structure of the self-consistent electromagnetic fields, $E$, $B$, and one-particle distribution functions, $f_s$, of a stationary dissipative unbounded $S$-species plasma, which satisfy a system of Maxwell–Vlasov–Fokker–Planck–Boltzmann equations with velocity-independent (effective) dissipation coefficients and applied constant electric field $\mathcal{E}$, is studied. It is proven that when the plasma is invariant along the direction of $\mathcal{E}$, then (a) the $f_s$ are given by $S$ drifting Maxwell–Boltzmann distributions, with densities satisfying self-consistent Poisson–Boltzmann equations of the type first considered by Bennett for $S=2$; (b) all smooth radial current profiles with normalized particle densities satisfy a generalized Bennett relation; (c) Bennett’s current profile is the only fully conformally invariant one; (d) there exist other radial solutions with profiles different from Bennett’s.

I. INTRODUCTION

Starting with Bennett’s seminal work the one-particle distribution functions of stationary current-carrying multi-component plasmas are frequently postulated to be of drifting Maxwell–Boltzmann type, e.g., Refs. 2–5. While such distribution functions satisfy the time-independent Vlasov equations, see, e.g., Ref. 2, they are not the only ones to do so and Vlasov theory, being nondissipative, does not discriminate between this and infinitely many other choices. Heuristic equilibrium statistical mechanical arguments do select the drifting Maxwell–Boltzmann distribution from other solutions, but the significance of this is unclear since the real plasma is not in thermal equilibrium, its current being a dissipative process in response to an external electric field.

In this paper we consider a system of $S$ stationary dissipative Maxwell–Vlasov–Fokker–Planck–Boltzmann equations with very simple dissipation terms in an externally imposed constant electric field $\mathcal{E}$. Similar dissipation terms are found in Refs. 8 and 9. Although somewhat simplistic, these equations are a convenient starting point for the mesoscopic modeling of spatially inhomogeneous current-carrying dissipative plasmas. We present the model equations in Sec. II. In Sec. III we give the proof that their $z$-invariant solutions are of the drifting Maxwell–Boltzmann form, with species-dependent mean velocity $c_s$ proportional to $\mathcal{E}$, and with Boltzmann factors given self-consistently by the solutions of a coupled pair of nonlinear Poisson–Boltzmann equations. At this point the problem is equivalent to the $S$-isothermal Bennett problem with its postulated mean drift speeds. The rest of the paper is devoted to the study of pinched normalizable solutions of the Poisson–Boltzmann equations in all space.

We derive in Sec. IV a priori conditions on the parameters for the existence of regular radial solutions with finite numbers of particles of species $s$ per unit length, $N_s$. In particular, we show that all such solutions have to satisfy certain sum rules, i.e., (36), from which we extract the “Bennett relation” (generalized to $S$ species),

$$c^{-2}l^2 - Q^2 = \sum_{s=1}^{S} N_s k_B T_s,$$

(1)

where $Q$ is the total charge per unit length along the invariant direction and $l$ the total current; $T_s$ is the temperature for species $s$; $c$ is the vacuum speed of light, and $k_B$ is Boltzmann’s constant. As pointed out by the referee, (1) corresponds to a radial virial identity; see Appendix A. Our derivation is based directly on the Poisson–Boltzmann equations. We then turn, in Sec. V, to Bennett’s pinch solution which was derived by him under the hypothesis that the ratio of the positive and negative species densities is constant in space. This required extra conditions on the parameters. These conditions are here shown to be precisely those under which the Poisson–Boltzmann equations are invariant under the full Euclidean conformal group. We further show that when such invariance holds, then for any $S>1$ all normalizable radial profiles are given by Bennett’s pinch. Bennett’s pinch thus follows from considerably weaker requirements than Bennett’s pointwise condition; cf. Refs. 1 and 2. In Sec. VI we address the general radial symmetric case. After mapping the problem to an equivalent Hamiltonian particle problem (see also Appendix B) in two dimensions, for $S=2$ solutions are shown to exist. Their general structure is different from Bennett’s solution, with positive and negative species having different asymptotic rate of decay. Concluding remarks are found in Sec. VII.

II. THE MODEL EQUATIONS

We consider a classical, nonrelativistic $S$-component plasma having charges $q_s$ and masses $m_s$. The time evolution of the one-particle distribution functions $f_s(x,v,t)$ are given by
we look for stationary normalizable solutions of (2) with φ, Eₖ, and A, B₀ independent of z, whence ∇ ⋅ B = 0. Maxwell's equations, stationarity, and our sign convention for εₖ then imply

\[ \frac{\partial E}{\partial t} = \frac{\partial B}{\partial x}, \quad \frac{\partial B}{\partial t} = -\nabla \times E \]  

\[ \nabla \cdot B = 0, \quad -\Delta A = 4 \pi c^{-1} \epsilon₀ \cdot j, \]  

\[ \nabla \times E = 4 \pi \epsilon₀ \cdot j. \]  

For simplicity we continue to use x now for the spatial variables in z direction. At |x| infinity, the particle densities are required to decay to zero so that they are R² normalizable, and the magnetic and electric fields approach their leading multipole values with no sources at infinity. We can thus solve (11) and (12) formally to obtain

\[ \phi - \phi₀ = -2 \ln r \rho; \quad c(\Delta - A₀) = -2 \ln r \mathbf{j}, \]  

where * denotes the standard convolution, r = |x|, j = j|e₀, and φ₀, A₀ are arbitrary gauge constants which are not determined by (11) and (12).

The z-independent stationary form of (2) and (6) can now be cast into the form

\[ m_s \partial f_s = 0 = m_s \partial f_s (\nabla f_s) - q_s \partial \epsilon_s \frac{1}{c} (\nabla (\phi - u_A A) + c^{-1} (\nabla - c_s) \times B) \]  

\[ + \partial \epsilon_s f_s \hat{F}_s (\nabla - c_s)] + m_s \epsilon_s (f_s, f_s), \]  

where

\[ c_s = (m_s \gamma_s)^{-1} q_s \gamma_s = cu_s e₀. \]  

A straightforward calculation verifies that the drifting Maxwell-Boltzmann distribution

\[ F_s (x, v) = N_s (\beta_s m_s / 2 \pi)^{3/2} \exp \left( -\beta_s \frac{1}{2} m_s (v - c_s)^2 \right) \]  

\[ + q_s (\phi(x) - u_s A(x)) \]  

is a normalized solution of (15) in which there are N_s particles of species s per unit length.

To show that (17) is the unique solution to (15), assume there exists another normalizable solution, f_s, of (15). The vanishing of the time derivative of its relative entropy (w.r.t. F_s) gives after some partial integrations (with vanishing boundary integrals)

\[ 0 = -\frac{\partial f_s}{\partial R^2} \left[ f_s \frac{\partial \epsilon_s [\phi(x) - u_s A(x)]}{F_s} \right] d^3x d^3v \]  

is a normalized solution of (15) in which there are N_s particles of species s per unit length.

III. TRANSLATION INVARIANT STATIONARY DISTRIBUTIONS

Setting

\[ E = -\nabla \phi + E₀ e₀, \]  

\[ B = \nabla A \times e₀ + B₀ e₀, \]  

we shall not consider such non-normalizable cases in the present work.
where we used the fact that $F_s$ annihilates the following three operators in (15) separately; the convective (Vlasov) derivative; the Fokker-Planck operator; the Boltzmann-type operator $C_s$. Since $C_s$ gives rise to a local $H$ theorem, the right-hand side of (19) is manifestly non-negative. It vanishes only if $f \cdot h_s(x) F_s$. But $f_s$ solves (15), whence $\nabla \cdot h_s = 0$, implying $h_s = \text{const}$. Therefore (17) is the only solution of (15) with the given normalization.

The problem of determining stationary, $z$-invariant solutions of (2)-(6) thus reduces to solving (11)-(13) with their right-hand side given by the zeroth and first velocity moments of (17). Then (13) becomes $\nabla B_s = 0$, whence

$$B_s = \mathcal{S} \cdot e_s$$

(20)

and (11) and (12) turn into the coupled pair of nonlinear integrodifferential equations

$$-\Lambda \phi = 4 \pi \sum_s N_s q_s Z_s^{-1} \exp[-\beta_s q_s (\phi - u_s A)],$$

(21)

$$-\Delta A = 4 \pi \sum_s N_s q_s u_s Z_s^{-1} \exp[-\beta_s q_s (\phi - u_s A)]$$

(22)

with asymptotic source free conditions (14). Any solution pair $(A, \phi)$ of (21) and (22) for parameters $N_s$, $\beta_s$, and $u_s$ given by (16), generates a stationary, translation invariant solution of Eqs. (2)-(6), and vice versa.

We will call (21) and (22) Bennett’s equations. Bennett originally derived a “semirelativistic” analog of (21) and (22), restricted to two species. He considered collisionless streams of charged particles with specified relativistic drift speeds, without thermal dispersion of the velocity component along the stream but with nonrelativistic species-dependent typical thermal motions across the stream. Expanding a normalized form of his formulas [Ref. 1, Eqs. (8) and (9)] to lowest significant order in $(u/c)^2$ gives our (21) and (22), except for the difference in the meaning of the temperatures [i.e., two-dimensional (2-D) versus three-dimensional (3-D) thermal motions]. In fact, relativistic modifications to (21) and (22) show up only in the parameters, whereas the mathematical structure of (21) and (22) obtains in Bennett’s semirelativistic treatment and also in a fully relativistic treatment, which couples the isothermal distributions of Jüttner [see Ref. 12, p. 46, Eq. (24)] self-consistently to Maxwell’s equations.

### IV. SOLVABILITY CONDITIONS

Bennett’s equations (21) and (22) are highly nonlinear and not readily susceptible to a general solution. Here we derive some a priori conditions on the parameters which have to be satisfied for a classical radial solution with normalized particle densities to exist. In particular, we prove that such solutions exist only if the generalized Bennett relation (1) holds. We mention without proof that it can be shown that (1) must be satisfied under mild integrability conditions for all classical solutions, without demanding radial symmetry.

We center a ball $B_r$ of radius $r$ at the origin and define

$$M_s(r) = \int_{\Omega_s} \exp[-\beta_s q_s (\phi - u_s A)] d^2x$$

(23)

so that

$$\lim_{r \to \infty} M_s(r) = 1.$$ (24)

Let a prime denote derivative w.r.t. $r$. By Newton’s theorem,

$$\phi' = -2 \sum_s N_s q_s r^{-1} M_s(r),$$

(25)

$$A' = -2 \sum_s N_s q_s u_s r^{-1} M_s(r).$$

(26)

By (24),

$$Q^{-1} \lim_{r \to \infty} r \phi'(r) = -2 = c I^{-1} \lim_{r \to \infty} r A'(r),$$

(27)

where

$$Q = \sum_s N_s q_s; \quad I = c \sum_s N_s q_s u_s.$$ (28)

By integration,

$$Q^{-1} \lim_{r \to \infty} \frac{\phi(r)}{\ln r} = -2 = c I^{-1} \lim_{r \to \infty} \frac{A(r)}{\ln r}$$

(29)

as leading-order behavior of normalizable solutions at infinity. Using (29) in the right-hand side of (21) and (22), we infer the asymptotic decrease of the particle density $n_s$ of species $s$, $n_s(r) \sim -2 \beta_s q_s (u_s c^{-1} - 1 - Q).$ (30)

Integrability now requires that

$$\beta_s q_s (u_s c^{-1} - 1 - Q) > 1$$

(31)

holds for each $s$.

To derive the generalized Bennett identity for radial solutions, we compute

$$M'_s(r) = 2 \pi r \exp[-\beta_s q_s (\phi(r) - u_s A(r))] / Z_s$$

(32)

and

$$M''_s(r) = -\beta_s q_s (\phi'(r) - u_s A'(r)) M'_s(r) + M'_s(r)/r.$$ (33)

then eliminate $\phi'$ and $A'$ in (33) with the aid of (25) and (26). Partial integration of $r M'_s(r)$ w.r.t. $r$ now gives

$$r M'_s(r) = 2 M_s(r) + 2 \sum_t N_t q_t \beta_t (1 - u_t u_s)$$

$$\times \int_0^r M_t(\xi) M'_s(\xi) d\xi.$$ (34)

By (30)

$$\lim_{r \to \infty} r M'_s(r) = 0.$$ (35)

Taking $r \to \infty$ in (34) therefore yields, for each $s$
\[ 1 = \sum_{i} N_{i} q_{i} \beta_{i} (u_{i} u_{i} - 1) \langle M_{i}(r) \rangle \]  
(36)  

with the definition

\[ (F)_{s} = \lim_{r \rightarrow 0} \int_{0}^{r} F(\xi) M_{s}(\xi) d\xi. \]  
(37)  

We multiply (36) by \( N_{s} k_{B} T_{s} \) and sum over \( s \), then add the same formula with indices \( s, t \) exchanged, yielding

\[ 2 \sum_{s} N_{s} k_{B} T_{s} = \sum_{s, t} N_{s} N_{t} q_{s} q_{t} (u_{s} u_{t} - 1) \times [\langle M_{s}(r) \rangle_{s} - \langle M_{s}(r) \rangle_{t}]. \]  
(38)  

By a partial integration and using (24)

\[ (f + f_{i}(r))_{s} = (M_{s}(r))_{s} = 1. \]  
(39)  

By (39) and (28) we see that (38) is identical to the generalized Bennett relation (1), which therefore must hold for all radial solutions of (21) and (22).

Physically acceptable \( |u_{s}| < 1 \) and \( T_{s} > 0 \) require a combination of positive and negative species to make the right-hand side of (38) positive. We thus recover the well-known fact that the attractive magnetic interactions cannot compensate the repulsive Coulomb interactions in a plasma with only positive (negative) species. For \( q_{s} q_{t} < 0 \) for at least one pair \((s, t)\) there is a solution manifold of (38).

V. BENNETT’S SOLUTION

A solution with normalized densities in all space was constructed by Bennett\(^1\) for \( S = 2 \). Bennett looked for radial solutions with a constant ratio \( n_{1}(r)/n_{2}(r) \) of particle densities throughout space. This is readily generalized to \( S \) species. It implies a constant ratio of current and charge densities, and of magnetic and electric potentials (after an appropriate choice of the irrelevant constants \( \phi_{0} \) and \( A_{0} \)).

In our nonrelativistic setup, his ansatz reads

\[ c \Gamma^{-1} A(x) = \chi(x) = Q^{-1} \phi(x) \]  
(40)  

which with (21) and (22) gives the conditions

\[ \beta_{q} q_{s} (u_{s} c^{-1} I - Q) = 2, \]  
(41)  

\( s = 1, \ldots, S \), on the parameters. If they are met, (40) maps both (21) and (22) into

\[ \psi(x) \rightarrow \psi(\chi/|x|^{2}) - \frac{1}{2} \ln |x|^{2}, \]  
(45)  

\[ A(x) \rightarrow A(\chi/|x|^{2}) - c^{-1} I \ln |x|^{2}. \]  
(46)  

This can only hold if (41) is satisfied, whence (41) is a set of necessary, and also sufficient conditions that (21) and (22) be invariant under the Euclidean conformal group. Because of (38) only \( S - 1 \) conditions of the type (41) have to be postulated for the parameters to obtain full conformal invariance.

Next, consider \( \Psi = I \phi - cQA \), which satisfies

\[ - \Delta \Psi = 4 \pi \sum_{s} N_{s} q_{s} (I - u_{s} c Q) \times \frac{\exp[-(\beta_{q} \phi + 2cA)/I]}{\int_{R^{2}} \exp[-(\beta_{q} \phi + 2cA)/I] d^{2}x} \]  
(47)  

with

\[ \Psi(x) \rightarrow 0; \quad |x| \rightarrow \infty, \]  
(48)  

which obtains from (21) and (22) after eliminating the products \( \beta_{q} u_{s} \) using (41). By (27) and (48), \( \nabla \Psi \) is square integrable, whence given any smooth \( A \) for which \( \int_{R^{2}} \exp(-2cA/I) d^{2}x < \infty \), a solution \( \Psi \) of (47) and (48) is a stationary point of the functional

\[ \int_{R^{2}} |\nabla \Psi|^{2} d^{2}x + 8 \pi \sum_{s} N_{s} k_{B} T_{s} I(I - u_{s} c Q) \times \ln \int_{R^{2}} \exp \left( -\frac{(\beta_{q} \phi + 2cA)}{I} \right) d^{2}x. \]  
(49)  

Because of (1) and \( |u_{s}| < 1 \) we have \( |I| > |u_{s} c Q| \), thus

\[ I(I - u_{s} c Q) > 0 \]  
(50)
which implies that (49) is convex. Therefore, given such A, a solution \( \Psi \) of (47) and (48) is unique. Using this and (41) we now verify that \( \Psi = \) 0 is the only solution of (47) and (48) for all such A. Since \( \Psi = \) 0 is equivalent to (40), up to a gauge constant, we are back to Liouville’s equation. Notice that in the above we did not require radial symmetry.

VI. OTHER SOLUTIONS

We consider now two species \((s = \pm)\) in the general case when (41) is not satisfied. This implies lack of inversion symmetry of (21) and (22), but invariance under the subgroups of dilations, translations, and rotations, as well as gauges still prevails. Looking for solutions which are radially symmetric around the origin leaves the dilation and gauge groups. We set \( Q = 0 \). The case \( Q \neq 0 \) can be obtained from it by a boost \(^{17}\) along \( e_z \), which leaves perpendicular scales and the mathematical structure of (21) and (22) unaffected and changes the particle densities only in their overall amplitude, preserving \( c^2 p^2 - j^2 \). We show the equivalence of (21) and (22) to a two-degrees-of-freedom Hamiltonian model and, using this, prove existence and find the structure of solutions in this case.

We transform to new dependent and independent variables, setting \( \tau = \ln r \), and

\[
\begin{align*}
x(\tau) &= \phi(e^\tau) + \bar{u} \tau T/c, \\
y(\tau) &= A(e^\tau) + \tau I/c,
\end{align*}
\]

where

\[
u = \frac{\sum_s (N_s T_s)^{-1} u_s}{\sum_s (N_s T_s)^{-1}}.
\]

It suffices to consider only the case \( \bar{u} < 0 \), the other being analogous. The full conformal case (41) obtains for \( \bar{u} = 0 \).

We now choose the gauge such that

\[
\int_{-\infty}^{\infty} \exp[-\beta_s q_s(x-u, y)]d\tau = 2
\]

for \( s = \pm \). Using this Eqs. (21) and (22) assume the form

\[
\begin{align*}
x(\tau) &= -\sum_s N_s q_s \exp[-\beta_s q_s(x-u, y)], \\
y(\tau) &= -\sum_s N_s u_s q_s \exp[-\beta_s q_s(x-u, y)],
\end{align*}
\]

dilation symmetry of (21) and (22) now means invariance of (55) and (56) under translations \( \tau \rightarrow \tau + \text{const} \). The finiteness of \( \phi(0) \) and \( A(0) \) and the asymptotic behavior (29) yield the asymptotic conditions

\[
\lim_{|\tau| \to \infty} |x(\tau)|/\bar{u} = 1/c = - \lim_{|\tau| \to \infty} |y(\tau)|.
\]

Upon reinterpreting \((x, y)\) as a position vector and \( \tau \) as a time variable, our problem becomes identical to that of a classical point particle which moves in a time-independent nonconservative force field. Despite the nonconservative character of the force field, we can construct an autonomous two-degrees-of-freedom Hamiltonian,

\[
H = \frac{1}{2}(p_x^2 - p_y^2) - U(x, y),
\]

where

\[
U(x, y) = \sum_s N_s k_BT_s \exp[-\beta_s q_s(x-u, y)].
\]

Equations (55) and (56) are equivalent to Hamilton’s equations of motion for this \( H \). Hamiltonians of this type with nonpositive definite kinetic energy term have a number of curious properties and have been considered recently by Pfirsch\(^{18}\) in a very different plasma-physical context.

A. Existence of solutions

We will establish existence of solutions by proving that among those trajectories in \((x, y)\) space which have the correct asymptotic incoming slope \( w_{\text{in}} = 1/\bar{u} \) in the quadrant \( \{x, y): x > 0, y < 0\} \), one can find one with the correct asymptotic outgoing slope \( w_{\text{out}} = -1/\bar{u} \) in the quadrant \( \{x, y): x < 0, y < 0\} \) if the incoming data for \( \tau \to -\infty \) fulfill (57). This suffices because \( H \) is a conserved integral of motion and \( U(x, y) \) decays asymptotically to zero along those trajectories, whence

\[
H = \frac{P^2}{2c^2} (\bar{u}^2 - 1) = \frac{1}{2} (p_x^2 - p_y^2)_{\text{in}} - \frac{1}{2} (p_x^2 - p_y^2)_{\text{out}},
\]

along such a trajectory. Together with the correct asymptotic slopes this implies the correct outgoing momenta when the correct incoming ones are given. Our procedure consists of first locating a corridor for the \( \tau \to -\infty \) asymptotic trajectory which has the following property: the \( \tau \to +\infty \) slopes of the two extremal trajectories which have their \( \tau \to -\infty \) asymptotes situated at one of the boundaries of the corridor sandwich the required future slope. The second step is a continuity argument that establishes the existence of the right outgoing slope for some data inside the corridor.

We compute the lines of force \( \{(x^F, y^F)\} \) in parameter representation

\[
x^F(\xi; \xi_0|x_0) = x_0 - k_B T_+ \left( 1 + \frac{N_+ T_+}{N_- T_-} \right)^{-1} \frac{\xi - \xi_0}{1 + u \bar{u}} \\
y^F(\xi; \xi_0|x_0) = \frac{k_B T_+}{q_+} \left( 1 + \frac{N_+ T_+}{N_- T_-} \right)^{-1} \frac{1 - u + \bar{u}}{(1 + u \bar{u})^2} \\
\times \ln \frac{1 - u + \bar{u} e^\xi}{1 + u - \bar{u} + (1 + u \bar{u}) e^\xi},
\]

and

\[
y^F(\xi; \xi_0|x_0) = -\left( \beta_+ q_+ u_+ - \beta_- q_- u_- \right)^{-1} \xi + x^F(\xi)/\bar{u}.
\]

For fixed \( \xi_0 \), the lines of force are labeled by \( x_0 = x^F(\xi = \xi_0) \). Some are shown in Fig. 1. All lines are asymptotically straight for \( \xi \to \pm \infty \). There is exactly one entirely straight line of force. We call it the critical line, \( C \). It is parallel to a family of straight lines, \( \{(x_v, y_v)\} \), on each of which

\[
\frac{dy^F}{dx} = \frac{1}{v} = \text{const}.
\]

They are given by
FIG. 1. Lines of force, critical (C) and cocritical (C') lines, and Ω regions for the equivalent particle model in (x,y) space. The flow of the force field is "downward" in Ω+, and from Ω_ into Ω_-

with

\[ \cos \theta = \frac{1}{\sqrt{1 + u^2}} \quad \sin \theta = -\frac{u}{\sqrt{1 + u^2}} \] (66)

so that their slope is \(1/\bar{u}\). In particular, the incoming data (57) thus point antiparallel to \((\bar{x}, \bar{y})\) along C, which is obtained for \(u = \bar{u}\). Similarly, at \(u = -\bar{u}\) there exists a second, cocritical line \(C'\), on which \((\bar{x}, \bar{y})\) points parallel to the outgoing data (57). The lines \(C\) and \(C'\) divide \((x,y)\) space into the three disjoint subspaces, \(\Omega_-\), \(\Omega_0\), \(\Omega_+\), shown in Fig. 1.

We infer from Fig. 1 that a particle launched on \(C\) stays on \(C\) and gets reflected downward. If a particle is launched on \(C'\) it gets deflected deeper into \(\Omega_-\), but clearly will aim too high to the left in the asymptotic future. These outgoing scenarios sandwich the required data. Clearly, if the launch is made in \(\Omega_-\) or \(\Omega_+\) close to \(C\) or \(C'\), respectively, the asymptotic future behavior will be farther away from the required (57). The launch has thus to take place in the corridor \(\Omega_0\), a transition region with a more complicated structure.

We now define the "launching parameter" \(b\) as the distance between \(C\) and the asymptote to the incoming trajectory (see Fig. 2), and \(b'\) as the distance between \(C\) and \(C'\). To prove existence of trajectories with the correct outgoing slopes, we have to show that the outgoing inverse slope \(1/w(b)\) varies continuously from \(1/w(\bar{u})\) to \(1/w(-\bar{u})\) when \(b\) is decreased from \(b'\) to some value \(b\). Since clearly \(1/w(b) > -\bar{u}\), and \(1/w(b = 0) = \bar{u} < 0\), one may be tempted to conclude the proof with this observation. However, the launching condition on the critical line itself is singular and we cannot yet conclude that \(\lim_{b, \tau, \tau_0} 1/w(b) = 1/w(0)\), which is precisely what we have to show.

To make the following discussion more transparent, we rotate and then translate the coordinates \((x,y)\) to a new coordinate system \((X,Y)\), such that the \(Y\) direction is parallel to the critical line of force, \(b = 0\) defines \(X = 0\), and \(b > 0\) means \(X < 0\). The transformation is

\[ \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} K \\ 0 \end{pmatrix} \] (65)

The asymptotic data imply now vertical upward motion for \(r \to \infty\), with \(X(\tau) = 0\), and positive slope \(\tan(2\theta)\) for \(r \to \infty\), situated in the lower left quadrant.

We have \(X(X, Y) < 0\) for \(X < 0\), and \(X \sim X_0\) for \(X > 0\), and \(X \to X_0\) for \(r \to 0\), for any given \(Y\). We also have \(Y(X, Y) > 0\) in a left neighborhood of \(X = 0\), and \(Y(X, Y)\) may or may not become positive for sufficiently negative \(X\). For given \(X > 0\), both accelerations increase exponentially in strength as \(Y \to \infty\), and decrease exponentially fast to zero as \(Y \to -\infty\).

We now show that for any given \(Y_0 < 0\) and small \(X_0 < 0\) we can find an \(\epsilon > 0\) so that there is a \(\tau_0\) such that \(Y(\tau_0) > Y_0\) and \(X(\tau_0) > X_0\) for the reflected trajectory, by choosing \(\epsilon > 0\). This implies that we can make the x-directed motion arbitrarily steep with arbitrarily negative \(X = 0\) arbitrarily close to the line \(X = 0\) at some finite time \(\tau_0\). Since the acceleration terms decrease exponentially with \(Y\) for \(Y \to \infty\), and \(X \to \infty\), the motion after time \(\tau_0\) can only contribute an exponentially small correction to the outgoing slope. This then proves the uniform approach of the outgoing slopes to the critical trajectory \(X = 0\) when we let \(X(\tau) \to 0\).

We now show that we can find a small positive \(\epsilon > 0\) such that for all \(X \in (X_0, 0)\) there exist some positive constants \(C_1(\delta)\) and \(C_2(\delta)\) such that

\[ X \geq C_1 F(Y) X; \quad Y \leq C_2 F(Y). \] (72)

We define \(X_*(\epsilon)\) and \(Y_*(\epsilon)\) by

\[ \begin{align*} X_* &= C_1 F(Y_*); \quad Y_* = C_2 F(Y_*). \end{align*} \] (73)

so that for \(r \leq \tau_*\), we have \(0 > X(\tau) > X_*(\epsilon)\) and \(Y(\tau) = Y_*(\epsilon)\) once we make sure this holds asymptotically for \(r \to 0\). \(\tau_*\) is the largest \(r\) for which \(X_*(\epsilon) > X_0\).

Both differential equations can be solved explicitly. Let

\[ \omega = q_+ \cos \theta \sqrt{N_x b_1(u + \bar{u}) C_2/2} \] (74)

and \(z(\tau) = \tanh(\omega \tau)\), then

\[ \begin{align*} Y_* &= [\beta q_+ (u + \bar{u}) \cos \theta]^{-1} \ln(1 - z^2[\tau]), \quad X_*(\tau) = -eP_s([-z(\tau)]), \end{align*} \] (75)

where
FIG. 2. Structure of the general trajectory of a solution in (x,y) space, together with its asymptotes (dashed lines) and the critical and cocritical lines. The “particle” comes in from lower right.

\[ P_\nu(-z) = K_\nu \int_0^\infty \frac{(\sinh \xi)^{2\nu+1}}{(\cosh \xi - z)^{\nu+1}} \, d\xi \]  \tag{77}

is a Legendre function of the first kind of order \( \nu \), with

\[ \nu = -\frac{1}{2} \left[ 1 - \sqrt{1 - 8C_1/C_2}\beta_+q_+(u_+-\hat{u})\cos \theta \right] \]  \tag{78}

and \( K_\nu \) a normalization constant. Since \( P_\nu(-z) \) is strictly positive for real \( z < 1 \) and diverges \( -\ln(1-z) \) for \( z \nearrow 1 \), we see that we can always choose an \( \xi_0^{-1} \) so that \( Y_\nu(\xi_0) = Y_0 \) and \( X_\nu(\xi_0) = X_0 \) for arbitrarily large negative \( Y_0 \) and arbitrary small negative \( X_0 \), with \( \xi_0 \) the larger of the two roots solving (75) with these data. Clearly, since this implies that the acceleration terms can be made exponentially small, a simple Picard iteration argument now shows that the motion for \( \tau \geq \tau_0 \) can pick up only an exponentially small change in slope, which a fortiori is true for the real trajectory as well. This proves the uniform approach of the asymptotic outgoing slope of the trajectory to \( \pm \infty \) as \( X(-\infty) \rightarrow 0 \). Therefore in \( (x,y) \) space the future inverse slope converges uniformly to \( 1/\bar{u} \) as \( h \rightarrow 0 \). By continuity there now exists a \( b_0 < \hat{b} \) such that \( w_{\text{out}}(b_0) \) matches the outgoing data (57). This concludes the proof.

**B. Spatial structure of solutions**

It remains to determine the qualitative structure of the solutions. Having shown that the asymptotic motion is linear with the right slopes from the shape of the lines of force we now readily infer the general structure of the trajectories in \( (x,y) \) space, (see Fig. 2).

In the fully conformal situation \( C \) would be vertical and the trajectory of the solution identical to some lower portion of \( C \).

The trajectory in Fig. 2 gives the general shape of a possible solution of (53) and (56). As a function of \( \tau \) \( x(\tau) \) first increases monotonically from \( -\infty \) reaches a maximum and then decreases again monotonically to \( -\infty \); \( x(\tau) \) decreases from \( +\infty \) in the infinite past to \( -\infty \) in the infinite future either monotonically or with at most one local minimum and one maximum in between. Taking the known asymptotic behavior into account, this now translates into the shapes \( \phi(r) \) and \( A(r) \) shown in Fig. 3. This completes the discussion of the qualitative radial profiles for the two-species case. A quantitative result can be obtained straightforwardly in the infinitesimal neighborhood of Bennett’s solution. In fact, an infinitesimally nearby solution has the mathematical structure (74)–(78) in \( X,Y \) space, with appropriate constants replacing \( C_1, C_2, \) and \( \bar{C} \).

Left for future studies is the case \( S \geq 3 \). It is readily seen that the transformation to the equivalent point particle analog is possible for arbitrary \( S \), but the dimensionality of the model is always 2, whereas the number of exponentials in the acceleration is \( S \), so that the mechanical analog will in general be nonautonomous. This means a significant complication.

**VII. CONCLUDING REMARKS**

In this paper we have investigated a simple kinetic model of an unbounded dissipative stationary \( S \) component plasma in a constant external driving electric field \( \mathcal{E} \). Our model shows that a dissipative treatment of a plasma will severely limit the number of possible stationary solutions as compared to dissipation free Vlasov theory; cf. Refs. 20 and 21 for related observations concerning dissipative versus ideal magnetohydrodynamics, and Refs. 22–26.
for ad hoc macroscopic variational approaches that select certain profiles from others. We did not, however, consider the time evolution for given initial data of the distributions and the fields. This means in particular that we do not know if or how a final stationary state with particle numbers \( N_i \) is achieved in the course of time. We also do not know how the final \( N_i \) are chosen. Since we are in an unbounded domain, the final \( N_i \) need not be the same as their initial values at \( t=0 \). For example, if the external electric field is too small to satisfy (1), we would expect to get, as \( t \to \infty \), only the trivial solution in which all \( N_i \) and therefore also \( I \) and \( Q \) go to zero. Less dramatically, the plasma may "shed" only some "excess" particles to "infinity." We hope to investigate this further later. Meanwhile we believe that our results point to the conclusion that typically a stationary current-carrying \( S \)-isothermal plasma will be non-neutral. Indeed, given that one ignores a neutral gas, the applied electric induction is unlikely to drive the individual species to exactly the "right" mean speeds for which neutrality of the stationary state would be observed in the laboratory frame. Non-neutral plasma beams are of course well known, but many theoretical beam profiles result from some special ansatz which allows for explicit solutions like in Bennett's original work.

As potential application, our model indicates that measurement of a few global quantities like current and charge, and the asymptotic decay of the density profiles, may reveal species individual information on effective transport coefficients via equations like (30). A further investigation of this aspect in more realistic effective dissipation models seems worth pursuing.

We finally notice that we have not investigated the stability of our solutions against perturbations which are invariant along \( \xi \); cf. Refs. 6, 7, and 27. It is also conceivable that some of our solutions are unstable to a macroscopic kink or sausage instability or a velocity space instability of Buneman type. The dissipation changes the problem considerably as compared to the Vlasov setting discussed in Ref. 2 for instance. We hope to address these questions as well as plasmas in confined geometry in future work.

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**APPENDIX A: VIRIAL IDENTITY**

The generalized Bennett relation (1) is a special case of a virial identity. Assuming cylindrical symmetry, taking the first velocity moment of (15) and summing over species yields, after a partial integrations, the momentum balance equation

\[
-\partial_r p + p E_r + r^{-1} j_r B_\phi = 0 \quad (A1)
\]

with \((r,\varphi,z)\) denoting cylinder coordinates and \(\partial_r p\) the radial pressure gradient, cf. Ref. 2. Multiplying (A1) by \(r^2\), expressing \(p\) and \(j_r\) through \(E_r\) and \(B_\phi\) using (4) and (5), then integrating over \(dr\) gives formally, after a partial integration,

\[
16 \pi \int_0^\infty p(r) r \, dr = \lim_{r \to \infty} (r^2 B_\phi^2 - r^2 E_r^2) \quad (A2)
\]

assuming that

\[
\lim_{r \to \infty} r^2 p(r) = 0 \quad (A3)
\]

To prove (A3) and to evaluate the right-hand side of (A2) one has to control \(\phi\) and \(A\) asymptotically by using the Poisson–Boltzmann equations, as in Sec. IV. Noting furthermore that here \(p = \sum_i k_i q_i n_i\), one arrives at (1).

Our derivation of (1) is done directly from the Poisson–Boltzmann equations, thus avoiding the detour via the first velocity moment of the stationary kinetic equation. However, the physically interesting connection with the more familiar notion of a virial identity is worth noting, and we are grateful to the anonymous referee for drawing our attention to the identity (A2).

**APPENDIX B: HAMILTON PRINCIPLE**

Since it may be useful for the discussion of certain aspects of radial solutions to (21) and (22) in the setting of (55)–(57) we show that they satisfy a variational principle. By Legendre transform from \(H(58)\) we obtain the Lagrangian

\[
L = \frac{1}{2} [(\dot{x} - \dot{x}_0)^2 + (\dot{y}^2 - y^2)] + U(x,y) \quad (B1)
\]

defined on the space of \(R^2\)-valued functions \( \tau \to \{x(\tau),y(\tau)\} \) for which both \(\dot{x}\) and \(\dot{y}\) are in \(\mathcal{C}^1(\mathbb{R})\), the Schwartz space of \(C^\infty\) functions with rapid decrease at infinity, for which \((\dot{x},\dot{y})\) satisfies the asymptotic conditions (57), and for which \(\dot{y} \neq 0\) (the latter because \(I>0\) as a consequence of \(e_\xi > 0\)). From (55)–(57) it immediately follows that, for a solution the asymptotic momenta are approached exponentially fast. Then the action integral

\[
\mathcal{S}[x,y] = \int_0^\infty L(x(\tau),y(\tau)) \, d\tau \quad (B2)
\]

exists and is non-negative on the specified function space. The equations of motion (55)–(57), i.e., including the asymptotic conditions are obtained from the Hamiltonian principle \(\mathcal{S}[\xi]=0\) where the variation is with respect to functions \(\{x(\tau),y(\tau)\}\) in the above function space.

The action functional \(\mathcal{S}\) is non-negative on the constructed function space, but a solution of our PDE system does not correspond to a global minimum of \(\mathcal{S}\). Without further constraints, one can construct a sequence of functions along which \(\mathcal{S}[\xi]\), but clearly 0 is not taken for any function in our function space. If the contraints (54) are taken into account, one can show existence of a global minimum on the space of continuously differentiable functions, but the minimizer is not twice continuously differentiable and hence does not solve our PDE system. Solutions instead correspond to critical points of \(\mathcal{S}\), which may be either a relative min-

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maximum or a saddle point; maxima do not exist. However, the global minimum may provide a useful lower estimate for $\mathcal{A}$.

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