On the Equivalence of Different Order Parameters for Ising Ferromagnets

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In a recent note Barber showed, for a spin-$\frac{1}{2}$ Ising system with ferromagnetic pair interactions, that some critical exponents of the "triplet order parameter" $\langle \sigma_i \sigma_j \sigma_k \rangle$ are the same as those of the magnetization $\langle \sigma_i \rangle$. Here we prove such results for all odd correlations and dispense with the requirement of pair interactions. We also prove that the critical temperature $T_c$, defined as the temperature below which there is a spontaneous magnetization, is for fixed "even" spin interactions $J_e$ independent of the way in which the "odd" interactions $J_o$ approach zero from "above." This is achieved by using only the "simplest," Griffiths-Kelley-Sherman (GKS), inequalities, which apply to the most general many-spin, ferromagnetic interactions.

KEY WORDS: Ising ferromagnets; different order parameters; critical exponents; inequalities.

We consider an Ising spin system with general (many-spin) ferromagnetic interactions; for a bounded region $\Lambda$ of a $d$-dimensional lattice the interaction energy of the spins in $\Lambda$ is

$$H(\sigma_{A}) = -\sum_{A \in \Lambda} J_{A} \sigma_{A}$$

$$= -\left( \sum_{i} h_{i} \sigma_{i} + \sum_{i,j} J_{ij} \sigma_{i} \sigma_{j} + \sum_{i,j,k} J_{ijk} \sigma_{i} \sigma_{j} \sigma_{k} + \cdots \right), \quad J_{A} > 0$$

(1)

Here $\sigma_{A} = \{ \sigma_{i} \}, i \in \Lambda, \sigma_{i} = \pm 1$, $A$ is any subset of $\Lambda$, and $\sigma_{A} \equiv \prod_{i \in A} \sigma_{i}$. On the right side of (1) we have written out $H(\sigma_{A})$ more explicitly according to

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the number of sites \(|A|\) in the set \(A\) and we have used the standard notation \(h_i\), the magnetic field at the site \(i\), for the one-body interaction \(J_A\), \(|A| = 1\).

When \(J_A > 0\) for all \(A\)—“positive” ferromagnetic case—the expectation values of the spins at temperature \(T = \beta^{-1}\),

\[
\langle \sigma_A \rangle = Z^{-1} \text{Tr}\{\sigma_A \exp[-\beta H(\sigma_A)]\}
\]

\[
= \partial (\ln Z)/\partial (\beta J_A); \quad Z = \text{Tr}\{\exp[-\beta H(\sigma_A)]\}
\]

satisfy the Griffiths-Kelley-Sherman (GKS) inequalities\(^{2,3}\)

\[
\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle = \partial \langle \sigma_A \rangle / \partial (\beta J_B) \geq 0, \quad J_C \geq 0 \quad \text{all } C
\]

(3)

It follows from (3) that if there exist some strictly positive interactions \(J_B > 0\) which “connect” all sites \(i\) in some set \(E\) such that these interactions alone would make the average of \(\sigma_E\) strictly positive at finite temperatures, then

\[
\langle \sigma_E \rangle \geq K_E > 0
\]

(4)

where \(K_E\) can be taken independent of \(T\) for all temperatures \(T \leq T_0 \leq \infty\). We shall assume from now on that this is true for all sets \(E, \; |E|\) even. As an example, let \(E\) consist of the sites \(k\) and \(l\) and assume that \(J_{ij} = J > 0\) for nearest-neighbor sites \(i\) and \(j\). Then

\[
\langle \sigma_k \sigma_l \rangle \geq [\tanh(\beta J)]^{|k - l|}
\]

(5)

where \(|k - l|\) is the length of the shortest “path” connecting \(k\) and \(l\). The right side of (5) is simply the expectation value of \(\langle \sigma_k \sigma_l \rangle\) when the only interactions are the nearest-neighbor ones along the path from \(k\) to \(l\).

Suppose now that we adjoin to the set \(E\) another site \(i\). We then have from (3) that

\[
\langle \sigma_i \sigma_E \rangle \geq \langle \sigma_i \rangle \langle \sigma_E \rangle
\]

(6)

\[
\langle \sigma_i \rangle = \langle \sigma_i \sigma_E \sigma_E \rangle \geq \langle \sigma_i \sigma_E \rangle \langle \sigma_E \rangle
\]

(7)

where we have used the fact that \(\sigma_E \sigma_E = 1\). Using (4), we then have

\[
K_E \langle \sigma_i \rangle \leq \langle \sigma_i \sigma_E \rangle \leq K_E^{-1} \langle \sigma_i \rangle
\]

(8)

which is our basic inequality.

We shall now use (8) to obtain information about critical exponents of “odd”-order parameters, e.g., the triplet \(\langle \sigma_1 \sigma_2 \sigma_3 \rangle\) considered by Wood and Griffiths,\(^{4}\) Baxter,\(^{5}\) and Barber.\(^{1}\) To do this we assume that the interactions are translation invariant, \(J_A = J_{A+x}\), where \(A + x\) is the set \(A\) translated by a lattice vector \(x\), i.e., \(h_i = h, J_{ij} = J^{(2)}(i - j)\), etc. Equation (1) now assumes the form

\[
H(\sigma_A, J_b, b_2) = \sum_{B \neq \{0\}} \sum_x J_B \sigma_{B+x}, \quad J_B \geq 0
\]

(1')
where \(\{0\}\) designates the origin and the sum over \(x\) goes over all \(x\) such that \(\{B + x\} \cap \Lambda\) is not empty, i.e., at least some of the sites in the set \(B + x\) are in \(\Lambda\). When not all sites of \(B + x\) are in \(\Lambda\), \(\sigma_{B + x} = \prod \sigma_i \prod \sigma_j, i \in \{B + x\} \cap \Lambda, j \in \{B + x\} \cap \Lambda^c\), then we either set \(\sigma_j = 0\) or \(\sigma_j = 1\) for all \(j \notin \Lambda\). These correspond, respectively, to zero and + boundary conditions, \(b_a = b_0\) or \(b_a = b_+\).\(^{(2,6)}\) We also assume that there is a "cutoff" in the range of the interactions\(^{(2)}\) so that we can let the region \(\Lambda\) increase to the whole lattice, i.e., take the thermodynamic limit, in a simple way.

It follows from the GKS inequalities that for \(J > 0\) the infinite-volume limit, \(\Lambda \to \infty\), of the correlation functions \(\langle \sigma_A \rangle(\beta, J; b_0, \Lambda)\) exist and are translation invariant.\(^{(2,6)}\) We shall denote these limits by \(\langle \sigma_A \rangle(\beta, J; b_\alpha), \alpha = 0\) or +, corresponding to \(b_0\) or \(b_+\) boundary conditions (b.c.). They are, by \((8)\), monotone functions of \(J\), for \(J > 0\). In particular, \(\langle \sigma_A \rangle(\beta, J; b_0) = m(\beta, J; b_0)\) is the magnetization per site. Letting \(J = (J_o, J_e)\) for "odd" and "even" interactions, then

\[
\langle \sigma_A \rangle(\beta, J_o, J_e; b_\alpha) = (-1)^{|A|} \langle \sigma_A \rangle(\beta, -J_o, J_e; b_\alpha')
\]

where \(b_0' = b_0\), \(b_+ = b_-,\) "minus" b.c.

We now consider the behavior of the odd correlation functions in the limit in which all the odd interactions \(J_o \to 0\) while \(J_e\) remains fixed, e.g., in the simplest case in which \(J_o = h\) we wish to know the behavior of \(\langle \sigma_A \rangle(\beta, h; b_\alpha)\) as \(h \to 0^+\). To answer this question we state the following lemmas, whose proofs follow from arguments similar to those used in the proofs of Lemmas 1 and 3 in Ref. 6.

**Lemma 1.** Let \(J' \geq 0\); then

\[
\lim_{J'/J} \langle \sigma_A \rangle(\beta, J'; b_0) = \langle \sigma_A \rangle(\beta, J; b_0)
\]

and

\[
\lim_{J'/J} \langle \sigma_A \rangle(\beta, J; b_+) = \langle \sigma_A \rangle(\beta, J; b_+).
\]

**Lemma 2.** Let \(J' = (J_{B'}, J), J_{B'} > 0, J \geq 0\); then

\[
\langle \sigma_B \rangle(\beta, J_{B'}, J; b_0) = \lim_{J_{B'} \to J_{B'}} (\partial \psi / \partial J_{B'})(\beta, J_{B'}, J)
\]

\[
= \lim_{J_{B'} \to J_{B'}} \langle \sigma_B \rangle(\beta, J_{B'}, J; b_+), \quad J_B > 0
\]

and

\[
\langle \sigma_B \rangle(\beta, J_{B'}, J; b_+) = \lim_{J_{B'} \to J_{B'}} (\partial \psi / \partial J_{B'})(\beta, J_{B'}, J)
\]

\[
= \lim_{J_{B'} \to J_{B'}} \langle \sigma_B \rangle(\beta, J_{B'}, J; b_0), \quad J_B \geq 0
\]
where \( \psi(\beta, J) = \beta^{-1} \lim_{\Lambda \to \infty} \ln Z/|\Lambda| \) is the free energy per site in the thermodynamic limit, which is independent of boundary conditions.(2)

By convexity the right and left derivatives of \( \psi \) exist and coincide for almost all values of \( J_B \) (the exceptions being denumerable). At these values of \( J_B \) we also have that

\[
\frac{\partial \psi(\beta, J_B, \mathbf{J})}{\partial J_B} = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \sum_x \langle \sigma_{B+x} \rangle (\beta, J_B, \mathbf{J}; b_X, \Lambda)
\]

where the sum over \( x \) is as in (1') with \( \{b_A\} = b \) a general set of boundary conditions(6,8) (corresponding to an arbitrary specification of \( \sigma_j \) for \( j \notin \Lambda \)), not just \( b_0 \) or \( b_+ \). At values of \( J_B \) where \( \partial \psi(\beta, J_B, \mathbf{J})/\partial J_B \) does not exist, we still have that \( \langle \tilde{\sigma}_B \rangle \) must lie between the left and right derivatives. Hence

\[
\langle \sigma_B \rangle (\beta, J_B, \mathbf{J}; b_0, \beta) \leq \langle \sigma_B \rangle (\beta, J_B, \beta) \leq \langle \sigma_B \rangle (\beta, J_B, \mathbf{J}; b_+) \quad \text{for } J_B > 0
\]

and the strict \( J_B > 0 \) is needed only for the left-side inequality.

Let us now consider first the case where \( J = (h, J_e) \), i.e., the only odd term in the energy is the one-body interaction, corresponding to an external field \( h \). We can then define, for given fixed \( J_e \), a spontaneous magnetization critical temperature \( \beta_c^{-1} \) by

\[
\lim_{h \to 0^+} m(\beta, h; b_a) = \lim_{h \to 0^+} \frac{\partial \psi(\beta, h)}{\partial h} = m^*(\beta) = \begin{cases} 0, & \beta < \beta_c \\ >0, & \beta > \beta_c \end{cases}
\]

where \( m^*(\beta) \) is, by Lemma 1, equal to \( \langle \sigma \rangle (\beta, J_e; b_-) \). It is known(2,7,8) that \( \beta_c \) must be greater than some minimum value \( \beta_0 > 0 \) and that for two and higher dimensions with positive (nonzero) nearest-neighbor pair interaction (this is sufficient but certainly not necessary) \( \beta_c < \infty \).

It can be proven in some cases (see review in Ref. 8) and is believed to be "generally" true that \( m^*(\beta_c) = 0 \), and that the behavior of \( m^*(\beta) \) near \( \beta_c \) is characterized by a critical index \( b \) (usually called \( \beta \)), \( m^*(\beta) \sim (\beta - \beta_c)^b \). It follows now from (8) that the same behavior is followed by all odd correlation functions.

**Lemma 3.** Let \( J = (h, J_e) \geq 0 \) be as in Eq. (1') and let (4) hold. Define \( \langle \sigma_A \rangle^*(\beta; b_a) = \lim_{h \to 0^+} \langle \sigma_A \rangle (\beta, h; b_a) \) with \( \alpha = 0, + \); then, for all odd \( |A| \),

\[
K_1 m^*(\beta) \leq \langle \sigma_A \rangle^*(\beta; b_a) \leq K_2 m^*(\beta), \quad 0 < K_1, K_2 < \infty
\]

In particular,

\[
\langle \sigma_A \rangle^*(\beta; b_a) = 0 \quad \text{for } \beta < \beta_c
\]
and

\[ \langle \sigma_A \rangle^*(\beta; b_a) \sim (\beta - \beta_c)^b \quad \text{for} \quad \beta \geq \beta_c \]  

where \( b \) is the critical exponent for \( m^*(\beta) \).

**Remark.** It can be readily shown, using Ginibre's proof of the GKS inequalities, that \( \langle \sigma_A \rangle^*(\beta, J'; b_{A'}, \Lambda) \leq \langle \sigma_A \rangle^*(\beta, J; b_+, \Lambda) \) whenever \( J_A \geq |J_{A'}| \). Hence for \( J = \{h, J_e\} \), \( \langle \sigma_A \rangle^*(\beta, h; b_{A'}, \Lambda) \leq \langle \sigma_A \rangle^*(\beta, |h|; b_+, \Lambda) \) for all \( b_{A'} \), and thus we must have that

\[ \limsup_{h \to 0+} \limsup_{A \to \infty} \langle \sigma_A \rangle^*(\beta, h; b_{A'}, \Lambda) = 0 \quad \text{for} \quad \beta < \beta_c, \quad |A| \text{ odd} \]

We can thus define \( \beta_c \) by the requirement that for \( \beta < \beta_c \) all equilibrium states have vanishing odd correlations when \( h = 0 \).

It should be noted here that for \( \beta > \beta_c \) we do not know, for general \( J_e \), whether \( \langle \sigma_A \rangle^*(\beta; b_0) \) equals \( \langle \sigma_A \rangle^*(\beta; b_+) = \langle \sigma_A \rangle(\beta; b_+) \) for \( |A| > 1 \). It is only when \( J_e \) is restricted to pair interactions, or more generally to interactions satisfying the Fortuin–Kasteleyn–Ginibre (FKG) inequalities, that one knows that \( \langle \sigma_A \rangle(\beta, h; b_0) = \langle \sigma_A \rangle(\beta, h) \), independent of all b.c. (not just \( b_0 \) or \( b_+ \)) whenever \( \partial \psi(\beta, h)/\partial h \) exists. Since this is true almost everywhere and \( \langle \sigma_A \rangle(\beta, h; b_0) \) is monotone decreasing in \( h \), the limits \( h \to 0+ \) of \( \langle \sigma_A \rangle(\beta, h; b_0) \) must coincide.

When \( J_e \) contains only pair interactions then by an extension of the Lee–Yang theorem we actually know that \( \langle \sigma_A \rangle(\beta, h) \) is analytic in \( h \) for Re \( h \neq 0 \). We may then define the zero-field susceptibility by

\[ \chi(A; \beta; b_a) = \lim_{h \to 0+} h^{-1} [\langle \sigma_A \rangle(\beta; h; b_a) - \langle \sigma_A \rangle^*(\beta; b_a)] \]  

It follows now from (8) and (18) that \( \chi(A; \beta, b_a) \) has the same critical behavior for all odd \( |A| \) as \( \beta \to \beta_c \) from below, e.g.,

\[ \chi(A; \beta; b_a) \sim (\beta_c - \beta)^{-\gamma}, \quad \beta_c \leq \beta \]

with \( \gamma \) independent of \( A \). Since \( \chi(i; \beta; b_a) \) is, by Lemma 2, the same for \( \alpha = 0, +, \gamma \) is also independent of \( b_a \).

The choice of the one-body potential (magnetic field \( h \)) as the sole symmetry-breaking interaction is of course arbitrary. It is not even clear a priori whether one can define for general \( J_e \) a unique critical temperature for symmetry breaking. This result follows, however, from our previous lemmas and the remark following Lemma 3.

**Theorem.** Let \( J_e \geq 0 \) be a (finite) set of fixed, translation-invariant, even Ising spin interactions such that (4) holds. Then:
(a) There exist a $\beta_c$ and positive constants $C_1$ and $C_2$ such that, for all $|\lambda|$ odd,

$$\langle \sigma_\lambda \rangle(\beta; b_+) = \begin{cases} 0, & \beta < \beta_c \\ >0, & \beta > \beta_c \end{cases}$$  \hspace{1cm} (21)

and (for all $\beta \geq \beta_0 > 0$)

$$C_1 m^*(\beta) \leq \langle \sigma_\lambda \rangle(\beta; b_+) \leq C_2 m^*(\beta)$$

where

$$m^*(\beta) = \langle \sigma_\lambda \rangle(\beta; b_+)$$  \hspace{1cm} (22)

(b) Let $J_o > 0$ be odd interactions; then for $J = (J_e, J_o)$, $J_e$ fixed, and $|\lambda|$ odd,

$$\langle \sigma_\lambda \rangle(\beta; b_+) = \lim_{\lambda \to 0} \langle \sigma_\lambda \rangle(\beta, J_o; b_+) = \lim_{\lambda \to 0} (\partial/\partial \lambda) \psi(\beta, J_o)$$  \hspace{1cm} (23)

where the last equality holds when $J_A > 0$, 1.

(c) For $\beta < \beta_c$ all infinite-volume equilibrium states for the interactions $J_e$ have vanishing odd correlations.

(d) If $J_B > 0$, $|B|$ odd, then

$$\lim_{\lambda \to 0} J_B^{-1} \langle \sigma_\lambda \rangle(\beta, J_o; b_+) = \chi_B(\lambda, \beta; b_+)$$  \hspace{1cm} (24)

satisfies the inequalities

$$C_1 \chi_B(i, \beta; b_+) \leq \chi_B(\lambda, \beta; b_+) \leq C_2 \chi_B(i, \beta; b_+)$$  \hspace{1cm} (25)

where, however, the values of $\chi_B$ may depend on the "path" $J_o \searrow 0$.

Remarks. (i) These results can be extended further if we are willing to make some assumptions about interchanges of limits $\lambda \to \infty$ and $J_o \searrow 0$, which are certainly justified for the case of pair interactions.\(^{(8)}\) We may then write

$$\lim_{\lambda \to 0} \partial \langle \sigma_\lambda \rangle(\beta, J_o; b_+)/\partial J_B = \sum_x <\sigma_\lambda \sigma_{B+x}(\beta; b_+) \langle \beta; b_+ \rangle \text{ for } \beta < \beta_c$$  \hspace{1cm} (26)

Writing now $A = E \cup \{l\}$ and $B = E' \cup \{j\}$, where $E$ and $E'$ are even sets, we have, by the same arguments as led to (8), that

$$K_B K_{E'} \langle \sigma_{i+j+x}(\beta; b_+) \langle \beta; b_+ \rangle \leq <\sigma_\lambda \sigma_{B+x}(\beta; b_+) \langle \beta; b_+ \rangle \leq (K_B K_{E'})^{-1} \langle \sigma_{i+j+x}(\beta; b_+) \langle \beta; b_+ \rangle$$  \hspace{1cm} (27)

We thus find that the different susceptibilities $\chi_B(\lambda, \beta; b_+)$ all have the same divergences as $\beta \to \beta_c$ from below.

(ii) When the FKG inequalities apply, e.g., only single and pair interactions, with $J(i - j) \geq 0$, then\(^{(6,8)}\) for $h > 0$,

$$<\sigma_\lambda(\beta; h; b) - <\sigma_\lambda(\beta, 0; b) \leq \sum_{i \in A} \{<\sigma_i(\beta; h; b) - <\sigma_i(\beta, 0; b)\}$$  \hspace{1cm} (28)
which can then be used in place of (7). One advantage of (28) is that it generalizes immediately, as does (6), to the case of "general" (non-spin-\( \frac{1}{2} \)) spins\(^{(13,14)}\) when the free measure is compact, e.g., \( \sigma \) uniformly distributed in the interval \([-1, 1]\). It also seems possible to extend some of our results to the unbounded spin case of interest in field theory and to some two- and three-component systems,\(^{(16)}\) but we shall not do that here.

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REFERENCES