AIP Journal of Applied Physics

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Citation: Journal of Applied Physics **114**, 233302 (2013); doi: 10.1063/1.4847957
View online: http://dx.doi.org/10.1063/1.4847957
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/114/23?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF APPLIED PHYSICS **114**, 233302 (2013)

Time evolution of electron flow in a model diode: Non-perturbative analysis

A. Rokhlenko and J. L. Lebowitz[a)]
Department of Mathematics, Rutgers University, Piscataway, New Jersey 08854-8019, USA

(Received 18 October 2013; accepted 30 November 2013; published online 18 December 2013)

Using a combination of Eulerian and Lagrangian variables we study the time evolution of the electron flow from a no-current state to a final state with the stationary current in a planar one-dimensional diode. The electrons can be injected externally or generated by the cathode via field emission governed by a current-field law. The initial zero current regime is replaced suddenly by injection or, in the case of field emission, by jumping the anode voltage from zero to a constant positive value. The case of equipotential electrodes and fixed injection is studied along with a positive anode potential. When the current is fixed externally, the approach to the stationary state goes without oscillations if the initial electron velocity is high enough and the anode can absorb the injected flow. Otherwise the accumulated space charge creates a potential barrier which reflects the flow and leads to its oscillations, but our method of analysis is invalid in such conditions. In the field emission case the flow goes to its stationary state through a train of decaying oscillations whose period is of the order of the electron transit time, in agreement with earlier studies based on perturbation techniques. Our approximate method does not permit very high cathode emissivity although the method works when the stationary current density is only about 10% smaller than the Child-Langmuir limit. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4847957]

I. INTRODUCTION

Stability of electron flows and their oscillations are of a great importance in technology, and they attract the interest of plasma researchers and are covered by numerous publications. While the main features of the steady flow in a diode have been known for a century,[1,2] an important step in studying time dependent states was made by Lomax[3] in 1960, who applied the Lagrange formulation of flow dynamics. This approach has been developed in many works, in particular, in Refs. 4–8, where the authors found conditions for flow stability and modes of oscillations. These works all used perturbation techniques, and in fact only the linear stability analysis describing small deviations from the steady flow has been fully studied. A nonlinear process in flow dynamics was studied in Ref. 9 in Eulerian variables by using direct numerical integration for solving Poisson equation. We describe here a first step toward considering far from stationary processes analytically for the turning on regime in a planar diode.

We study the electron flow in the transient state when the external parameters, such as injected current or acceleration voltage, are rapidly changed at $t = 0$ and then stay fixed. We consider the space charge limited one-dimensional (1D) flow, produced by field emission (regime I) or externally (regime II), in conditions which forbid applying linearization techniques.[3–8] In particular, we study the transition from no-flow initial state to a stationary current bearing state. The one-dimensional system should approximate a planar diode whose sizes in the two transversal dimensions are much larger than the inter-electrode distance. Our emission law is not realistic but is qualitatively of a right form, i.e., the stronger the cathode field the larger the emission, and it can be used for developing more practical models. The results cannot be quantitatively compared yet with the behavior of real diodes in the transient regimes, and this makes it possible for simplicity to use dimensionless units without mapping them onto realistic ones.

II. MAIN EQUATIONS AND SETUP OF 1D FLOW MODEL

The cathode is placed at $x = 0$, anode at $x = x_a$, the potential, flow velocity, and current density are denoted as $\varphi(x,t)$, $v(x,t)$, $j(x,t)$, respectively, as functions of x and time t. We assume that the current at the cathode is determined by a function of the cathode electric field $f(t)$ (regime 1) or fixed externally (regime 2), i.e.,

$$j(0,t) \equiv j(t) = F[f(t)], \quad \text{or} \quad j(0,t) = j,$$
$$f(t) = \frac{\partial\varphi}{\partial x}(0,t) = E(0,t), \quad v(0,t) = \omega, \tag{1}$$

where $E(x, t)$ is the electric field at x. The initial electron velocity ω is fixed. Let $\rho(x, t)$ is the charge density, then the current density is $j(x, t) = \rho(x, t)v(x, t)$, and the system is governed by the following set of equations in Euler coordinates:

$$\frac{\partial^2\varphi}{\partial x^2}(x,t) = \rho(x,t), \; \frac{\partial j}{\partial x}(x,t) = -\frac{\partial\rho}{\partial t}(x,t),$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial\varphi(x,t)}{\partial x}, \tag{2}$$

setting the electron charge be 1, its mass be 2, and thus the electron kinetic energy is $v^2(x, t)$. The cathode potential $\varphi(0,t)$ is always 0.

[a)]Also at Department of Physics, Rutgers University, Piscataway, New Jersey 08854-8019, USA.

For $t<0$ our system is free of charge, $\rho=0$, and all its parameters have been time independent. The initial conditions (IC) at $t=0$ are

$$j(x,0)=0,\ \begin{cases}\varphi(x_a,0)=0,\ f(x,0)=0, & \text{regime I},\\ \varphi(x_a,0)=V_a, f(x,0)=V_a/x_a, & \text{regime II}.\end{cases} \tag{3}$$

We consider processes at times $t>0$ assuming that in regime I the anode voltage at $t=0$ jumps to $\varphi(x_a,t)=V_a=const$, the cathode field to $f(0,0)=V_a/x_a$, and then the cathode field and current will evolve as $f(0,t)\equiv f(t)$ and $j(0,t)\equiv j(t)=F[f(t)]$, respectively. In the regime II the cathode current jumps at $t=0$ from 0 to $j(0,t)=j=const$.

Along with the Eulerian variables x, t above we use the Lagrangian variables τ, t for an electron emitted at time τ and observed at time t. In this way all flow parameters, which depend on both x, t, will be functions of τ and t. Such functions as $f(t)$, $j(t)$, and $\varphi(0,t)$ describe the regime at the cathode, and they depend on t only. We assume that a pair τ, t has a one-to-one correspondence with some x, t (The new external parameters for $t>0$ can be such that the one-to-one correspondence becomes impossible and our approach fails, see later). This allows to find uniquely the electric field $E(\tau,t)$ as well as $\varphi(\tau,t)$ and current density $j(\tau,t)$ at the point x, t (This somewhat sloppy use of the same notation in the new units will not lead to confusion). A straightforward analysis in these variables[3] shows that if T is the time needed for an electron, emitted at τ, to cross the diode then this electron is located at

$$x(\tau,t)=\int_\tau^t\left[j(t')\frac{(t-t')^2}{4}+f(t')\frac{t-t'}{2}\right]dt' + \omega(t-\tau),\quad \tau\le t\le T, \tag{4a}$$

with $x(\tau,\tau+T)=x_a$. The electron velocity is the t–derivative of Eq. (4a)

$$v(\tau,t)=\frac{\partial x}{\partial t}(\tau,t)=\omega+\frac{1}{2}\int_\tau^t[j(t')(t-t')+f(t')]dt'. \tag{4b}$$

The electric field (i.e., the force) at the point $x(\tau,t)$ is the product of the acceleration and mass

$$E(\tau,t)=2\frac{\partial^2 x}{\partial t^2}(\tau,t)=f(t)+\int_\tau^t j(t')dt', \tag{5}$$

where the partial derivative is taken in the Lagrange variables, i.e., for fixed τ.

For an electron, which hits the anode at time t, then Eq. (4a) takes the form

$$x_a=\int_{t-T(t)}^t\left[j(t')\frac{(t-t')^2}{4}+f(t')\frac{t-t'}{2}\right]dt'+\omega T(t), \tag{6}$$

where $T(t)$ is the transit time for this electron, assuming $t-T(t)\ge 0$. In the Euler variables the boundary condition (BC) for $\varphi(x_a,t)$ has the following form:

$$V_a=\varphi(x_a,t)=\int_0^{x_a}E(x,t)dx.$$

In the Lagrangian variables using Eq. (5), it can be rewritten as

$$V_a=2\int_t^{t-T(t)}\frac{\partial^2 x}{\partial t^2}(\tau,t)\frac{\partial x}{\partial\tau}(\tau,t)d\tau, \tag{7}$$

where we have used the assumption[3] that x is unique for each given t and emission time τ; therefore, $dx=\frac{\partial x}{\partial\tau}d\tau$. Equations (6) and (7) should hold for all $t\ge T(t)$.

One can find an explicit expression for the charge density at τ, t

$$\rho(\tau,t)=\frac{\partial E}{\partial x}=\frac{\partial E}{\partial\tau}\Big/\frac{\partial x}{\partial\tau} =\frac{j(\tau)}{j(\tau)(t-\tau)^2/4+f(\tau)(t-\tau)/2+\omega}, \tag{8a}$$

where we have taken partial derivatives of E and x given in Eqs. (4) and (5). We see that ρ is a decreasing function of t if $f(t)\ge 0$; otherwise, it can be non-monotonic. The total space charge between the electrodes at a moment t in the Eulerian variables

$$\varrho(1,t)=\int_0^1\rho(x,t)dx=E(1,t)-E(0,t). \tag{8b}$$

This can be rewritten using Eq. (5) in the form

$$\varrho(t-T,t)=E(t-T,t)-E(t,t)=\int_{t-T(t)}^t j(t')dt', \tag{8c}$$

in terms of the Lagrangian coordinates leading to simple integration of the emitted current. We will use later an Eulerian relationship for electrons emitted at $t=0$ and located within the area $0<x<X(t)<1$

$$\varrho(X,t)=\int_0^X\rho(x,t)dx=E(X,t)-E(0,t)=\int_0^t j(t')dt'. \tag{8d}$$

Note that Eqs. (8b) and (8c) define the same function if t is fixed.

Evaluating the derivatives via Eq. (4), one can substitute them into Eq. (7), apply Eq. (6), and come to the following BC which as Eq. (6) should hold for all $t>0$

$$V_a=x_af(t)+\int_{t-T(t)}^t dt'\left[j(t')\frac{(t-t')^2}{4}+f(t')\frac{t-t'}{2}+\omega\right] \times\int_{t'}^t j(t'')dt''. \tag{7a}$$

We see that Eqs. (6) and (7a) can be considered from now on in the usual Eulerian variables because t is the same in both systems and variable τ is absent.

In the case when the cathode current is a function of the cathode field $j=F(f)$, such as field emission or Child-Langmuir (CL) flow, the system (6) and (7a) has the following form:

$$x_a = \int_{t-T(t)}^{t} \left\{ F[f(t')]\frac{(t-t')^2}{4} + f(t')\frac{t-t'}{2} + \omega \right\} dt',$$

$$V_a = x_a f(t) + \int_{t-T(t)}^{t} dt' \left\{ F[f(t')]\frac{(t-t')^2}{4} + f(t')\frac{t-t'}{2} + \omega \right\} \times \int_{t'}^{t} F[f(t'')]dt''. \quad (9)$$

There are two unknown functions $f(t)$ and $T(t)$ in Eq. (9) whose initial values at $t=0$ are determined by the IC.

One could consider a situation with a non-zero voltage in the initial state $V\,(t-)$ which makes a sudden jump ΔV at $t=0+$ and a simultaneous jump $\Delta V/x_a$ of the cathode field. This corresponds in our non-relativistic approximation to the superposition of the initial diode electric field and the homogeneous field, i.e., $E(x,0+) = E(x,0-) + \Delta V/x_a$.

Mathematically, Eq. (9) should be sufficient for determining $f(t)$ and $T(t)$, but a clear strategy for solving them is not seen. Keeping intact the main features of a field emitted current we will simplify Eq. (9) to some degree and use other functions of the flow found above, like Eqs. (4b), (8a), and (8c), to help us find an approximate solution. In particular, using Eqs. (4b) and (8a) the time dependence of the anode current can be written as

$$j_a(t) = \frac{j(t-T)v(t-T,t)}{j(t-T)T^2/4 + f(t-T)T/2 + \omega}, \quad T = T(t), \quad t \geq T(t). \quad (10)$$

In the stationary state, when f and j are constant, the denominator of Eq. (10) is the velocity at the anode and j_a is equal to the cathode current j as it should.

III. METHOD OF SOLUTION

Our method of attacking this time dependent problem is to study the flow evolution at discrete time steps which correspond to successive time intervals. If

$$\Theta_i = \sum_{k=1}^{i} T_k, \quad \Theta_0 = 0,\ i = 1,2,\ldots,$$

then these intervals are $\Theta_{i-1} < t < \Theta_i$, where $i=1,\ 2,\ldots$. Here T_i is the transit time of the electrons which leave the cathode at $t = T_{i-1}$. In particular, the first electrons starting the process at $t=0$ need T_1 for their travel to the anode. When the flow can approach asymptotically to a stationary state the most important time interval is the first one $0<t<T_1$, where the changes are dramatic, then in our computations only several intervals 4–6 are needed for reaching the stationary state with a very high precision.

This method of discrete steps can be based on Eq. (9) or using the following scheme. There is a possibility to derive a different set of equations in Eulerian variables implied[11] by a relationship for $x_a=1$

$$\frac{df}{dt}(t) = \frac{\partial \varphi}{\partial t}(1,t) + \int_0^1 [j(x,t) - j(0,t)]dx, \quad (11)$$

which fits to the situation with discrete time intervals. During the first time interval when the first electrons did not yet reach the anode but are located at $X(t)<1$ the current density $j(x,\ t)=0$ for all $x>X(t)$. Taking into account that the anode potential is fixed we use Eq. (8a) to rewrite Eq. (11) in the Lagrange variables

$$j(t) + \frac{df}{dt}(t) = \int_0^{X(t)} j(x,t)dx = \int_t^0 v(\tau,t)\frac{\partial E}{\partial \tau}(\tau,t)d\tau$$

and substitute $v(\tau,\ t)$ from Eq. (4b) and the τ-derivative of E. The result is

$$j(t) + \frac{df}{dt}(t) = \omega\int_0^t j(t')dt' + \frac{1}{2}\int_0^t dt'[j(t')(t-t') + f(t')] \times \int_0^{t'} j(t'')dt''. \quad (12)$$

This provides a closed equation for finding $f(t)$ when $j(t)$ is given or $j(t)=F(f)$ and $t\leq T_1$. Equations corresponding to Eq. (12) for the time intervals $\Theta_{i-1} < t < \Theta_i$ can be useful only for $j(t)=const$ for better precision when $t>T_1$. We apply here a more universal scheme with Eq. (9), which provide sufficiently accurate results.

This approximation is based on representing the function $f^i(t)$ on each time interval as a series

$$f^i(t) = \sum_{n=0} c_n^i \left(\frac{t-\Theta_{i-1}}{T_i}\right)^n \quad (13)$$

and truncating it to a finite sum with $n\leq N$. The number of terms N will be chosen in accord with the available number of equations for determining c_n^i, $n=0..N$ and the corresponding transition time T_i.

A. Regime I with linear $F[f]$, $\omega=0$, and $V_a(t>0)=1$

Let $F[f(t)] = af(t)$, $a>0$ and at $t=0-$ we have $x_a=1$, and the Eulerian variables $\varphi(x,0)=0$, $\rho(x,0)=0$ are changed with

$$\varphi(x,0) = x, \quad f(x,0) = 1 \text{ for } t = 0+. \quad (14)$$

In this way we study the flow after the diode is turned on by applying the voltage $V_a=1$ suddenly. When $\omega\neq 0$ all our equations become only slightly more complicated, we consider the simplification $\omega=0$ keeping in mind mainly high applied voltages when ω can be neglected.

Thus we come to a problem of finding two unknown functions $f(t)$ and $T(t)$ which satisfy Eq. (9) in the form

$$\frac{1}{4}\int_{t-T}^{t} f(t')[a(t-t')^2 + 2(t-t')]dt' = 1, \quad (15a)$$

$$f(t) + \frac{a}{4}\int_{t-T}^{t} dt' f(t')[a(t-t')^2 + 2(t-t')]\int_{t'}^{t} f(t'')dt'' = 1, \quad (15b)$$

for all $t\geq T$, where T is the transit time $T=T(t)$ for the electron which hits the anode at time t. Here as before $f(t)$ is the electric field at the emitter. Note that Eq. (15a) is already implemented in Eq. (15b) as well as in Eq. (17b) below.

Equations (9) or (15) with proper BC and IC obviously uniquely determine the flow evolution in our system because they are valid for all $t \geq T(t)$ in Eqs. (15), but finding $f(t)$ from them is a difficult problem. Approximate methods of solving them in our approach will involve the use of discrete sets of values of t, which leads to losing some information. As Eqs. (15) are identities valid for all t we supplement Eqs. (15) with three derivative of these equations. Thus we conserve at least some information to get additional identities of the same nature as Eqs. (15). The derivative of the integral in Eq. (15a) is zero, and therefore

$$\left(1-\frac{dT}{dt}\right)f(t-T)(aT^2+2T) = 2\int_{t-T}^{t} f(t')[a(t-t')+1]dt'. \tag{16}$$

This equation will be used for evaluating the derivative $T'(t)$ in computations below. After some manipulations one can derive from Eqs. (15b) and (16) the following relation:

$$\frac{df}{dt}+af(t) = \frac{a}{2}\int_{t-T}^{t} dt' f(t')[a(t-t')+1]\int_{t-T}^{t'} f(t'')dt'', \tag{17a}$$

which as well as Eqs. (15) is valid for all $t \geq T(t)$ but does not involve the derivative $T'(t)$ explicitly. We add to Eqs. (15) and (17a) the derivative of Eq. (17a)

$$\frac{d^2f}{dt^2}(t)+a\frac{df}{dt}(t) = \frac{af(t)}{2}\int_{t-T}^{t} f(t')dt' + \frac{a^2}{4}\left[\int_{t-T}^{t} f(t')dt'\right]^2 - \frac{a}{aT^2+2T}\left\{\int_{t-T}^{t} f(t')[a(t-t')+1]dt'\right\}^2, \tag{17b}$$

which can be rewritten using Eqs. (4b) and (8c) as

$$\frac{d^2f}{dt^2}(t)+a\frac{df}{dt}(t) = \frac{f(t)}{2}\varrho(t)+\frac{\varrho^2(t)}{4}-\frac{4a}{aT^2+2T}v^2(t).$$

Then we calculate the time derivative of Eq. (17b) and obtain the following relation in the Eulerian variables:

$$\begin{aligned}\frac{d^3f}{dt^3}(t)+a\frac{d^2f}{dt^2}(t)-\frac{\varrho(t)}{2}\frac{df}{dt}(t) &= [f(t)+\varrho(t)]\left[\frac{af(t)}{2}-\frac{6av(t)}{aT^2+2T}\right] \\ &\quad + 8av^2(t)\frac{aT+1}{(aT^2+2T)^2}\left[3-\frac{4v(t)}{aT^2+2T}\right],\end{aligned} \tag{17c}$$

for the moment $t = T(t)$ when the electrons emitted at $t = 0$ reach the anode.

The set of Eqs. (15) and (17) will be used in our calculations. Using Eq. (4b), Eq. (10) for the anode current density can be now simplified to the form

$$j_a(T) = \frac{4av(T)}{aT^2+2T}, \tag{18}$$

where the electron velocity at the anode v and T are both functions of a single variable t and can be treated in the Eulerian units. For this one should calculate $T(t)$ by the Lagrangian approach to find j_a as a continuous function of time. Equations (16) and (18) imply a useful relationship $j_a(T) = af(0)(1-T')$ for the anode current.

In the stationary case $f(t) = f = const$ the electron velocity at the anode $v_a = 1$ and Eqs. (8a) and (17a) imply $j = af = v_a\rho_a = \rho(T+\tau,\tau) = 4a/(aT^2+2T)$. Thus $f = 4/(aT^2+2T)$ and from Eq. (6) $f = 12/(aT^3+3T^2)$. Therefore

$$T_{st} = \frac{4}{\sqrt{1+2a/3+a^2}-a+1},$$
$$f_{st} = \frac{1+3a^2}{2}+\frac{1-3a}{2}\sqrt{1+2a/3+a^2},$$

and, in particular, the well known results $T = 3$, $j = 4/9$, $f = 0$ if $a = \infty$ for the CL flow, and $T = 2$ when $a = 0$ (no current). These formulas are in agreement with corresponding relations found in Refs. 8 and 10.

On the initial time interval $0 < t < T_1$ using the approximation Eq. (13) with a finite N we have

$$f(T_1) = \sum_{i=0}^{N} c_i^1, \quad \varrho(T_1) = aT\sum_{i=0}^{N}\frac{c_i^1}{i+1},$$
$$v(T_1) = \frac{T_1}{2}\sum_{i=0}^{N}\frac{c_i^1}{i+1}\left(1+\frac{aT_1}{i+2}\right).$$

Using these notations the straightforward but tedious calculations allow to rewrite Eqs. (15a), (15b), and (17a)–(17c), respectively, in the following form:

$$\frac{1}{2}\sum_{i=0}^{N}\frac{c_i^1T_1^2}{(i+1)(i+2)}\left(1+\frac{aT_1}{i+3}\right) = 1, \tag{19a}$$

$$f(T_1)+\varrho(T_1)-\frac{a}{2}\sum_{i,j=0}^{N}\frac{c_i^1c_j^1T_1^3}{(i+1)(i+j+2)(i+j+3)} \times\left(1+\frac{aT_1}{i+j+4}\right) = 1, \tag{19b}$$

$$2\sum_{i=0}^{N}c_i^1\left(1+\frac{i}{aT_1}\right) = \sum_{i,j=0}^{N}\frac{c_i^1c_j^1T_1^2}{(i+1)(i+j+2)}\left(1+\frac{aT_1}{i+j+3}\right), \tag{19c}$$

$$\frac{1}{T_1}\sum_{i=0}^{N}c_i^1 i\left(1+\frac{i-1}{aT_1}\right) = \frac{2f(T_1)+\varrho(T_1)}{4a}\varrho(T_1) - \frac{4v^2(T)}{aT_1^2+2T}, \tag{19d}$$

$$8\frac{v^2(T_1)(aT_1+1)}{(aT_1^2+2T_1)^2}\left[3-\frac{4v(T_1)}{aT_1^2+2T_1}\right]+[f(T_1)+\varrho(T_1)]\times\left[\frac{f(T_1)}{2}-\frac{6v(T_1)}{aT_1^2+2T_1}\right]=\frac{1}{aT_1^3}\sum_{i=0}^{N}c_i^1 i(i-1)(i-2+aT_1)-\frac{\varrho(T_1)}{2aT}\sum_{i=0}^{N}ic_i^1. \quad (19e)$$

Equations (19) are five algebraic equations for $f(t)$ and T_j on each consequent interval $\Theta_{j-1}\le t\le T_j+\Theta_{j-1}$, and by differentiation one can have as many equation as he wants if their practical use is reasonable. Our method of using a polynomial form (13) neglects the higher derivatives of $f(t)$, and we think that to go beyond d^3f/dt^3 (in Eq. (17c)) is not sensible.

Note that Eq. (12) can be rewritten using the expansion (13) of $f(t)$ on the first interval $0\le t<T_1$ as following:

$$\sum_{i=0}\frac{c_i^1}{T_1^i}(at^i+it^{i-1})=a\sum_{i=0}\frac{c_i^1}{T_1^i}\frac{t^{i+1}}{i+1}\left[\omega+\frac{1}{2}\sum_{j=0}\frac{c_j^1t^{j+1}}{T_1^j(i+j+2)}\left(1+\frac{at}{i+j+3}\right)\right].$$

This relationship and the BC make it possible to evaluate all the coefficients of the series (13). The first of them are

$$c_0^1=1,\quad \frac{c_1^1}{T_1}=-a,\quad \frac{c_2^1}{T_1^2}=\frac{\omega a+a^2}{2},\quad \frac{c_3^1}{T_1^3}=a\frac{1-4\omega a-2a^2}{12},\ldots, \quad (20)$$

and the computation of the higher ones is more laborious but straightforward. The general term $b_k=c_k^1/T_1^k$ can be computed from the following equation:

$$(n+1)b_{n+1}=-ab_n+a\omega\frac{b_{n-1}}{n}+\frac{a}{4}\sum_{i=0}^{n-2}\frac{b_ib_{n-2-i}}{(i+1)(n-1-i)}+\frac{a^2}{4n}\sum_{i=0}^{n-3}\frac{b_ib_{n-3-i}}{(i+1)(n-2-i)}.$$

This provides an important information about the first coefficients $c_0^1,c_1^1,\ldots$, but the convergence of the series is very slow in this case (regime I). We use only these two coefficient then model $f(t)$ with a finite sum and apply Eqs. (19) for this model of the field emission.

Two more remarks. A simple analysis of Eqs. (1) and (2) allows also to find the upper bound for T_1 in the general case. By integrating twice the Maxwell equation in Eq. (2) and using BC we have first

$$\frac{\partial\varphi}{\partial x}(x,t)=f(t)+\int_0^x\rho(x',t)dx', \quad (21a)$$

and then the potential in the form

$$\varphi(x,t)=xf(t)+\int_0^x(x-x')\rho(x')dx'. \quad (21b)$$

Let us consider the moving boundary $X(t)$ between the space charge and vacuum and thus $\rho(x>X)=0$. Now we exhibit Eq. (21b) at the anode $x=1$ and Eq. (21a) at $x=X(t)$

$$1=f(t)+\int_0^{X(t)}(1-x')\rho(x')dx',\qquad \frac{\partial\varphi}{\partial x}(X,t)=f(t)+\int_0^{X(t)}\rho(x',t)dx'. \quad (21c)$$

Combining Eq. (21c) we derive

$$\frac{\partial\varphi}{\partial x}(X,t)=1+\int_0^{X(t)}x'\rho(x',t)dx'. \quad (22)$$

Equation (22) shows explicitly that the frontal electrons are always accelerated by the field stronger than the vacuum field, i.e., 1, and therefore the crossing time T_1 is less than 2, which is consistent with the intuition. One more important statement relates to the continuity of the cathode field at $t=T_1$. As $\varphi(1,t)$ is constant while the cathode field $f(t)$ and current $j(0,\ t)$ are continuous functions, the derivative df/dt in Eqs. (17) is continuous at T_1 because $X(t)$ approaches and stays equal to 1. This justifies our technique of solving Eqs. (17) on the second and the following step and keeping continuity of $f(t)$ with its first derivative at $t=T_j$. It is easy to see that the second derivative of f at T_1 is discontinuous in general.

IV. FLOW EVOLUTION FOR DIFFERENT CATHODE EMISSIVITY

Using Eqs. (19) for $0<t<T_1$ and Eq. (20) we evaluate $c_3^1,\ldots,c_6^1$ and T_1. Then we will study the flow on the interval $T_1<t\le T_1+T_2$, where T_2 is the transition time for electrons emitted at $t=T_1$, and so on step by step until the current stabilization. On each subsequent interval the IC for $f(T_i)$ and its first derivative will be matched with the previous values found by calculations. Our 5 equations are Eqs. (15a), (15b) and Eqs. (17a), (17b), (17c); this implies that on all intervals we can evaluate corresponding T_i and four coefficients $c_2^i\ldots c_5^i$ because two coefficients c_0^i and c_1^i are determined by the IC at $t=\Theta_{i-1}$ mentioned above. The technique of using Eqs. (19) cannot be applied on the later intervals because the upper limit of the integral in (22) must be equal to 1 there. It turns out that relatively small current variations at later time

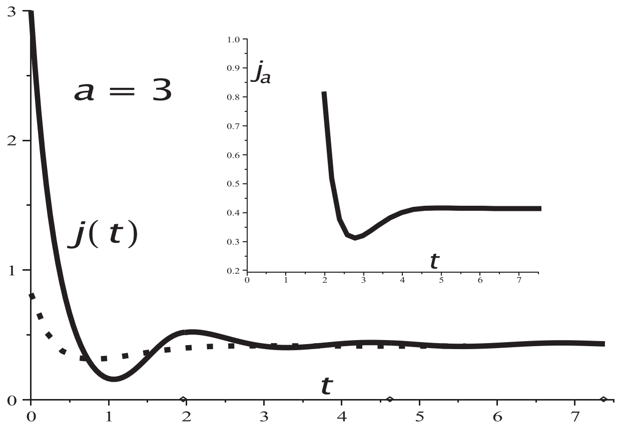

FIG. 1. Evolution of cathode and anode currents for $a = 3$. Inset shows anode current j_a. Dotted curve for j_a shifted left by T_1. On t-axis are shown points for T_1, Θ_2, Θ_3.

$t > T_1$ are well described without using higher derivatives of $f(t)$.

Taking $a = 3$ this scheme can be realized, and the time dependence of the cathode current density is exhibited in Fig. 1.

The stabilization process almost finishes after $t > T_1 + T_2$ and $j(t)$ asymptotically approaches to the stationary[10] value. For $0 \leq a \leq 3.5$ the asymptotic values of j for $a = 1$, 2, 3, 3.5 are found at corresponding $t = T_1 + T_2 + T_3$ to be 0.367, 0.417, 0.432, 0.434, respectively. They differ from the currents in the steady states at most by 0.4% for $a = 3.5$. However, unfortunately larger values of a cannot be considered by this approximation because the minimum current (see Fig. 1) becomes negative. This contradicts the initial assumption and physics: the flow becomes 0 when $f \leq 0$, i.e., ($j \neq af$). The minimum of $j(t)$ at $t \approx 0.1$, in Fig. 1, 0.02 is already close to zero. Note that the time of the flow stabilization is in agreement with the linear theory[7] for small perturbations of the stationary state. Using Eq. (10) we exhibit also the anode current evolution which clearly starts at $t = T_1$. On the first step the transition time $T_1 = 1.98$ is close to 2, which corresponds to the case when an electron moves without space charge in the same diode. Surprisingly T_1 is quite close to 2 though the space charge pushes forward the first bunch of electrons while the anode attracts them. Starting from the second step all T_j stay close ($T_2 = 2.66$, $T_3 = 2.75$) to the asymptotic stationary value $T_{as} = 2.732$ (Ref. 10) as well as the current density $j_{as} \rightarrow 0.431$. These quantities for $a = 3$ are not far away from the Child-Langmuir ones: $T_{CL} = 3$, $j_{CL} = 0.444$ when the emissivity is infinite, $a = \infty$.

Using Eq. (10) one can find the time dependence of the anode current which is shown in the inset of Fig. 1 for the case when $a = 3$. It appears only for $t \geq T_1$. We plot the cathode current together with the anode current shifted for the sake of illustration to the left by T_1 (the electrons emitted at $t = 0$ reach the anode at $t = T_1$). In order to have a sufficient number of points for $j_a(t)$ we solve Eqs. (15) not only for $\tau = 0$ and $\tau = T_j$, $j = 1, 2, \ldots$ but also in intermediate points with smaller intervals like $\tau_k = 0.1k$, $k = 1, 2, \ldots$. This allows to find also corresponding values for the cathode current j and T.

The asymptotic behavior of both graphs in Fig. 1 is identical. Interestingly the initial anode current is lower than the cathode one as a dense space charge is formed temporarily in front of the electrons which were emitted later. It decreases their acceleration and thus makes slower their speed near the anode.

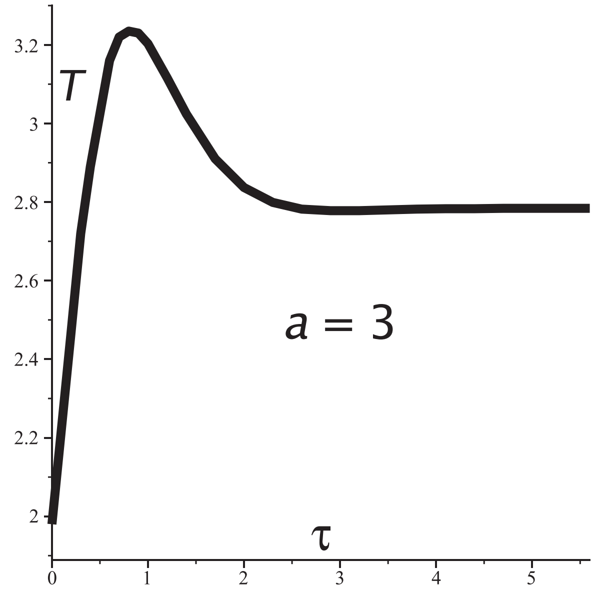

FIG. 2. Transition time T for $a = 3$ as function of τ for $0 < \tau < 5.6$.

In Fig. 2 is shown the time dependence of the transition time T starting from $T = T_1$. The frontal electrons move a little faster than in vacuum and they push back the layer of the space charge following behind them; thus, it moves even slower than in the stationary conditions.

Fig. 3 exhibits the flow properties when $a = 1$. The stationary values in this case are $T_a = 2.449$ and $j_a \rightarrow 0.367$ while the oscillations are rather weak, i.e., when $a < 1$ the diode comes to its stationary state with a short delay (about one half of the transition time in free space) and almost monotonically.

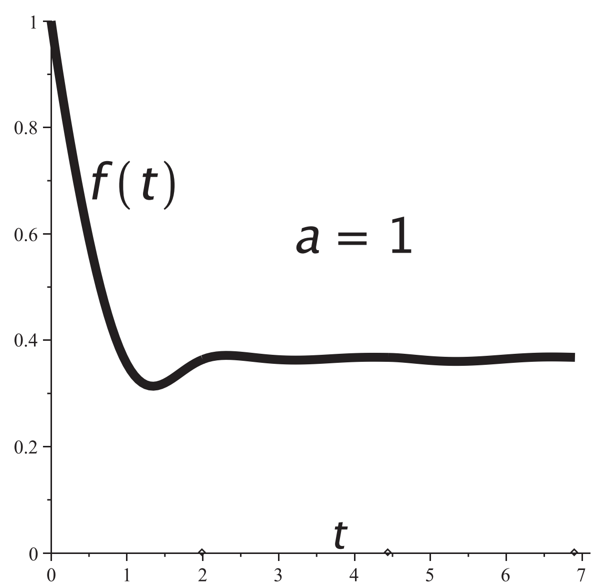

FIG. 3. Evolution of cathode field for $a = 1$ on interval $0 < t < \Theta_3$. Points on t-axis correspond to $T_1 = 1.998$, $T_2 = 2.449$, $T_3 = 2.458$.

A. Regime II with fixed injected current

This setup is much simpler than the case of the field emission (even for the present emission model) because Eq. (12) becomes linear for the unknown function $f(t)$. In this case Eqs. (9) and (19) are valid, and they are linear for c_n^i and therefore much simpler.

1. $V_a = 1$

Let us substitute $j = Const$ into Eq. (12) and perform the integration

$$\frac{df}{dt}(t) - \frac{j}{2}\int_0^t t' f(t')dt' = -j + \omega jt + \frac{j^2 t^3}{12}. \tag{23}$$

Equation (23) is exact for $t \le T_1$. It can be solved (see Appendix) explicitly with the help of Bessel functions, but the solution in the form of a series similar to Eq. (16)

$$f(t) = \sum_{n=0} b_n t^n$$

provides excellent precision and a faster numerical procedure. We substitute it into Eq. (23), apply the IC $f(0) = 1$, and obtain the following relations which define the cathode field on the first time interval

$$b_0 = 1, \quad b_1 = -j, \quad b_2 = \frac{\omega j}{2}, \quad b_3 = \frac{b_0 j}{12}, \quad b_4 = -\frac{j^2}{48},$$
$$b_{n+3} = \frac{j b_n}{2(n+2)(n+3)}, \quad (n \ge 2).$$

The series converges rapidly for not very large j and ω, and 12-15 terms guarantee a good precision but we used about 30 terms.

For $t > T_1$ one needs to construct the following set of equations for $t = T(t)$ analogous to Eqs. (17) for each consequent step

$$\int_{t-T}^{t}\left[j\frac{(t-t')^2}{4} + f(t')\frac{t-t'}{2} + \omega\right]dt' = 1, \tag{24a}$$

$$f(t) + j\int_{t-T}^{t}\left[j\frac{(t-t')^3}{4} + f(t')\frac{(t-t')^2}{2} + \omega(t-t')\right]dt' = 1, \tag{24b}$$

$$\frac{df}{dt}(t) - \frac{jT}{2}\int_{t-T}^{t} f(t')dt' + j\int_{t-T}^{t} f(t')(t-t')dt' = 0, \tag{24c}$$

$$\frac{d^2f}{dt^2}(t) + j\frac{[Tf(t-T) - \int_{t-T}^{t} f(t')dt']^2}{jT^2 + 2Tf(t-T) + 4\omega} = jT\frac{f(t) + f(t-T)}{2}. \tag{24d}$$

Using the approximation (16), Eqs. (24) are converted into a set similar to Eq. (19)

$$\frac{jT^3}{12} + \frac{T^2}{2}\sum_{i=0}^{N}\frac{c_i}{(i+1)(i+2)} + \omega T = 1, \tag{25a}$$

$$\sum_{i=0}^{N} c_i + \frac{j\omega T^2}{2} + \frac{j^2 T^4}{16} + jT^3\sum_{i=0}^{N}\frac{c_i}{(i+1)(i+2)(i+3)} = 1, \tag{25b}$$

$$\sum_{i=1}^{N} i c_i\left[1 - \frac{jT^3}{2(i+1)(i+2)}\right] = 0, \tag{25c}$$

$$\sum_{i=2}^{N} i(i-1)c_i = \frac{jT^3}{2}\left(f_0 + \sum_{i=0}^{N} c_i\frac{i-1}{i+1}\right) - \frac{jT^4}{4\omega + 2Tf_0 + jT^2}\left(f_0 - \sum_{i=0}^{N}\frac{c_i}{i+1}\right)^2, \tag{25d}$$

where f_0 denotes the cathode field at the start of the corresponding step.

The results of such computation for $\omega = 0.5$ with $j = 1$ and $j = 1.5$ are shown in Fig. 4. The current becomes stable without oscillations in this case and after 5-8 transition times (which vary approximately from 1.2 to 1.5) the cathode field differences from the quantities, evaluated in Ref. 7 for the stationary flows, are about 10^{-4}%.

When $j = 1$ there is no virtual cathode and $f > 0$, but with the same ω the stronger current $j = 1.5$ creates a virtual cathode and f becomes negative. The transition times on the consequent steps are 1.211, 1.355, 1.367, 1.368, 1.368 for $j = 1$ and 1.201, 1.437, 1.482, 1.491, 1.493 for $j = 1.5$, and they practically become stabilized later. The initial anode current densities at $t = T_1$ are 0.814 and 1.127 for $j = 1$ and 1.5, respectively. Note that the frontal electron velocities at the anode are 1.199 and 1.234, i.e., larger than they would be in vacuum $\sqrt{1+\omega^2} = 1.118$, while the anode current

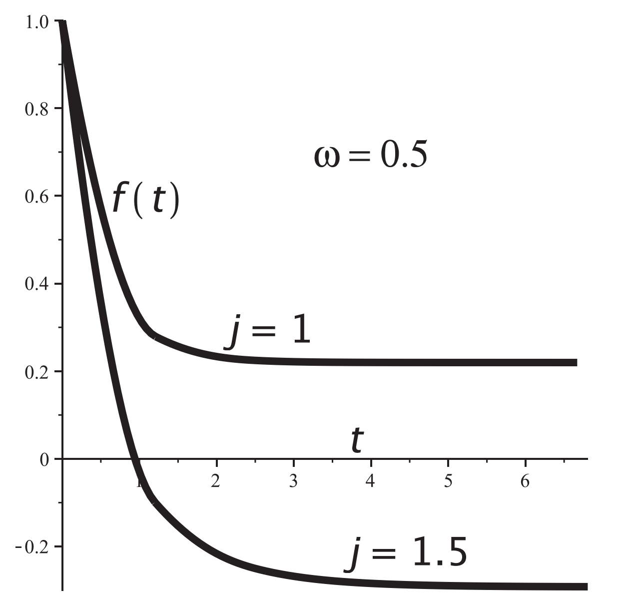

FIG. 4. Cathode electric field evolution when $j = 1$ and $j = 1.5$ with $\omega = 0.5$.

densities are lower than corresponding values of j. The frontal electrons in vacuum would need $T = 1/\omega$ for crossing the diode, but on the first interval $t < T_1$ they move faster being pushed by the field of particles behind them.

2. $V_a = 0$

We consider now the closed system with equipotential electrodes. The injected current creates the space charge, and the lowest (negative) potential - virtual cathode - in the diode occurs somewhere at $0 < x < 1$.

In Fig. 5 we show the cathode field behavior when the injected current is fixed for the particle initial velocity ω of three different values 0.8, 1.0, and 1.4. It is clear that the larger is ω the smaller is the charge density, and when $\omega = 0.7$ we did not reach the flow stabilization because the anode cannot absorb the incoming flow and probably the charge density inside the diode grows infinitely; there are possibilities also for some chaotic behavior which is beyond our present computational techniques. Clearly the same can happen also with a positive anode when the current density is large while ω is small.

For $\omega = 0.8$ the field f tends to -1.15, it becomes closer to zero when ω increases. The cathode electric field always goes to its asymptotic value without oscillations (see Fig. 5) and gets stabilized relatively fast. The values of transition times $T_k, k = 1..n$ vary from 0.8 to 1.3, there are about five of them in Fig. 5. Electrons after entering the diode meet an already formed space charge and transition times get longer depending on its density.

In Fig. 6 is shown the behavior of the space charge boundaries $X(t)$ and the locations of the virtual cathode $y(t)$ for $\omega = 0.8$ and 1.4 on the corresponding first time intervals $0 < t < T_1$ (≈1.19 and 0.72, respectively).

As seen in Fig. 6, plotted for $j = 1.5$, $\omega = 0.8$, 1.4, location $X(t)$ of the frontal electrons moves almost by a linear in t law $X(t) \approx \omega t$, the maximum deviation is only a few percents. The construction of Fig. 6 is simple: the minimum

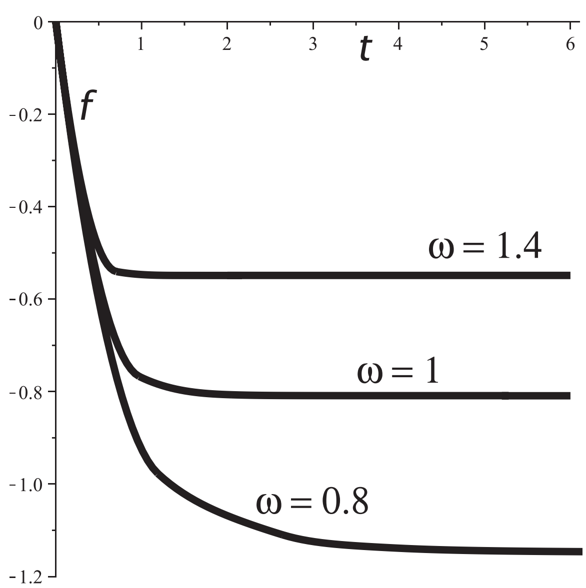

FIG. 5. Cathode electric field evolution when $j = 1.5$ and $\omega = 0.8$, 1, 1.4.

corresponds to zero electric field, and Eq. (5) allows to evaluate τ for each $t \le T_1$ as soon as $f(t)$ is computed. Then y is evaluated by Eq. (4a) which is also used for $X(t)$ when τ is replaced by zero. A similar graph can be made for the case $V_a = 1$ above when the virtual cathode is realized there and $f(t) < 0$, say for the case illustrated by the lower curve in Fig. 4.

The evolution of the potential of the virtual cathode is shown in right figure. For smaller initial velocity the space charge is denser and thus the potential is more negative. On the next time intervals the potential goes deeper and approaches to its stationary value. In Fig. 6 we see that a larger initial velocity ω makes T_1 shorter and curves stop at corresponding values T_1.

V. ON THE VALIDITY OF THE METHOD USED

We present now a simple sufficient condition which roughly indicates the limitation for the method used here of analyzing the flow when j is fixed externally. Let us consider at time t two electrons emitted at τ_1 and $\tau_2 > \tau_1$ and assume that $x(\tau_1, t) \le x(\tau_2, t)$, i.e., a later emitted electron can overcome an earlier emitted one. Thus Eq. (4a) implies for them

$$x(\tau_1, t) - x(\tau_2, t) = \int_{\tau_1}^{\tau_2} \left[j\frac{(t-t')^2}{4} + f(t')\frac{t-t'}{2} \right] dt' + \omega(\tau_2 - \tau_1) \le 0,$$

or

$$j\frac{(t-\tau_1)^3 - (t-\tau_2)^3}{6} + 2\omega(\tau_2 - \tau_1) \le -\int_{\tau_1}^{\tau_2} f(t')(t-t')dt'. \tag{26}$$

In particular Eq. (26) means that there are exist two close τ_1, τ_2 (such that $\delta = \tau_2 - \tau_1 \ll t$), which satisfy this inequality. Then Eq. (26), by keeping only linear in δ terms, can be reduced approximately to

$$j\frac{t-\tau}{2} + 2\frac{\omega}{t-\tau} \le -f(\tau), \quad \text{where} \quad \tau = (\tau_2 + \tau_1)/2. \tag{27}$$

Equation (27) can be satisfied even on the initial time interval $t < T_1$.

This situation is illustrated in Fig. 7 where the current $j = 1.5$ is formed by the flow with the initial velocity only $\omega = 0.3$.

In this case $T_1 = 2.557$ and when, for example, $\tau = 1.1$, Eq. (27) holds for the emission time on the interval $1.022 < \tau < 2.37$. If $\tau = 1.022$ the curve only touches the line $-f(\tau)$ at $t = 1.86$. In the case $\tau = 1.1$ in Fig. 7 the electrons pass some of them emitted earlier when $1.7 < t < T_1$. Such regime cannot be treated by our method. The violation of our assumption on this subject takes place also for ω significantly larger than 0.3, especially for $t > T_1$, we show here only a simplest situation. Note that the left side of Eq. (27) has its minimum $2\sqrt{j\omega}$ when $t - \tau = 2\sqrt{\omega/j}$ which means that if

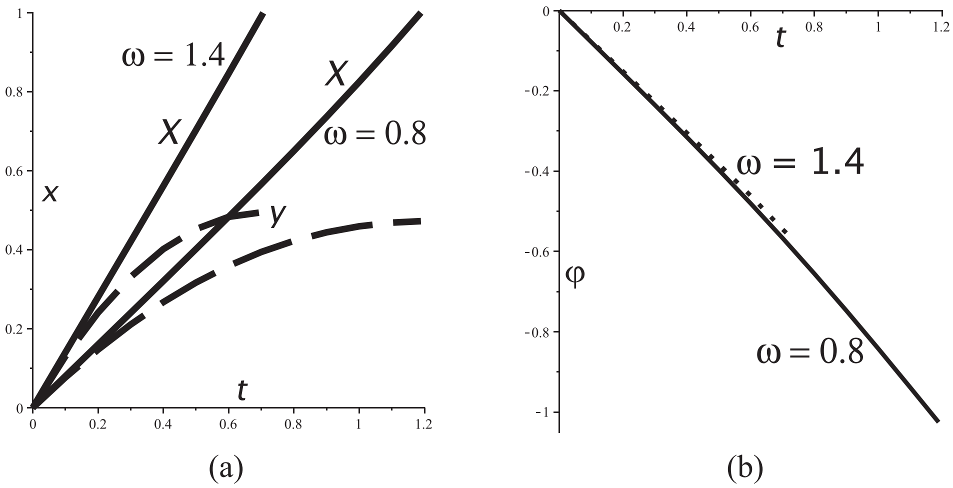

FIG. 6. (a) Plots of $X(t)$, $y(t)$. (b) Virtual cathode potential $\varphi(y,t)$.

$$-f(t) < 2\sqrt{j\omega}, \quad 0 < t < \infty, \tag{28}$$

the technique of this work is valid.

Otherwise the inequality (27) can hold for some t, τ where the point (x, t) in the Eulerian variables corresponds to at least two different points (τ_1, t) and (τ_2, t) when the flow is described by the Lagrangian coordinates. This undermines our method for the case with $j = const$.

VI. DISCUSSION

In conclusion we note that the approximate treatment of the large scale flow variation developed above is based on computations within short time intervals $T_1, T_2, \ldots$. In each of them we have five equations which make possible the approximate evaluation of four flow parameters and provide the IC for the next interval. Computations on the first time interval $0 < t < T_1$ can be performed in closed form using Eq. (12) which makes them more precise especially in the case of an injected current $j = const$. Our techniques are illustrated only for the case when the system was turned on, but its generalization for any given initial state is straightforward: exactly as it was performed in the paper in transitions between the time intervals Θ_{n-1} and Θ_n. The field emission model is more difficult: errors of computations are larger on the first interval where $f(t)$ varies significantly and Eq. (12) is not very helpful; the results for later times are more accurate.

All our work was performed in dimensionless units which easily can be converted into the physical variables using corresponding equations in Refs. 7 and 8. The limitations of our techniques are outlined and some estimates are performed. The method of flow evolution is realized in our paper for an externally injected current and for a simple model of the field emission law, but it clearly can be extended to more realistic emission dependences, and we are working presently in this direction.

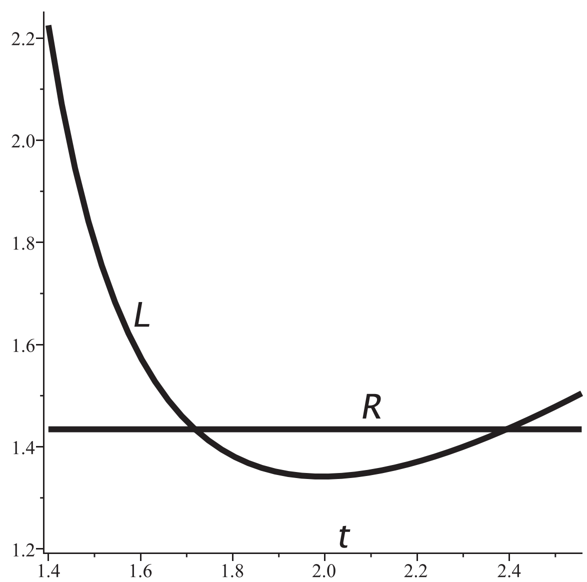

FIG. 7. Left L and right R sides of Eq. (27) for electrons emitted at $\tau = 1.1$.

ACKNOWLEDGMENTS

Work was supported in part by AFOSR Grant No. FA 9550-10-1-0131 and by NSF Grant No. DMR-1104501. The motivation of this study was initiated by our discussions with N. J. Fisch to whom we express our thanks. We also benefited from discussions with J. Luginsland.

APPENDIX: EXACT SOLUTION FOR j = const

Evolution of the cathode electric field in the case of externally injected current is described by Eq. (23) which can be solved analytically for $t \le T_1$. By differentiating Eq. (23) we have

$$\frac{d^2 f}{dt^2}(t) - \frac{jt}{2} f(t) = \omega j + \frac{j^2 t^2}{4} \equiv h(t). \tag{A1}$$

If $x_a = V_a = 1$ the IC for $f(t)$ found earlier

$$f(0) = 1, \quad f'(0) = -j. \tag{A2}$$

Using standard procedures,[12] we find[13] two independent solutions of the homogeneous equation

$$f_1(t) = \sqrt{t} K_{1/3}\left(\sqrt{2j} t^{3/2}/3\right), \quad f_2(t) = \sqrt{t} I_{1/3}\left(\sqrt{2j} t^{3/2}/3\right),$$

where $K_\nu(z), I_\nu(z)$ are the modified Bessel functions, whose Wronskian is

$$W\{K_\nu(z), I_\nu(z)\} = 1/z.$$

By a simple calculation we find

$$W\{f_1, f_2\} = f_1(t)f_2'(t) - f_2(t)f_1'(t) = 3/2. \tag{A3}$$

Using Eq. (A3) the solution of Eq. (A1) can be written[12] in the form

$$f(t) = C_1 f_1(t) + C_2 f_2(t) + \frac{2}{3}\int_0^t [f_2(t)f_1(s) - f_1(t)f_2(s)]h(s)ds. \tag{A4}$$

We satisfy the IC (Eq. (A2)) for $V_a = 1$ by taking

$$C_1 = \frac{\sqrt{3}\Gamma(2/3)}{\pi}\left(\frac{j}{18}\right)^{1/6},$$
$$C_2 = \Gamma(2/3)\left(\frac{j}{18}\right)^{1/6} - \Gamma(4/3)(18j^5)^{1/6}.$$

Calculation show that $-f(t)$ can be large if ω is small. These Maple calculations by the polynomial approximation for Eq. (23) are faster than using the exact Eq. (A4). Both functions f_1 and f_2 are non-negative and $t \geq s$ in Eq. (A4). It is easy to show that the integral term in Eq. (A4) is positive: $f_2(t)/f_1(t) - f_2(s)/f_1(s) \geq 0$ because the ratio $f_2(t)/f_1(t)$ is an increasing function as its derivative has the same sign as $W\{f_1, f_2\}$ (see Eq. (A3)). When $V_a = 0$ we have

$$C_1 = 0, \ C_2 = -\Gamma(4/3)(18j^5)^{1/6}.$$

This is the case studied in Ref. 14 by numerical simulations where however $j(t)$ increased linearly before becoming constant.

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