STATISTICAL MECHANICS OF CONTINUOUS SPIN SYSTEMS

by

J.L. LEBOWITZ AND E. PRESUTTI

U.S.A. ITALY

Abstract: We present results relating to the existence and uniqueness of the free energy and equilibrium states for classical continuous spin systems with superstable interactions.

We consider the lattice $\mathbb{Z}^d$ at each site of which there is a vector spin variable $S_x$, $x \in \mathbb{Z}^d$, $S_x \in \mathbb{R}^d$. We denote by $S \in \{S\}$ a spin configuration on $\mathbb{Z}^d$. Each $S_x$ has associated with it an intrinsic positive measure $\mu(dS)$, the same for all sites, such that $\mu(dS) e^{-\alpha S^2} < \infty$ for $\alpha > 0$. The energy of a given spin configuration $S_\Lambda$ in $\Lambda \subseteq \mathbb{Z}^d$ consists of both pair and self interactions and satisfies the following conditions:

a) **Superstability** There exists $A > 0$, $C \in \mathbb{R}$ such that

$$U(S_\Lambda) \geq \sum_{x \in \Lambda} [A S_x^2 - C]$$

where $S_\Lambda$ is a configuration in $\Lambda$.

b) **Regularity** If $\Lambda_1$, $\Lambda_2$ are disjoint then their interaction energy $W(S_{\Lambda_1}, S_{\Lambda_2}) = U(S_{\Lambda_1} \cup S_{\Lambda_2}) - U(S_{\Lambda_1}) - U(S_{\Lambda_2})$ has the bound

$$\left| W(S_{\Lambda_1}, S_{\Lambda_2}) \right| \leq \frac{1}{2} K \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} |S_x| |S_y| |x-y|^{-\nu-\delta}$$

where $|x| = \max_{1 \leq i \leq d} |x_i|$, $|S| = \left[ \sum_{i=1}^{d} (S^i)^2 \right]^{\frac{1}{2}}$.

For $\Lambda$ bounded in $\mathbb{Z}^d$ we consider the restriction, $S_\Lambda^c$ of $S$ to $\Lambda^c$ and define the partition function $Z(\Lambda|S_\Lambda^c)$ and free energy per site $F(\Lambda|S_\Lambda^c)$ with 'boundary conditions' (b.c.) determined by $S$ as

$$Z(\Lambda|S_\Lambda^c) = \int \mu(\Lambda)(dS_\Lambda) \exp \left[-U(S_\Lambda) - W(S_\Lambda|S_\Lambda^c)\right]$$

$$F(\Lambda|S_\Lambda^c) = |\Lambda|^{-1} \log Z(\Lambda|S_\Lambda^c)$$

where $\mu(\Lambda)(dS_\Lambda) = \prod_{x \in \Lambda} \mu(dS_x)$, $|\Lambda| = \# $ of sites in $\Lambda$. (The dependence on temperature and magnetic field is included in $U$ and $W$; it will be made explicit when necessary.)
Theorem 1. Let (1) and (2) hold and let $S \in \mathcal{H}_a$:

\[ \mathcal{H}_a = \{ S \in \mathcal{H} \mid S_y^2 < \lambda_n |y| \text{ for } |y| > 1 \} \]. Let $\{ \Lambda_n \}$ be a sequence of increasing domains tending to $Z^0$ in the sense of Van Hove $[1]$ then

\[ \lim_{n \to \infty} F(\Lambda_n | S^{\Lambda_n}) = F \] exists and is independent of the sequence $\{ \Lambda_n \}$ and of the b.c.

\[ S^{\Lambda_n} \]

Remark: "Zero" b.c. correspond to $S = 0$ for $x \in \mathcal{A}^c$. The thermodynamic limit of the "periodic" b.c. free energy can also be shown to exist and be equal to $F$.

A probability measure $\nu$ on the configuration space $\{ S \}$ is said to be regular if it satisfies the following condition: There exists $\gamma > 0$, $\delta > 0$, such that for every $\Lambda$ bounded in $Z^0$ and $N^2 > 0$ the following holds:

\[ \nu[B(N^2|\Lambda)] \leq \exp[-|\Lambda| (\gamma N^2 - \delta)] \]

where

\[ B(N^2|\Lambda) = \{ S \mid \sum_{x \in \Lambda} S_x^2 \geq N^2 |\Lambda| \} \]

For $\Lambda$ bounded in $Z^0$ we denote by $F(\Lambda|\nu)$ the free energy in $\Lambda$ for boundary conditions specified by the measure $\nu$ as

\[ F(\Lambda|\nu) = \int \nu(dS) F(\Lambda|S^{\Lambda}) \]

Theorem 2. Let (1) and (2) hold and let $\Lambda_n$ be as in Theorem 1 then for $\nu$ regular $\lim_{n \to \infty} F(\Lambda_n | \nu) = F$.

The finite volume equilibrium measure with boundary conditions $S^{\Lambda_n}$

\[ \nu_{\Lambda_n}(dS_{\Lambda_n} | S^{\Lambda_n}) \]

is given by

\[ \nu_{\Lambda_n}(dS_{\Lambda_n} | S^{\Lambda_n}) = Z^{-1}(\Lambda|S^{\Lambda_n}) \mu(dS_{\Lambda_n}) \exp[-U(S_{\Lambda_n}) - W(S_{\Lambda_n} | S^{\Lambda_n})] \]  \hspace{1cm} (3)

a measure $\nu$ on $\{ S \}$ is said to be an equilibrium measure (for our system) if its conditional probabilities $\nu(dS_{\Lambda_n} | S^{\Lambda_n})$ satisfy the Dobrushin, Lanford and Ruelle (DLR) $[2]$ equations, i.e. eq. (3).

Theorem 3. Let the conditions of Theorem 1 be satisfied and let $\nu_{\Lambda_n}(dS_{\Lambda_n} | S^{\Lambda_n})$ be finite volume equilibrium states then it is always possible to choose subsequences $\{ n_1 \}$ (which may depend on the b.c.) such that $\nu_{\Lambda_{n_1}}(dS_{\Lambda_{n_1}} | S^{\Lambda_{n_1}})$ is a regular equilibrium measure on $\{ S \}$, $[3]$.

The one component spin system, $S_x \in \mathbb{R}$, will be called ferromagnetic with translation invariant interactions if

\[ U(S_{\Lambda_n}) = -\frac{1}{2} \sum_{x \neq y \in \Lambda_n} J(x,y) S_x S_y - \sum_{x \in \Lambda_n} \psi(S_x) - h \sum_{x \in \Lambda_n} S_x \cdot \mathbf{1}(x) \geq 0. \]

Theorem 4. Let $\nu$ be a regular equilibrium measure of a ferromagnetic system in an external field $h$ whose interactions satisfy (1) and (2) then $\nu$ is unique (and hence translation invariant) whenever the infinite volume free energy $F(h)$...
is differentiable with respect to $h$ [4].

Acknowledgments

Work supported by NSF Grant #MPS 75-20638. We would like to thank Prof. David Ruelle for very valuable discussions.

References

J.L. Lebowitz
Belfer Graduate School of Science
Yeshiva University
New York, N.Y.

E. Presutti
Istituto Matematico Universita
dell'Aquila, L'Aquila, Italy