Thermal Conductivity and Weak Coupling

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We want to consider models that are locally chaotic, that means models constituted by 'simple' chaotic systems coupled between them by an interaction, either by a smooth potential or by collisions. The uncoupled chaotic systems can be of deterministic dynamics nature or be perturbed by some energy conserving noise. All the exchange of energy between the system are regulated by the hamiltonian mechanism. From this point of view there is, at least conceptually, not so much difference between a chaotic deterministic or a stochastic systems.

These models are more approachable than systems like FPU, where even though non-linearity is important, only the large coupled system will have some chaotic properties. Furthermore one can study weak coupling limits in this locally chaotic situation.

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We want to study how energy diffuse **macroscopically** in these models.

Smooth coupled dynamics

$p_x \in \mathbb{R}, , q_x \in \mathbb{M},\ x \in \Lambda,\ |\Lambda| = N$ or $\Lambda = \mathbb{Z}$.

$$\mathcal{H}_{\varepsilon} = \sum_x \left[\frac{p_x^2}{2} + U(q_x) + \varepsilon V(q_x - q_{x-1}) \right]$$
$$= \sum_x \left[e_x + \varepsilon V(q_x - q_{x-1}) \right] = \sum_x e_x^{\varepsilon}$$

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$$\begin{aligned}\dot{q}_x &= p_x \\ \dot{p}_x &= \varepsilon \nabla V(q_{x+1} - q_x) - \varepsilon \nabla V(q_x - q_{x-1}) - \text{local dynamics}\end{aligned}$$

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Local chaotic or stochastic dynamics conserve $\{e_x\}_x$, one parameter family of equilibrium measures:

$$d\mu_\beta = \frac{e^{-\beta \mathcal{H}_\varepsilon}}{Z_\beta} \prod_x dp_x dq_x \qquad \beta = T^{-1} > 0$$

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$$\frac{d}{dt} e_x^{\varepsilon} = \varepsilon \left(\jmath_{x-1,x} - \jmath_{x,x+1} \right)$$ local conservation of energy.

$$\jmath_{x,x+1} = -p_x V'(q_{x+1} - q_x)$$ hamiltonian energy currents

Non-stationary behavior

We would like to prove that

$$\frac{1}{N}\sum_x G(x/N) e_x^{\varepsilon}(N^2 t) \underset{N\to\infty}{\longrightarrow} \int G(y) u(t,y) dy$$

with $u(t,y)$ solution of the nonlinear heat equation:

$$\partial_t u = \partial_y \left(\mathcal{D}_{\beta(u)} \partial_y u \right)$$

with the thermal diffusivity defined by the *Green-Kubo formula*:

$$\boxed{\mathcal{D}_\beta(\varepsilon) = \varepsilon^2 \chi_\beta^{-1} \sum_{x\in\mathbb{Z}} \int_0^\infty \langle \mathrm{J}_{x,x+1}(t) \mathrm{J}_{0,1}(0) \rangle_\beta \; dt}$$

$$\chi_\beta = \sum_x (< e_x^{\varepsilon} e_0^{\varepsilon} >_\beta - < e_0^{\varepsilon} >_\beta^2) > 0, \qquad\qquad \beta = \beta(u)$$

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Not clear under which initial conditions such limit would be true

Equilibrium Fluctuations: Linear response

Here is a theorem that has a clear and precise mathematical statement:
Assuming that the corresponding limits exist, we have

$$\boxed{\mathcal{D}(\varepsilon) = \frac{1}{\chi_\beta} \lim_{t\to\infty} \frac{1}{t} \sum_{x\in\mathbb{Z}} x^2 \left[< e_x^\varepsilon(t) e_0^\varepsilon(0) >_\beta - < e_0^\varepsilon >_\beta^2 \right]}$$

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Notice that

$$\frac{\left[< e^{\varepsilon}_x(t) e^{\varepsilon}_0(0) >_\beta - < e^{\varepsilon}_0 >^2_\beta\right]}{\sum_x (< e^{\varepsilon}_x e^{\varepsilon}_0 >_\beta - < e^{\varepsilon}_0 >^2_\beta} = p_t(0,x), \qquad \sum_x p_t(0,x) = 1$$

if positive can be seen as a transition probability of a random walk, whose $\mathcal{D}$ is the asymptotic variance.

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Conjecture:

$$\boxed{N p_{N^2 t}([Nx],[Ny]) \underset{N\to\infty}{\longrightarrow} (2\pi\mathcal{D})^{-1/2} \exp\left(-\frac{(x-y)^2}{2t\mathcal{D}}\right)}$$

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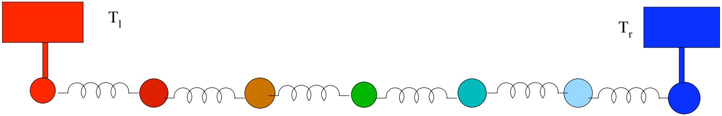
this is more challenging than proving existence for $\mathcal{D}$.

Stationary states: Fourier's Law

Many approach try to derive directly the stationary Fourier Law from the stationary state:



$$\lim_{N\to\infty} \frac{N \left\langle J_{x,x+1} \right\rangle_{ss}}{\Delta T} = \kappa = \frac{\chi}{T^2}\mathcal{D}$$

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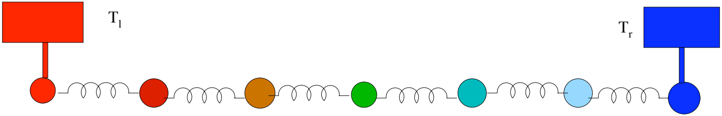

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This is mathematically more difficult, since the space-time sclae is hiddel in the stationary state.

Weak coupling: $\varepsilon \to 0$

A two step approach. First a Van Hove type limit:
$\varepsilon \to 0$ and $t \longrightarrow \epsilon^{-2} t$:

$$J_{x.x+1}(t) = \epsilon \int_0^{\epsilon^{-2} t} \mathrm{J}_{x,x+1}(s) ds$$

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Results for some local caotic dynamics:

- Liverani-Olla (JAMS 2012) Anharmonic oscillators + stochastic perturbation acting independently on each particle conserving $|p_x|^2$, dimension $\nu \geq 2$.
- Liverani-Dolgopiat (CMP 2012) Uniformly hyperbolic dynamics ('geodesic flow in negative curvature manifolds': deterministic).

Weak coupling Limit

Autonomous stochastic evolution of the energies:

Theorem

$$\mathcal{E}_x^{\varepsilon}(t) = e_x^{\epsilon}(\epsilon^{-2} t) \xrightarrow[\varepsilon \to 0]{\textit{in law}} \mathcal{E}_x(t)$$

solution of the system of SDE:

$$d\mathcal{E}_x(t) = d\tilde{J}_{x-1,x} - d\tilde{J}_{x,x+1}$$

$$d\tilde{J}_{x,x+1} = \alpha(\mathcal{E}_x(t), \mathcal{E}_{x+1}(t))\, dt + \gamma(\mathcal{E}_x(t), \mathcal{E}_{x+1}(t))\, dw_{x,x+1}(t)$$

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$\{w_{x,x+1}(t)\}_x$ independent Wiener processes,

$$\alpha(\mathcal{E}_x, \mathcal{E}_{x+1}) = e^{\mathcal{U}(\mathcal{E})}(\partial_{\mathcal{E}_{x+1}} - \partial_{\mathcal{E}_x})\left[e^{-\mathcal{U}(\mathcal{E})}\gamma^2(\mathcal{E}_x, \mathcal{E}_{x+1})\right]$$

$$\gamma^2(\mathcal{E}_0, \mathcal{E}_1) = \int_0^\infty \langle j_{0,1}(t) j_{0,1}(0)\rangle_{\epsilon=0,\mathcal{E}_0,\mathcal{E}_1}\,dt$$

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$$\iint_{\mathbb{R}\times\mathbb{M}} F\left(\frac{p^2}{2} + U(q)\right) \frac{e^{-\beta(\frac{p^2}{2}+U(q))} dp\, dq}{Z^{-1}(\beta)}$$

$$= \int_0^\infty F(\mathcal{E}) e^{-\mathcal{U}(\mathcal{E}) - \beta\mathcal{E}} d\mathcal{E}$$

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$$\prod_x e^{-\mathcal{U}(\mathcal{E}_x)-\beta\mathcal{E}_x} d\mathcal{E}_x, \qquad \beta > 0$$

reversible stationary probabilites on $\mathbb{R}_+^{\mathbb{Z}}$ for the energy stochastic dynamics.

Macroscopic Diffusion from the energy model

A further diffusive space-time scaling can be effectued in order to obtain the **heat equation** from this GL stochastic dynamics (another non-gradient stochastic dynamics, but reversible)
(C. Liverani, S. Olla, M. Sasada, in progress)

$$< \mathcal{E}_{Nx}(N^2 t)\mathcal{E}_{Ny}(0) >_\beta - \bar{\mathcal{E}}^2 \underset{N\to\infty}{\longrightarrow} \tilde{\chi}_\beta \frac{e^{-\frac{(x-y)^2}{2t\tilde{\mathcal{D}}_{vhl}}}}{\left(2\pi\tilde{\mathcal{D}}_{vhl}\right)^{-1/2}}$$

$$\tilde{\mathcal{D}}_{vhl} \neq \mathcal{D}$$

Convergence of Green-Kubo formula

This should be a less ambitious program: just prove the convergence of the Green-Kubo formula for the thermal conductivity:

$$\mathcal{D} = \varepsilon^2 \chi_\beta^{-1} \sum_{x\in\mathbb{Z}} \int_0^\infty \langle \mathrm{J}_{x,x+1}(t) \mathrm{J}_{0,1}(0)\rangle_\beta \ dt \ ,$$

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No result for deterministic hamiltonian models.

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C. Bernardin, S.O., JSP 2011

Theorem

If the hamiltonian dynamics is perturbed by a velocity flip random dynamics, then we have the existence of

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This is also true for more general stochastic perturbation with generator $L = A + S$, with $Sp_x = -\gamma p_x$ and conserving parity in p.

Strategy for Green-Kubo for the deterministic dynamics

joint work (in progress) with

C. Bernardin, F. Huveneers, J. Lebowitz, C. Liverani.

- Take a system of ε-coupled chaotic dynamics (Sinai's billiards, geodesic flows ...) where we *control* the exponential decay of correlations (when non-interacting).

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- This gives the convergence of the GK formula. Try now an expansion in the coupling $\varepsilon V(q_{x+1} - q_x)$:

$$\mathcal{D}(\varepsilon, \zeta) = \varepsilon^2 \mathcal{D}_2(\zeta) + \varepsilon^3 \mathcal{D}_3(\zeta) + \dots$$

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The hope is that, in this situations, one obtains the expansion of the thermal conductivity of the deterministic system.

Small coupling expansion of TC

Even for the stochastically perturbed model, the expansion is quite complicate, and has some surprises:

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the first term is the macroscopic diffusivity of the energy GL model, obtained through the weak coupling limit:

$$\begin{aligned} \mathcal{D}_2(\zeta) &= \left\langle \gamma_{0,1}^2 \right\rangle_\beta - \int_0^\infty \sum_x \langle \alpha(\boldsymbol{\mathcal{E}}_x(t), \boldsymbol{\mathcal{E}}_{x+1}(t)) \alpha(\boldsymbol{\mathcal{E}}_0(0), \boldsymbol{\mathcal{E}}_1(0)) \rangle_\beta \\ &= \tilde{D}_{VHL} \end{aligned}$$

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Similar expansion for the thermal conductivity defined through the NESS of the N system attached to Langevin heat bath at different temperatures.

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In the Dolgopiat-Liverani case the WC limit is taken with $\zeta = 0$, so one has $\tilde{D}_{VHL} = \mathcal{D}_2(0)$.

(Weak)-coupling of integrable systems

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- **Chain of Rotors** $V(q_x - q_{x-1}) = \cos(q_x - q_{x-1})$:

$$\mathcal{D}_2(\zeta) \sim O(1)$$

but $\gamma_\zeta^2(e_x, e_{x+1}) \to 0$ as $\zeta \to 0$. *Resonances*.