Universal conductivity properties in many body physics

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- Several quantum many body systems exhibit remarkable universality properties, which one would like to mathematically derive starting from microscopic models.
- I will present rigorous universality results for non integrable quantum spin chains and for graphene.
- Such results are based on Renormalization Group which appear naturally when universality is involved.
- It is remarkable that RG can be made fully rigorous. Technical problems include the convergence of the expansions, role of irrelevant terms,cancellations, emerging symmetries...

Non integrable Quantum Spin chains

- The Heisenberg XXZ spin chain $H_0 =$

$$-\sum_{x=1}^{L-1}[JS_x^1 S_{x+1}^1 + JS_x^2 S_{x+1}^2 + J_3 S_x^3 S_{x+1}^3 - hS_x^3]$$

where $S_x^\alpha = \sigma_x^\alpha/2$ for $i = 1, 2, \ldots, L$ and $\alpha = 1, 2, 3$, σ_x^α being the Pauli matrices ($J = 1$).

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- The above model can be solved by Bethe ansatz, and it is interesting to add a next-to-nearest neighbor interaction breaking exact solvability, that is consider

$$H = H_0 + H_1$$

$$H_1 = -\lambda \sum_{x=1}^{L-1}[S_x^1 S_{x+2}^1 + S_x^2 S_{x+2}^2 + S_x^3 S_{x+2}^3]$$

Linear response theory

- By the Peierls substitution $j_x = S_x^1 S_{x+1}^2 - S_x^2 S_{x+1}^1 + \lambda F_x$ where F_x is an expression *quartic* in the spin operators.

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- If $\rho_x = S^3_x - \frac{1}{2}$ and $(j^0_x, j^1_x) = (\rho_x, j_x)$

$$K^{\mu,\nu}_{\beta,\lambda}(p_0, p) = \int_0^\beta dx_0 e^{-ip_0 x_0} < \hat{j}^\mu_{x_0,p} \hat{j}^\nu_{x_0,p} >_{\beta,T}$$

and $< O >_\beta = \frac{\mathrm{Tr} e^{-\beta H} \mathcal{T}}{\mathrm{Tr} e^{-\beta H}}$, $O_{x_0} = e^{Hx_0} O e^{-Hx_0}$, T denotes truncation and $\mathcal{T}$ denotes time ordering.

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- The susceptibility is defined as $\kappa_\lambda = \lim_{p \to 0} \lim_{p_0 \to 0} \lim_{\beta \to \infty} K_{\beta,\lambda}^{00}(\mathbf{p})$.
- Using the Jordan-Wigner transformation it can be written in terms of fermions $a_x^\pm$.

Conductivity

- According to Kubo formula the conductivity at $T = 0$ is

$$\sigma_\lambda(\omega) = \lim_{\delta \to 0} \lim_{p \to 0} \lim_{\beta \to \infty} \frac{D_{\beta,\lambda}(\mathbf{p})}{ip_0}\Big|_{ip_0 \to \omega + i\delta}$$

where $\mathbf{p} = (p_0, p)$ and

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  is called Drude weight.
- An ideal conductor has a non vanishing D_λ (*infinite* dc conductivity); a normal conductor has a finite non vanishing $\sigma(0)$ while an insulator has vanishing $\sigma(0)$ (in both cases the Drude weight is vanishing)

Conductivity in the XXZ chain

- In the XXZ chain($J_3 \neq 0, \lambda = 0$), Bethe ansatz provides exact formulas (Yang-Yang '66)

$$D_0 = \frac{\pi}{\bar{\mu}} \frac{\sin \bar{\mu}}{2\mu(\pi - \bar{\mu})}$$

$$\kappa_0 = \frac{\bar{\mu}}{2\pi} \frac{1}{(\pi - \bar{\mu})} \sin \bar{\mu} \qquad v_{s,0} = \frac{\pi}{\bar{\mu}} \sin \bar{\mu}$$

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- If $\lambda \neq 0$ is the conductivity still infinite? Is the universal relation still true?

- Benfatto, Falco, Mastropietro Comm. Math.Phys. 2009; PRL 2011; Mastropietro PRE 2013

Theorem. *There exists* $\varepsilon < 1$ *such that, if* $|J_3|, |\lambda| \leq \varepsilon$ *the zero temperature Drude weight is non vanishing and analytic in* J_3, λ*; moreover*

$$D_\lambda = K \frac{v_{s,\lambda}}{\pi} \quad \kappa_\lambda = \frac{K}{\pi v_{s,\lambda}}$$

with $K =$

$$1 - \frac{1}{\pi v_{s,\lambda}}[(J_3 + 2\lambda)(1 - \cos 2p_F) + \lambda(1 - \cos 4p_F) + F]$$

and $v_s = \sin(p_F) + \tilde{F}$, $\sin p_F = h$ *and* $|F| \leq C\varepsilon^2, |\tilde{F}| \leq C\varepsilon.$

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- The above theorem implyes infinite conductivity at $T = 0$ for small J_3, λ.
- The theorem implies the universal relation

$$\frac{D_\lambda}{\kappa_\lambda} = v_{s,\lambda}^2$$

which was conjectured by Haldane (1980), extending previous ideas by Kadanoff (1971).

Conductivity in the non integrable case

- D_λ is also connected to the critical exponents by exact relations; for instance if X is the exponent of $< S^3_{\mathbf{x}} S^3_{\mathbf{0}} >$ then

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- Other exponents are determined by X using the Kadanoff relations which can be proven to be true in this model

Sketch of the proof

- Ward Identites

$$-ip_0 < \hat{\rho}_{\mathbf{p}} \hat{a}^-_{\mathbf{k}} \hat{a}^+_{\mathbf{k}+\mathbf{p}} >_{\beta,T} + p < \hat{j}_{\mathbf{p}} \hat{a}^-_{\mathbf{k}} \hat{a}^+_{\mathbf{k}+\mathbf{p}} >_{\beta,T} =$$
$$[\left\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \right\rangle_{\beta,T} - \left\langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \right\rangle_{\beta,T}]$$

$$-ip_0 \hat{K}^{0,0}_{\beta,\lambda}(\mathbf{p}) + p\hat{K}^{10}_{\beta,\lambda}(\mathbf{p}) = 0$$
$$-ip_0 \hat{K}^{0,1}_{\beta,\lambda}(\mathbf{p}) + pD_{\beta,\lambda}(\mathbf{p}) = 0$$

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- This implies

$$\hat{K}^{00}_{\lambda}(p_0, 0) = 0, \quad D_\lambda(0, p) = 0$$

Relation between regularity of the FT of correlations and conductivity; for instance if the FT is continuous the Drude weight is vanishing (what is not not in the case).

Sketch of the proof

- We perform an rigorous RG analysis and we get that the current-current correlation $K^{0,1}_{\beta,\lambda}(\mathbf{p})$ can be naturally decomposed as sum of two terms where the second contains also the irrelevant terms (Umklapp, non linear bands)

$$K^{1,1}_{\lambda}(\mathbf{x}) = K^{(a)1,1}_{\lambda}(\mathbf{x}) + K^{(b)1,1}_{\lambda}(\mathbf{x})$$

and

$$|K^{(a)1,1}_{\lambda}(\mathbf{x})| \le \frac{C}{1+|\mathbf{x}|^2}$$

$$|K^{(b)1,1}_{\lambda}(\mathbf{x})| = \frac{C}{1+|\mathbf{x}|^{2+\theta}}, \qquad \theta > 0$$

Used the Gram bounds for fermionic expectations; they imply the convergence of the series expansion (Constructive QFT tools as multiscale analysis, Gallavotti trees and Battle-Brydges-Federbush formula).

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- The bound for $K_\lambda^{(a)1,1}(\mathbf{x})$ are not sufficient to say the the FT is bounded; moreover the contribution of the irrelevant terms is $O(1)$.
- We need to exploit the idea of emerging symmetries introducing a QFT model descibing massless Dirac femions with a momentum regularization and a non local quartic interaction.
- We can tune by implicit function theorem the parameters so that $K_\lambda^{(a)1,1}(\mathbf{x})$ is equal to the correlations of this effective model up to constants.

- This implies an exact expression for $K^{(a)}$

$$\hat{K}_\lambda^{(a)1,1}(\mathbf{p}) = \frac{1}{4\pi v_s Z^2}\frac{(\tilde{Z}^{(1)})^2}{1-\tau^2}[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau]$$

$$\hat{K}_\lambda^{(a)0,0}(\mathbf{p}) = \frac{1}{4\pi v_s Z^2}\frac{(\tilde{Z}^{(0)})^2}{1-\tau^2}[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau]$$

where $\tau = \frac{\lambda_\infty}{4\pi v_s}$, $D_\omega(\mathbf{p}) = -ip_0 + \omega v_s p$. In order to get that it is essential that we can study both models via multiscale analysis.

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- $\tilde{Z}^{(0)} \neq \tilde{Z}^{(1)}$ as irrelevant terms breaks Lorentz symmetry.

Sketch of the proof

- On the other hand the parameters are not all independent; the condition $D_{\beta,\lambda}(0,p) = 0$ fixes the value of $\hat{K}_{\lambda}^{(b)1,1}(0)$.

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- From the WI of the effective model

$$\tilde{Z}[-ip_0\frac{1}{\tilde{Z}^{(0)}}\hat{<}\hat{\rho}_{\mathbf{p}}\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}> +pv_s\frac{1}{\tilde{Z}^{(1)}}<\hat{j}_{\mathbf{p}}\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}>] =$$

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$$=\frac{1}{1-\tau}[<\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}}>-<\hat{a}^+_{\mathbf{k}+\mathbf{p}}\hat{a}^-_{\mathbf{k}+\mathbf{p}}>]$$

- The bare parameters are not independent but fixed by the lattice WI

$$\frac{1}{1-\tau}\frac{\tilde{Z}^{(0)}}{\tilde{Z}}=1 \qquad \frac{v_s\tilde{Z}^{(0)}}{\tilde{Z}^{(1)}}=1$$

- In conclusion

$$\hat{K}^{00}_{\lambda}(\mathbf{p}) = \frac{K}{\pi v_s} \frac{v_s^2 p^2}{p_0^2 + v_s^2 p^2} + O(\mathbf{p})$$
$$\hat{D}_{\lambda}(\mathbf{p}) = \frac{K v_s}{\pi} \frac{p_0^2}{p_0^2 + v_s^2 p^2} + O(\mathbf{p})$$

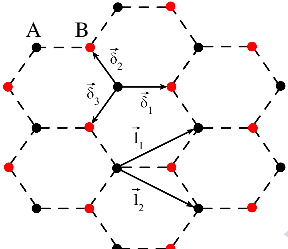
  with $K = \frac{1-\tau}{1+\tau}$, and the theorem follows
- Hopefully an extension of this non perturbative RG analysis at $\beta < \infty$ is possible.

Hubbard model on the honeycomb lattice

$$H_U = -t \sum_{\vec{x}\in\Lambda, i=1,2,3} \sum_{\sigma=\uparrow\downarrow} \left(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right) +$$

$$U \sum_{\substack{\vec{x}\in\Lambda \\ i=1,2,3}} \sum_{\sigma,\sigma'} \left(a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma} - \frac{1}{2} \right) \left(b^+_{\vec{x}+\vec{\delta}_i,\sigma'} b^-_{\vec{x}+\vec{\delta}_i,\sigma'} - \frac{1}{2} \right)$$

$a^\pm_{\vec{x}}$, $b^\pm_{\vec{x}}$ fermionic operators,
$\vec{\delta}_1 = (1,0)$, $\quad \vec{\delta}_2 = \frac{1}{2}(-1, \sqrt{3})$, $\quad \vec{\delta}_3 = \frac{1}{2}(-1, -\sqrt{3})$, $\Lambda \equiv \Lambda_A$
periodic triangular lattice



Physical observables

1. $\Psi^{\pm}_{\vec{x},\sigma} = \left(a^{\pm}_{\vec{x},\sigma}, b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}\right)$, $\Psi^{\pm}_{\mathbf{x},\sigma} = e^{Hx_0}\Psi^{\pm}_{\vec{x},\sigma}e^{-Hx_0}$ with $\mathbf{x} = (x_0, \vec{x})$ and $x_0 \in [0, \beta]$, for some $\beta > 0$.

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2. If $S(\mathbf{x} - \mathbf{y}) = \langle \Psi^{-}_{\mathbf{x}} \Psi^{+}_{\mathbf{y}} \rangle_{\beta}$ we denote by $\hat{S}(\mathbf{k})$ the F.T., $\mathbf{k} = (k_0, \vec{k})$, $k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})\ :\ n_0 \in \mathbb{Z}$, $\vec{k} \in \mathcal{B}$ the first Brillouin zone.

The 2-point function for $U = 0$

1. $$S_0(\mathbf{k}) = \frac{1}{k_0^2 + |v_F^{(0)}\Omega(\vec{k})|^2} \begin{pmatrix} ik_0 & -v_F^{(0)}\Omega^*(\vec{k}) \\ -v_F^{(0)}\Omega(\vec{k}) & ik_0 \end{pmatrix},$$

$$v_F^{(0)}\Omega(\vec{k}) = t\sum_{i=1}^{3} e^{i\vec{k}(\vec{\delta}_i - \vec{\delta}_1)} = t(1 + 2e^{-i3/2k_1}\cos\frac{\sqrt{3}}{2}k_2).$$

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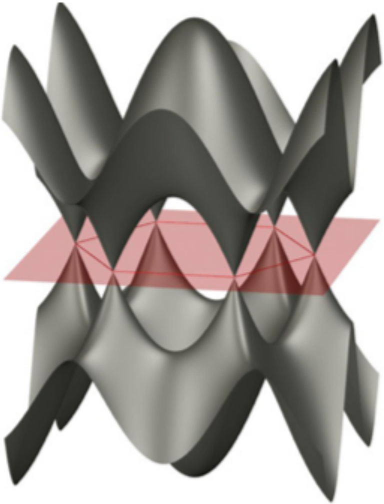
2. If $\vec{p}_F^{\,\pm} = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$, $v_F^{(0)} = \frac{3}{2}t$ close to a Dirac propagator (massless Dirac in $2+1$ while in the previous case $1+1$)

$$S_0(\mathbf{k} + \mathbf{p}_F^{\pm}) \sim \begin{pmatrix} ik_0 & v_F^{(0)}(ik_1' \mp k_2') \\ v_F^{(0)}(-ik_1' \mp k_2') & ik_0 \end{pmatrix}^{-1},$$

The dispersion relation



- The currents are (spin is understood)

$$\vec{\hat{J}}_{\vec p} = iet \sum_{\substack{\vec x \in \Lambda \\ ,j}} e^{-i\vec p \vec x} \vec\delta_j \eta^j_{\vec p} \left(a^+_{\vec x} b^-_{\vec x + \vec\delta_j} - b^+_{\vec x + \vec\delta_j} a^-_{\vec x} \right) = v_F^{(0)} \vec{\hat{j}}_{\vec p}$$

with $\eta^j_{\vec p} = \frac{1 - e^{-i\vec p \vec\delta_j}}{i\vec p \vec\delta_j}$; sum of the three bond currents

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- The conductivity at imaginary frequencies by Kubo formula is $\omega = \frac{2\pi}{\beta} n$

$$\sigma^\beta_{lm}(i\omega) = -\frac{2}{3\sqrt{3}} \frac{e^2}{\hbar\omega} \left[(v_F^{(0)})^2 < \hat{j}_{l,\omega,0}; \hat{j}_{m,-\omega,0} >_\beta + \Delta^\beta_{lm} \right],$$

where $3\sqrt{3}/2$ is the area of the hexagonal cell, $< \hat{j}_{l,\omega,\vec{p}}; \hat{j}_{m,-\omega,\vec{p}} >= FT(< \hat{j}_{l,x_0,\vec{p}}; \hat{j}_{m,y_0,-\vec{p}} >)$.

THE OPTICAL CONDUCTIVITY FOR $U = 0$: THEORETICAL PREDICTIONS

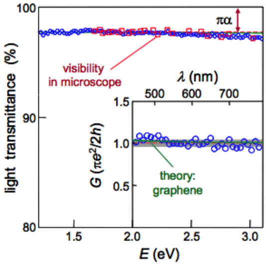
- Stauber, Peres, Geim PRB (2008)

$$\lim_{\omega\to 0}\lim_{\beta\to\infty}\sigma^{\beta}_{lm}(\omega + i0^{+}) = \delta_{lm}\sigma_0 \qquad \sigma_0 = \frac{\pi e^2}{2h}$$

Universal conductivity (t independent) for ω small and greater than β^{-1}. *Finite* as the density of states is vanishing.

The optical conductivity: experiments

Nair et al. Science (2008). The conductivity in a frequency range $\beta^{-1} << \omega << t$ is $\sigma_0 = \frac{\pi e^2}{2h}$ (universality) up a few percent (In the same range the conductivity for N-layer graphene is $\sigma_0 = N\frac{\pi e^2}{2h}$ up a few percent.)



They measure the transparency T of light and from that the conductivity $T(\omega) = 1/[(1 + 2\pi\sigma(\omega)]^2$ (in the fig. called $G((\omega))$. Between 2 and 3 eV $\frac{\sigma(\omega)}{\sigma_0} = 1.01 \pm 0.03$

Experiments and some puzzle

- The electron-electron interaction is large $e^2/\hbar v_F^0 \sim 2.18$
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Why the conductivity is universal, that is there is no an essential many body renormalization in the conductivity?
- Exacerbating the problem, in other experiments the interaction appear. Ellis et al Nat. Mat. (2011): the Fermi velocity is strongly enlarged by the interactions at low frequencies.
- There is a large debate in current times on the graphene conductivity. In particular some people have found interaction dependent corrections while others objects that these are spurious effects due to the uv regularizations.

Universality of the conductivity

- Giuliani, Mastropietro. CMP 293,301 (2010); PRB(R)79, 201403 (2009); Giuliani, Mastropietro, Porta. PRB 83, 195401 (2011); CMP 311,317 (2012).

Theorem

For $|U| \leq U_0$ and any fixed ω, $\sigma^{\beta}_{lm}(i\omega)$ is analytic in U uniformly in β and

$$\lim_{\omega \to 0^+} \lim_{\beta \to \infty} \sigma_{lm}(i\omega) = \frac{e^2}{h} \frac{\pi}{2} \delta_{lm} \ .$$

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- While the Fermi velocity and the wave function renormalization are renormalized $v_F(U) > v_F(0)$ the conductivity is protected: radiative corrections cancel out.

PROOF.

- The correlation is then written as a convergent (due to Gram bounds) tree expansion at weak coupling and , if $\hat{K}_{lm}(\mathbf{p})$ is the FT of $\langle J_{l,\mathbf{x}}; J_{m,\mathbf{y}}\rangle$ and $\hat{K}_{0m}(\mathbf{p})$ is the FT of $\langle \rho_{\mathbf{x}}; J_{m,\mathbf{y}}\rangle$, from the bound

$$|K_{\mu,\nu}(\mathbf{x})| \leq \frac{C}{1+|\mathbf{x}|^4} ,$$

$\hat{K}_{\mu\nu}(\mathbf{p})$ is continuous at $\mathbf{p} = \mathbf{0}$

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$$|K_{\mu,\nu}(\mathbf{x})| \leq \frac{C}{1+|\mathbf{x}|^4} ,$$

  $\hat{K}_{\mu\nu}(\mathbf{p})$ is continuous at $\mathbf{p} = \mathbf{0}$
- Now the WI implies that the Drude weight is vanishing

$$\sigma_{lm} = -\frac{2}{3\sqrt{3}} \lim_{\omega \to 0^+} \lim_{\beta \to \infty} \frac{1}{\omega} \left[\hat{K}_{lm}(\omega, \vec{0}) - \hat{K}_{lm}(\mathbf{0}) \right] .$$

  $K_{l,m}(\mathbf{p})$ is even: if the derivative were continuous the conductivity vanishes. But is not. (CFR $1D$ $\hat{K}_{l,m}$ non continuous $\sigma(0) = \infty$)

The current-current function

- As a result of the Renormalization Group analysis and tree expansion

$$\hat{K}_{lm}(\mathbf{p}) = \frac{Z_l Z_m}{Z^2} \langle \hat{j}_{\mathbf{p},l}; \hat{j}_{-\mathbf{p},m} \rangle_{0,v_F} + \hat{R}_{lm}(\mathbf{p})$$

where $\langle \cdot \rangle_{0,v_F}$ is the average associated to a non-interacting system with Fermi velocity

$$v_F(U) = \frac{3}{2}t + dU + .. \quad Z_\mu = 1 + aU + bU^2 + ..$$

and

$$|R_{lm}(\mathbf{x},\mathbf{y})| \leq \frac{C}{1 + |\mathbf{x} - \mathbf{y}|^{4+\theta}}$$

with $0 < \theta < 1$ (power counting improvement due to irrelevance), so that $\hat{R}_{lm}(\omega, \vec{0})$ is continuous and differentiable at $\mathbf{p} = \mathbf{0}$.

Implications of WI

1. By the lattice WI and the fact that the 2 point and vertex functions is equal to the free one up to a renormalization of the parameters plus a vanishing corrections at the Fermi points

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3. Note that $\hat{K}_{lm}(\mathbf{p})$ is even

Universality of the conductivity

- Finally

$$\sigma_{11} = -\frac{2}{3\sqrt{3}} \lim_{\omega \to 0^+} \frac{1}{\omega} \Big[\big(\hat{R}_{11}(\omega, \vec{0}) - \hat{R}_{lm}(0, \vec{0}) \big)$$

$$+ \Big(v_F^2 \big\langle \hat{j}_{(\omega,\vec{0}),l} ; \hat{j}_{(-\omega,\vec{0}),m} \big\rangle_{0,v_F} - v_F^2 \big\langle \hat{j}_{\mathbf{0},l} ; \hat{j}_{\mathbf{0},m} \big\rangle_{0,v_F} \Big) \Big] \, .$$

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$$\sigma_{11} = -\frac{2}{3\sqrt{3}} \lim_{\omega \to 0^+} \frac{1}{\omega}\Big[\big(\hat{R}_{11}(\omega,\vec{0}) - \hat{R}_{lm}(0,\vec{0})\big)$$
$$+\big(v_F^2 \langle \hat{j}_{(\omega,\vec{0}),l}; \hat{j}_{(-\omega,\vec{0}),m}\rangle_{0,v_F} - v_F^2 \langle \hat{j}_{\mathbf{0},l}; \hat{j}_{\mathbf{0},m}\rangle_{0,v_F}\big)\Big] \,.$$

- The first term is differentiable and even hence vanishing, while the first term is identical to the free one so it does not depend from v_F ∎

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- This is consequence of the Fermi velocity divergence, a rather unphysical phenomenon.
- However if we take into account retardation effects, there is emergence of Lorentz symmetry and the Fermi velocity flows to the light velocity (Giuliani Mastropietro Porta Ann. Phys. 2012).
- In this case the conductivity is different from the non interacting one, but still (Herbut-Mastropietro 2013) does not depend from the material parameter.

Conclusion

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- Their use allows the proof of several universality properties.
- (Non trivial) extensions would include finite temperature effects and disorder.