life in extreme dimensions

predrag cvitanović

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Hour of the Wolf

1. dynamics in ∞ dimensions
2. who's afraid of symmetry
3. knowing when to stop

our hero

Max von Sydow

a physicist so brilliant he has not read a paper since grad school

our heroine

Liv Ulmann

understands it all but cannot save him

theirs is a life in extreme dimensions

since 1822 have Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{R}\nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0,$$

since 1883 Osborne Reynolds experiments

the most fundamental outstanding problem of classical physics :

turbulence!

numerical challenges

computation of turbulent solutions

requires 3-dimensional volume discretization
$\rightarrow$ integration of 10^4-10^6 coupled ordinary differential equations

challenging, but today possible

computation of turbulent solutions

requires 3-dimensional volume discretization
$\rightarrow$ integration of 10^4-10^6 coupled ordinary differential equations

challenging, but today possible

typical simulation

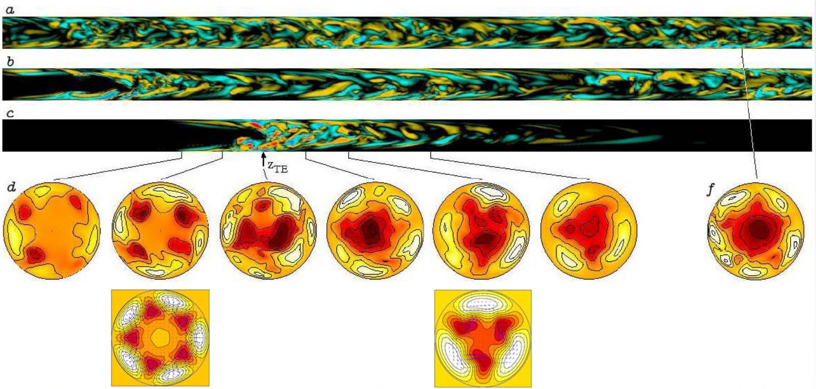
each instant of the flow $>$ Megabytes
a video of the flow $>$ Gigabytes

Max the man :

he can do it all

example : pipe flow

amazing data! amazing numerics!



- here each instant of the flow $\approx$ 2.5 MB
- videos of the flow $\approx$ GBs

turbulence.zip

or ‘equation assisted’ data compression:

replace the ∞ of turbulent videos by the best possible

small finite set

of videos encoding all physically distinct motions of the turbulent fluid

Liv Ullmann to Max von Syddow :

please, look at it in

the state space!

E. Hopf 1948, Ya. Sinai 1972 :

identify templates, partition it!

!!! THE POINT OF THIS TALK !!!

UNLEARN: 3-d VISUALIZATION	THINK: ∞-d PHASE SPACE
instant in turbulent evolution:	instant in turbulent evolution:
a 3-d video frame, each pixel a 3-d velocity field	a unique point
	theory of turbulence = geometry of the state space

[E. Hopf 1948]

dynamical description of turbulence

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$x(t) \in \mathcal{M}$
a state of physical system at instant in time

deterministic dynamics

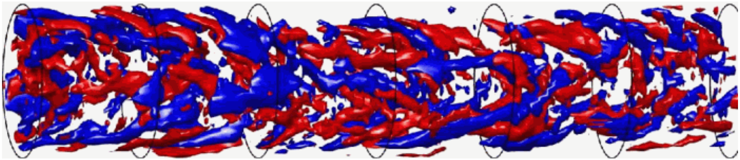
trajectory $x(\tau) = f^\tau(x_0)$ = representative point time τ later

today's experiments

example of a representative point

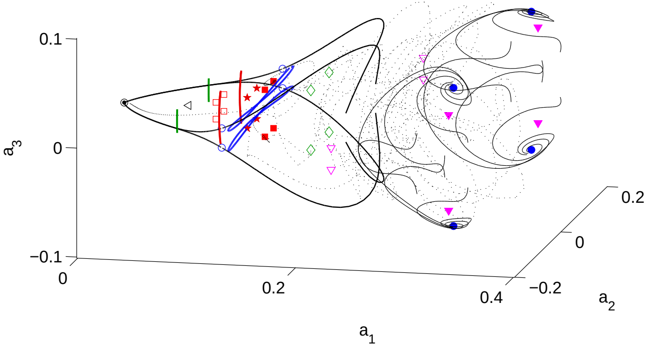
$x(t) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-d velocity field over the entire pipe[1]



[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

can visualize 61,506 dimensional state space of turbulent flow

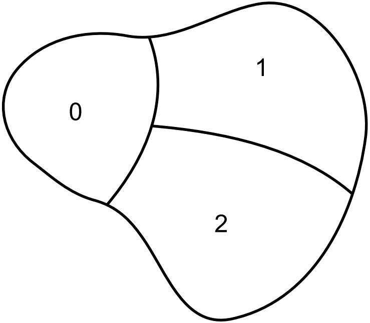


equilibria of turbulent plane Couette flow,
their unstable manifolds, and
myriad of turbulent videos mapped out as one happy family

for movies, please click through ChaosBook.org/tutorials

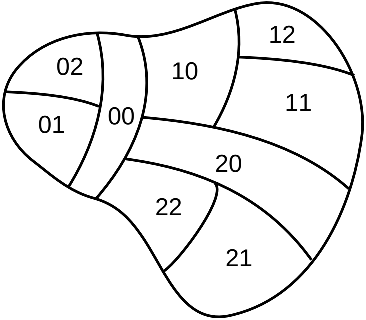
deterministic partition into regions of similar states

1-step memory partition



$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$
ternary alphabet
$\mathcal{A} = \{1, 2, 3\}$.

2-step memory refinement



$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine ‘words’
$\{00, 01, 02, \cdots, 21, 22\}$.

the problem with symmetry :

nature loves symmetry

or does she?

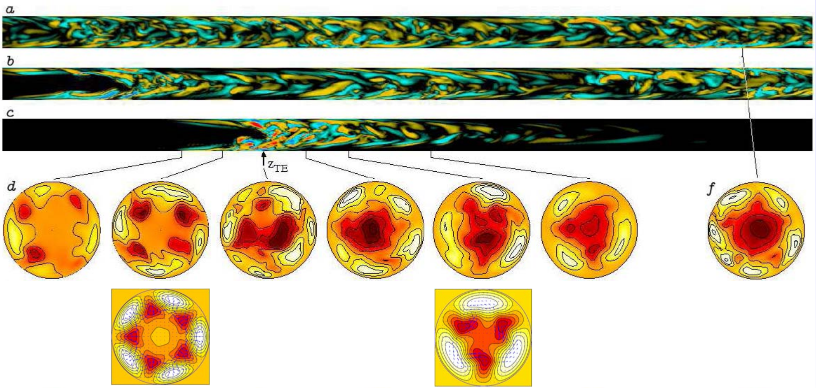
problem

physicists like symmetry more than Nature

Rich Kerswell

nature : turbulence in pipe flows

top : experimental / numerical data
bottom : theorist's solutions



Nature, she don't care : turbulence breaks all symmetries

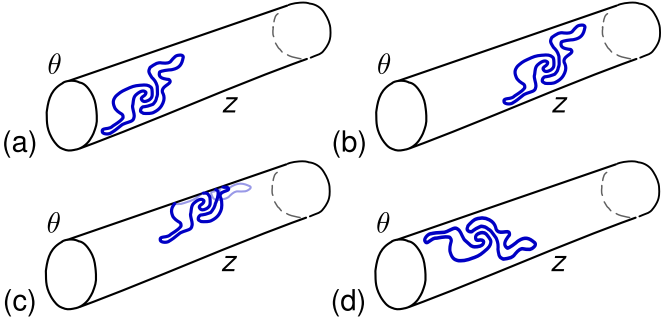
Liv to you :

please, use symmetries!

H. Poincaré 1899, Elie Cartan 1926 :

section it, slice it!

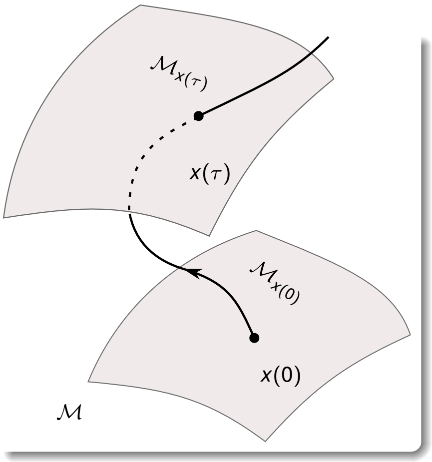
example : $\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$ symmetry of pipe flow



a fluid state, shifted by a stream-wise translation, azimuthal rotation g_p is a physically equivalent state

- **b)** stream-wise
- **c)** stream-wise, azimuthal
- **d)** azimuthal flip

group orbits



group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\, x \mid g \in G\}$$

group orbits are NOT circles

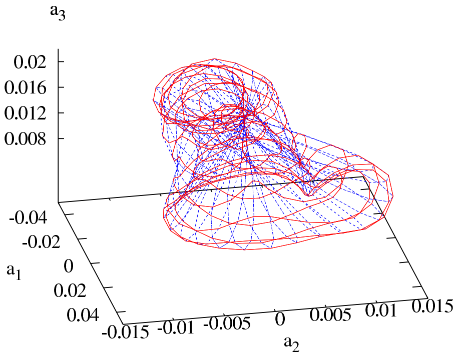
nature couples many Fourier modes

group orbit manifolds of highly nonlinear states are smooth, but not nice

example : group orbit of a pipe flow turbulent state

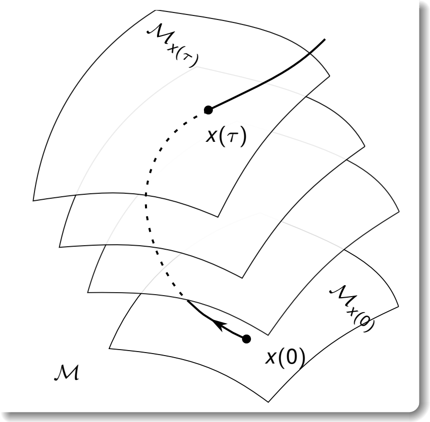
$\mathbf{SO}(2) \times \mathbf{SO}(2)$ symmetry
$\Rightarrow$ group orbit is topologically 2-torus, but a mess in any projection

a turbulent state



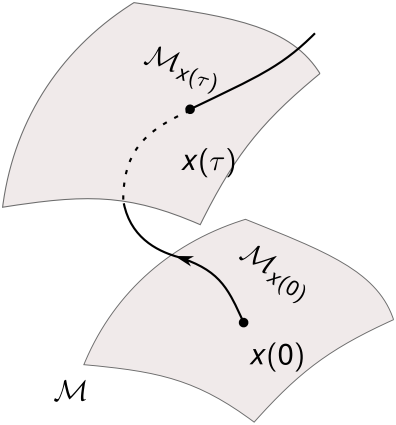
group orbits of highly nonlinear states are topologically tori, but highly contorted tori

foliation by group orbits

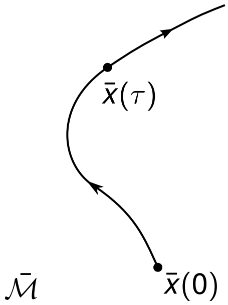


actions of a symmetry group foliates the state space $\mathcal{M}$ into a union of group orbits $\mathcal{M}_x$

full state space



reduced state space



replace each group orbit by a unique point in a lower-dimensional

symmetry reduced state space $\mathcal{M}/G$

inspiration : pattern recognition

you are observing turbulence in a pipe flow, or your defibrillator has a mesh of sensors measuring electrical currents that cross your heart, and

you have a precomputed pattern, and are sifting through the data set of observed patterns for something like it

here you see a pattern, and there you see a pattern that seems much like the first one

how ‘much like the first one?’

take the first pattern

‘template’ or ‘reference state’

a point $\bar{x}'$ in the state space $\mathcal{M}$

and use the symmetries of the flow to

slide and rotate the ‘template’

act with elements of the symmetry group G on $\bar{x}' \to g(\theta)\,\bar{x}'$

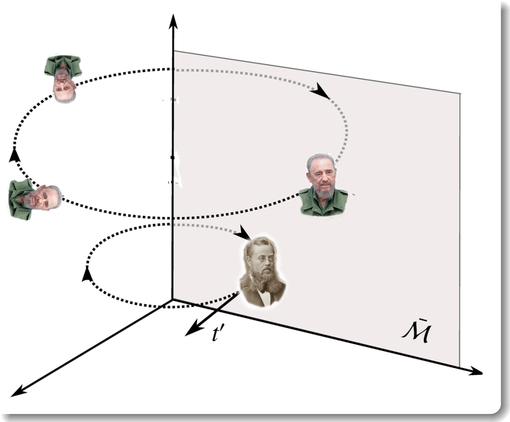
until it overlies the second pattern (a point x in the state space)

distance between the two patterns

$$|x - g(\theta)\,\bar{x}'| = |\bar{x} - \bar{x}'|$$

is minimized

idea: the closest match



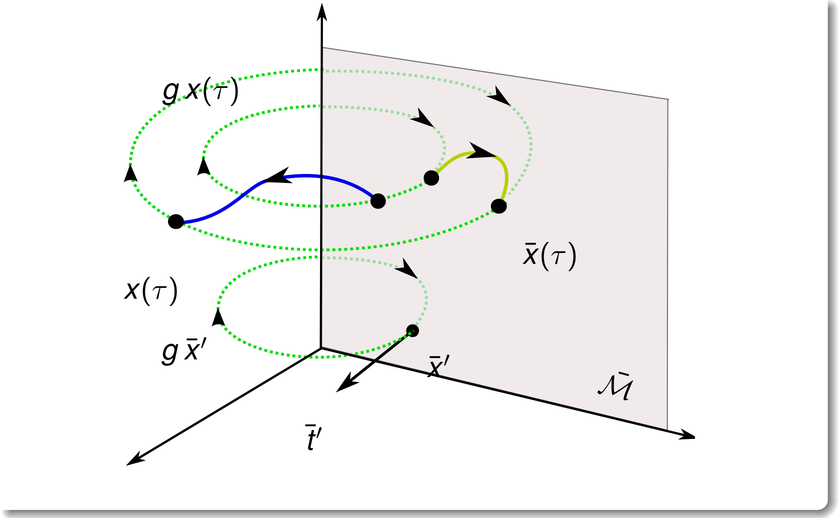
template: Sophus Lie

(1) rotate face x traces out group orbit $\mathcal{M}_x$

(2) replace the group orbit by the closest match $\bar{x}$ to the template pattern $\bar{x}'$

the closest matches $\bar{x}$ lie in the $(d-N)$ symmetry reduced state space $\bar{\mathcal{M}}$

flow within the slice



full-space trajectory $x(\tau)$
rotated into the reduced state space $\bar{x}(\tau) = g(\theta)^{-1}x(\tau)$
by appropriate *moving frame* angles $\{\theta(\tau)\}$

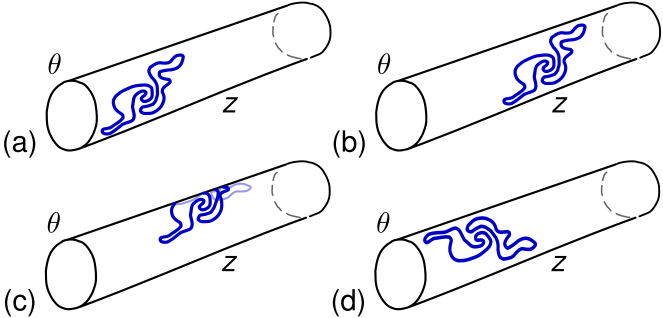
take home :

if you have a symmetry, reduce it!

your quandry

mhm - seems this would require extra thinking
what's the payoff?

$SO(2)_z \times O(2)_\theta$ relative periodic orbits of pipe flow

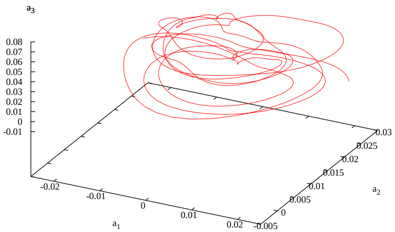


relative periodic orbit : recurs at time T_p, shifted by a streamwise translation, azimuthal rotation g_p

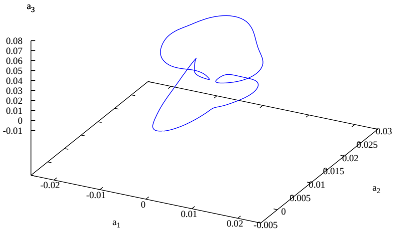
- **b)** stream-wise recurrent
- **c)** stream-wise, azimuthal recurrent
- **d)** azimuthal flip recurrent

example : pipe flow relative periodic orbit

3 repeats, full space

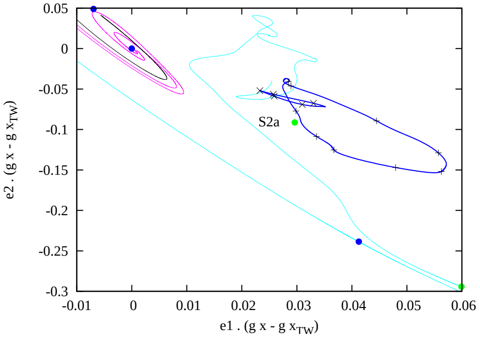


reduced space



it works : all pipe flow solutions in one happy family

could not find without symmetry reduction :



first pipe flow relative periodic orbits embedded in turbulence!

take-home message

if you have a symmetry

use it!

without symmetry reduction, no understanding of fluid flows, nonlinear field theories possible

Liv to Max :

deterministic partitions are no good!

deterministic dynamics: partitioning can be arbitrarily fine

requires exponential # of exponentially small regions

deterministic dynamics: partitioning can be arbitrarily fine

requires exponential # of exponentially small regions

yet

in practice

every physical problem must be coarse partitioned

Liv to Max :

please, know **when to stop!**

Laplace 1810, A. Lyapunov 1892 :

noise frees us from the shackles of determinism!

knowing when to stop

[click here for an example of a fluid in motion]

need the 3D velocity field at every (x, y, z)!

motions of fluids : require ∞ bits?

numerical simulations track millions of computational degrees of freedom; observations, from laboratory to satellite, stream terabytes of data, but how much information is there in all of this?

knowing when to stop

motions of fluids : require ∞ bits??

that cannot be right...

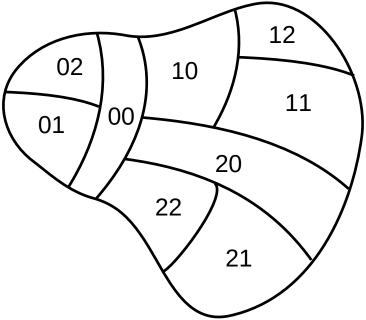
knowing when to stop

> Science originates from curiosity and bad eyesight.
>
> — Bernard de Fontenelle,
> *Entretiens sur la Pluralité des Mondes Habités*

in practice

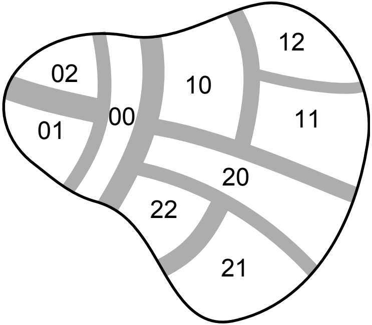
every physical problem is coarse partitioned and finite

deterministic vs. noisy partitions



deterministic partition

can be refined
ad infinitum



noise blurs the boundaries

when overlapping, no further refinement of partition

mathematician's idealized state space

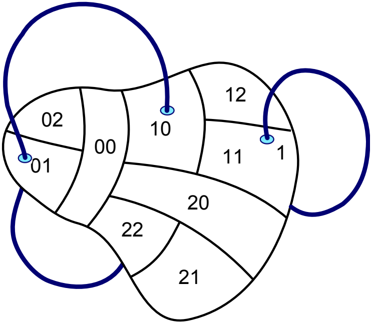
a manifold $\mathcal{M} \in \mathbb{R}^d$: d continuous numbers determine the state of the system $x \in \mathcal{M}$

noise-limited state space

a 'grid' $\mathcal{M}'$: N discrete states of the system $a \in \mathcal{M}'$, one for each noise covariance ellipsoid Q_a

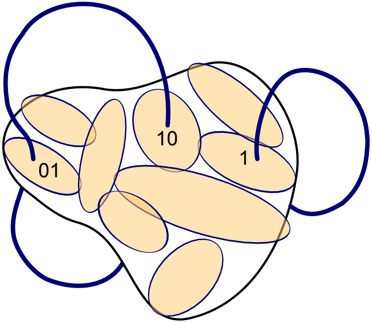
periodic orbit partition

deterministic partition



some short periodic points:
fixed point $\overline{1} = \{x_1\}$
two-cycle $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition

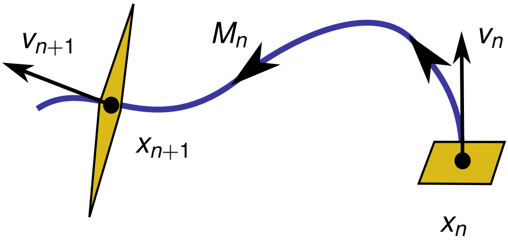


periodic points blurred by noise
into cigar-shaped densities

challenge : knowing when to stop

determine the finest possible partition for a given noise

linearized deterministic flow



$$x_{n+1} + z_{n+1} = f(x_n) + M_n\, z_n\,, \quad M_{ij} = \partial f_i/\partial x_j$$

in one time step a linearized neighborhood of x_n is

(1) advected by the flow and

(2) mapped by the Jacobian matrix M_n into a neighborhood whose size and orientation are given by the M eigenvalues and eigenvectors

covariance advection

let the initial density of deviations z from the deterministic center be a Gaussian whose covariance matrix is

$$Q_{jk} = \left\langle z_j z_k^T \right\rangle$$

a step later the Gaussian is advected to

$$\begin{aligned} \left\langle z_j z_k^T \right\rangle &\to \left\langle (M\,z)_j \,(M\,z)_k^T \right\rangle \\ Q &\to M\,Q\,M^T \end{aligned}$$

add noise, get the next slide

covariance evolution

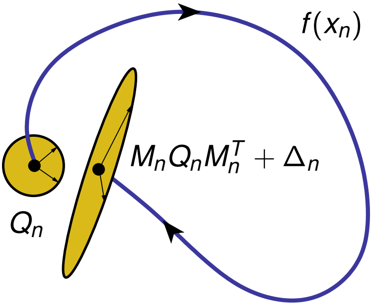
$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n$$

(1) advect deterministically
local density covariance matrix $Q \to MQM^T$

(2) add noise covariance matrix Δ

covariances add up as sums of squares

roll your own cigar



in one time step
a Gaussian density distribution with covariance matrix Q_n is

(1) advected by the flow

(2) smeared with additive noise

into a Gaussian 'cigar' whose widths and orientation are given by the singular values and vectors of Q_{n+1}

noisy dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow ALWAYS induces a local, history dependent effective noise

example : noise and a single attractive fixed point

if all eigenvalues of M are strictly contracting, all $|\Lambda_j| < 1$

any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies

Lyapunov equation: time-invariant measure condition

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov doctoral dissertation 1892]

example : Ornstein-Uhlenbeck process

width of the natural measure concentrated at the attractive deterministic fixed point $z = 0$

$$\rho_0(z) = \frac{1}{\sqrt{2\pi\, Q}} \exp\left(-\frac{z^2}{2\, Q}\right) , \qquad Q = \frac{\Delta}{1 - |\Lambda|^2} ,$$

- is balance between contraction by Λ and noisy smearing by Δ at each time step

local problem solved: can compute every cigar

a periodic point of period *n* is a fixed point of *n*th iterate of dynamics

global problem solved: can compute all cigars

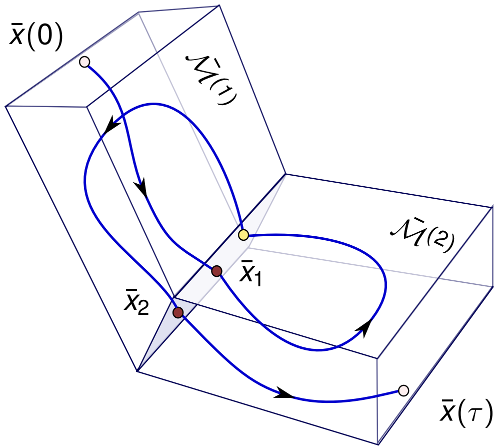
more algebra: can compute the noisy neighborhoods of all periodic points

charting the state space

for turbulent/chaotic systems an atlas - a set of charts - is needed to capture the dynamics

templates $\bar{x}'^{(j)}$ should be representative of the dynamically dominant patterns seen in the solutions of nonlinear PDEs

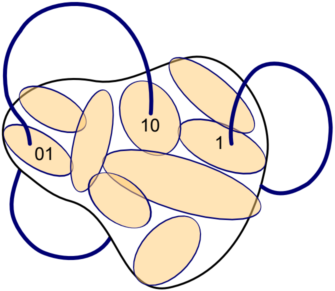
each chart $\bar{\mathcal{M}}^{(j)}$ captures a neighborhood of a template $\bar{x}'^{(j)}$



two charts drawn as two $(d-1)$-dimensional slabs

shaded plane : the ridge, their $(d-2)$-dimensional intersection

optimal partition hypothesis



optimal partition:

the maximal set of resolvable periodic point neighborhoods

the payback for your patience

claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise

the payback for your patience

claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Markov graphs

the payback for your patience

claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Markov graphs
- finite matrix calculations $\Rightarrow$ optimal estimates of long-time observables (Lyapunov exponents, mean temperature in Chicago and its variance, etc.)

Liv & Max :

can this ever work?

example: representative solutions of fluid dynamics

- Professor Zweistein, from the back of auditorium:
  - **(1)** she has already done all this in 1969
  - **(2)** you must be kidding, it cannot be done for turbulence

example: representative solutions of fluid dynamics

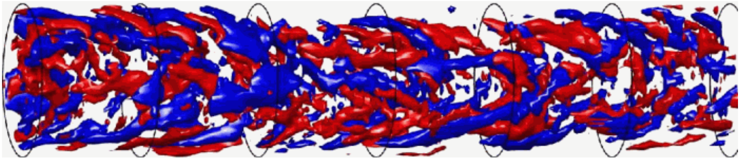
- Professor Zweistein, from the back of auditorium:
  - **(1)** she has already done all this in 1969
  - **(2)** you must be kidding, it cannot be done for turbulence
- OK, OK, we have about 50 state space cell centers

  [click here for examples of frozen fluid states]

  [click here for examples of a fluid in periodic motions]

  and we have their Jacobians (that was hell to get)

- Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,



is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating these solutions?

- disclosure

  we have not yet tested the method on fluid dynamics data sets.

- Georgia Tech Center for Nonlinear Science is looking for several brave postdocs to help us really ‘zip’ turbulence

- Georgia Tech Center for Nonlinear Science is looking for several brave postdocs to help us really ‘zip’ turbulence
- the brave candidates: step up after the talk