Nonlocal Dissipation

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Rutgers, May 13

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C-Majda-Tabak: analogies to 3D Euler.

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For Burgers: criticality is real.

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$$\Lambda f(x) = cP.V. \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+1}}dy$$

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4. C-Vicol: nonlinear maximum principle, stability of the "only small shocks" condition.

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Properties of Solutions

Short time existence proof guarantees that solutions which start in $\theta_0 \in \dot{H}^1$ become $S(t_0)\theta_0 \in C^\alpha \cap \dot{H}^1$ instantly ($t_0 > 0$, size depends badly on t_0) for small $\alpha > 0$.

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(CTV '13) Let $\theta_0 \in \dot{H}^1(\mathbb{T}^2)$, $f \in L^\infty(\mathbb{T}^2) \cap \dot{H}^1(\mathbb{T}^2)$. There exists $\alpha = \alpha(f) \in (0,1)$ and constants $C_\infty = C_\infty(f)$ and $C_\alpha = C_\alpha(f)$ depending only on $\|f\|_{H^1} + \|f\|_{L^\infty}$ such that

$$\|S(t)\theta_0\|_{L^\infty} \le C_\infty,$$

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holds for all $t \ge \tau$, with $\tau = \tau(\theta_0, f)$ bounded on bounded sets of initial data in $\dot{H}^1$.

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(C-Glatt-Holtz-Vicol) Let θ have mean zero, let $p \geq 2$ be even. There exists a p-independent constant C so that

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$1 \leq p \leq \infty$, *c = p-independent.*

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Ideas of Proofs: C^α, continued

Lemma

Let $f \in \dot{H}^1 \cap L^\infty$. Let $\theta_0 \in L^\infty \cap \dot{H}^1$. There exist $\alpha > 0$, C_α, depending only on $\|\theta_0\|_{L^\infty}$ and $\|f\|_{L^\infty}$, such that

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This is a new proof of regularity, using the nonlinear maximum principle and obtaining directly the De Giorgi improvement.

Ideas of Proofs: C^α, continued

Lemma

Let $f \in \dot{H}^1 \cap L^\infty$. Let $\theta_0 \in L^\infty \cap \dot{H}^1$. There exist $\alpha > 0$, C_α, depending only on $\|\theta_0\|_{L^\infty}$ and $\|f\|_{L^\infty}$, such that

$$\|S(t)\theta_0\|_{C^\alpha} \le C_\alpha$$

for all $t \ge \tau$, with $\tau > 0$ bounded uniformly on bounded sets of initial data.

This is a new proof of regularity, using the nonlinear maximum principle and obtaining directly the De Giorgi improvement. Once the uniform C^α bound is obtained, it is possible to obtain uniform bounds for higher regularity, in particular

$$T^{-1}\int_\tau^{T+\tau} \|S(t)\theta_0\|_{H^1}^2 dt \le C$$

and backward uniqueness (injectivity of $S(t)$).

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$$T^{-1}\int_\tau^{T+\tau} \|S(t)\theta_0\|_{H^1}^2 dt \leq C$$

and backward uniqueness (injectivity of $S(t)$). These are used then to prove existence of the universal attractor and its finite dimensionality.