Leo Kadanoff and Critical-Point Scaling

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Static Phenomena Near Critical Points: Theory and Experiment

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This paper compares theory and experiment for behavior very near critical points. The primary experimental results are the "critical indices" which describe singularities in various thermodynamic derivatives and correlation functions. These indices are tabulated and compared with theory. The basic theoretical ideas are introduced via the molecular field approach, which brings in the concept of an order parameter and suggests that there are close relations among different phase transition problems. Although this theory is qualitatively correct it is quantitatively wrong, it predicts the wrong values of the critical indices. Another theoretical approach, the "scaling law" concept, which predicts relations among these indices, is described. The experimental evidence for and against the scaling laws is assessed. It is suggested that the scaling laws provide a promising approach to understanding phenomena near the critical point, but that they are by no means proved or disproved by the existing experimental data.

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SCALING LAWS FOR ISING MODELS NEAR T_c*

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Abstract

A model for describing the behavior of Ising models very near T_c is introduced. The description is based upon dividing the Ising model into cells which are microscopically large but much smaller than the coherence length and then using the total magnetization within each cell as a collective variable. The resulting calculation serves as a partial justification for Widom's conjecture about the homogeneity of the free energy and at the same time gives his result $s\nu' = \gamma' + 2\beta$.

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Spin-Spin Correlations in the Two-Dimensional Ising Model (*).

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Summary. — The Onsager solution to the two-dimensional Ising model is phrased in the language of thermodynamic Green's functions. The description is quite closely analogous to a theory of noninteracting fermions, except that displacements along one of the lattice directions replace the time displacements of the standard fermion theory. This formulation is used to provide new derivations of the well-known results for the partition function and the zero-field magnetization. Aside from these formulational points, the main new result of this paper is an evaluation of the spin-spin correlation function in the limit of large, but not infinite, spatial separations between the spins. For the square lattice with nearest neighbor interactions and all coupling constants identical, the correlation functions respectively for T just greater than or just less than T_c are shown to be of the form $\langle[\sigma_{j_1,k_1}-\langle\sigma\rangle][\sigma_{j_2,k_2}-\langle\sigma\rangle]\rangle=\varepsilon^{\frac{1}{4}}f_{\gtrless}(\varepsilon R)$, where $\varepsilon=4|K-K_c|$ is a measure of the distance from the critical temperature and $R=[(j_1-j_2)^2+(k_1-k_2)^2]^{\frac{1}{2}}$ is the spatial separation of the spin sites. This result holds when $\varepsilon\to 0$ but εR remains finite. The functions $f(x)$ are evaluated in the asymptotic limit of large x and shown to be $f_<(x)=e^{-2x}x^{-2}\pi^{-1}2^{-21/8}$ and $f_>(x)=e^{-x}(\pi x)^{-1/2}2^{-3/8}$.

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1. KADANOFF'S SCALING THEORY[2)]

It is convenient here to use the language appropriate to the Ising model of a ferromagnet, rather than the language of a fluid. At the critical point, the correlation length ξ of the spin-spin correlation function $G(r) = \langle s(0)s(r)\rangle$ is infinite. Near the critical point it is finite, but much greater than the lattice spacing, so it is possible to find a number L which is much greater than 1 yet such that L lattice spacings is still much less than ξ. Imagine the Ising lattice divided into cells L lattice spacings on a side, each containing L^d spins (figure 1). It was supposed by Kadanoff that the net magnetization in

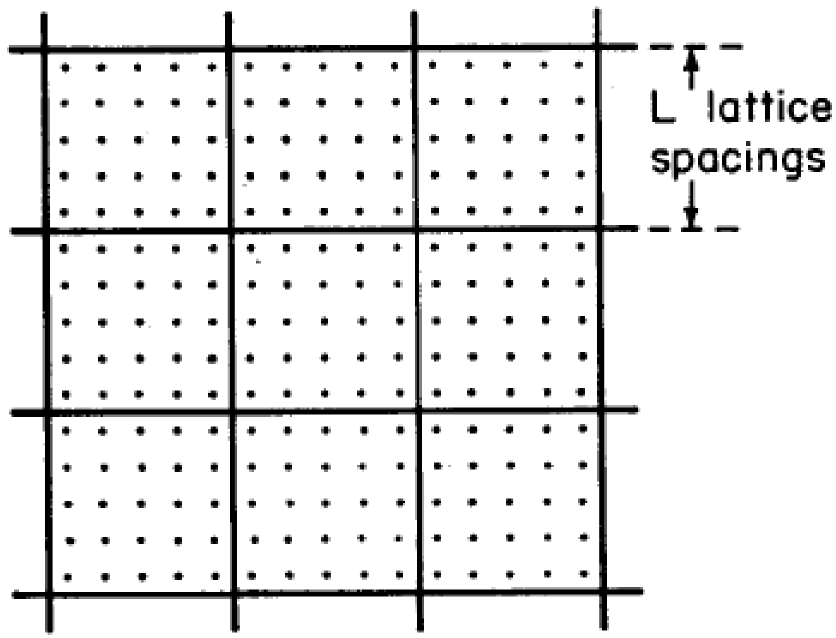
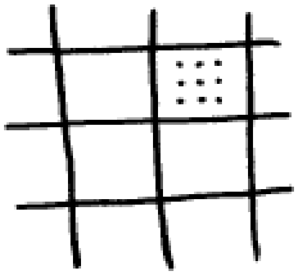


Figure 1

Kadanoff theory of origin of homogeneity and scaling



In Ising-model language — near crit. pt., where $\xi \gg a$, look at blocks of spins of linear dimension L (each block containing L^d spins) with $L \ll \xi$. Kadanoff hypothesis is that blocks interact with each other as though they were again Ising spins, so that system may be viewed either as the original Ising model or, alternatively, as a rescaled version of the original with parameters related to those of the original by scale factors. Theory does not calculate contribution to free energy from within blocks; that is non-singular because blocks are finite; i.e., because $L \ll \xi$. Theory calculates only the singular part of the free energy, associated with critical phenomena and coming from the spin-spin interactions: $F(t,H)$ is singular part of free energy per spin, with $t = \frac{T-T_c}{T_c}$.

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interactions: $F(t,H)$ is singular part of free energy per spin, with $t = \frac{T-T_c}{T_c}$.

Then contribution to the singular part of the free energy made by the spins in a block is $L^d F(t,H)$. But it is also $\boxed{F(L^y t, L^x H),}$ the singular part of the free energy per spin in a new Ising model, which in temperature is distant $L^y t$ from critical and in field by $L^x H$. Will have x, y both >0 because L is larger fraction of ξ than the original lattice spacing is, so the rescaled Ising model has to be further from its critical point than the original from its. Then

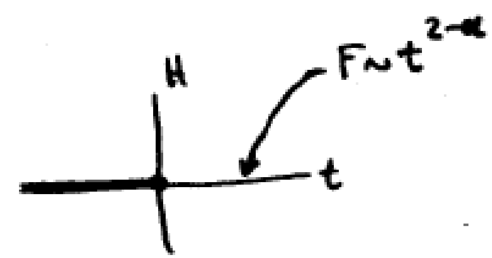
$$L^d F(t,H) \equiv F(L^y t, L^x H) \quad \text{for all } L \ll \xi .$$

$\therefore$ F homogeneous in t, $H^{y/x}$ of degree $\frac{d}{y}$:

$$\boxed{F(t,H) = t^{d/y} f\left(\frac{t}{H^{y/x}}\right)}.$$

On critical "isochore" $H=0$, so for $t>0$ at $H=0$,

$$F(t,0) = t^{d/y} \underbrace{f(\infty)}_{\text{assumed finite, non-0}}.$$



Then identify

$$\boxed{\frac{d}{y} = 2 - \alpha.}$$

Also,

$$\boxed{F(t,H) = t^{d/y} f\left(\frac{t}{H^{y/x}}\right)} = t^{d/y}\left(\frac{t}{H^{y/x}}\right)^{-d/y} g\left(\frac{t}{H^{y/x}}\right) = H^{\frac{d}{x}} g\left(\frac{t}{H^{y/x}}\right),$$

so on critical isotherm $t=0$,

$$F(0,H) = \underbrace{g(0)}_{\text{assumed non-0}} H^{d/x}.$$

But $\left(\frac{\partial F}{\partial H}\right)_T = -M$ [analog of $\left(\frac{\partial p}{\partial \mu}\right)_T = \rho$; cf. p. 42: $M = \langle N_\uparrow - N_\downarrow \rangle = \frac{kT}{Z_I}\left(\frac{\partial Z_I}{\partial H}\right)_T$; $F = -kT \ln Z_I$].

Then $M \sim H^{\frac{d}{x}-1}$,

so

$$\boxed{\frac{d}{x} = 1 + \frac{1}{\delta}.}$$

Have now identified x, y in terms of α and δ (and d).

Have now identified x, y ...

Lecture 19

By differentiating F to give M get earlier homogeneity of H as function of $t, M^{1/\beta}$ [i.e., $\mu(\rho,T)-\mu(\rho_c,T)$ as function of $T-T_c$ and $T_c-\tau(\rho)$], and know that that homogeneity alone gives $\delta = 1+\frac{\gamma}{\beta}$ and $\alpha+2\beta+\gamma=2$.

Correlation function (at $H=0$ for simplicity) — for some p,

$$\boxed{h(t,r) \equiv L^{p}\, h\left(L^{y}t,\ \frac{r}{L}\right)} \quad \text{for all } L \ll \xi .$$

Then $h(t,r)$ homogeneous in $t^{-1/y}, r$ of degree p; so

$$\boxed{h(t,r) = r^{p}\, G\left(\frac{r}{t^{-1/y}}\right).}$$

$\therefore$ identify

$$\boxed{p = -(d-2+\eta), \quad \frac{1}{y} = \nu.}$$

Then homogeneity of h, as before, implies $(2-\eta)\nu = \gamma$ from OZ relation $\rho kT\chi = 1+\rho\int h(r)\,d\tau$ while $\frac{1}{y}=\nu$, with earlier $\frac{d}{y} = 2-\alpha$, gives

$$\boxed{d\nu = 2-\alpha,}$$

the key d-dependent exponent relation.

This Kadanoff theory is direct inspiration for renormalization-group theory, next major topic.