Universality of the second mixed moment of the characteristic polynomials of the 1D Gaussian band matrices

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RBM. Model definition

$\{H_n\}$ are Gaussian hermitian (or real symmetric) matrices whose entries are numerated by indices $i, j \in [-n, n]^d \subset \mathbb{Z}^d$ and:

$$E\{H_{jk}\} = 0, \quad E\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}J_{ij}, \quad J_{ij} = \left(-W^2\Delta + 1\right)^{-1}_{ij},$$

where Δ is the discrete Laplacian on $[-n, n]^d$:

$$\Delta f(j) = \sum_{|j-k|=1} (f(k) - f(j)).$$

Note that for 1D random band matrices $J_{ij} \approx W^{-1} \exp\{-C|i-j|/W\}$, and so the variance of the matrix elements is exponentially small when $|i-j| \gg W$. Hence W can be considered as the width of the band.

Varying W we can see that random band matrices are natural interpolations between random Schrödinger matrices

$$H_{RS} = -\Delta + \lambda V$$

and mean-field random matrices such as $N \times N$ Wigner matrices, i.e. hermitian random matrices with i.i.d elements.

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The key physical parameter of these models is the localization length ℓ, which describes the typical length scale of the eigenvectors of random matrices.

Mirlin, Fyodorov (1991):

for 1D RBM the localization length $\ell \approx W^2$

- $W \gg \sqrt{N}$ the eigenvectors are expected to be delocalized
- $W \ll \sqrt{N}$ localization

In terms of eigenvalues: the local eigenvalue statistics in the bulk of the spectrum change from Poisson, for $W \ll \sqrt{N}$, to GUE (hermitian matrices with i.i.d Gaussian elements), for $W \gg \sqrt{N}$.

At the present time only some upper and lower bound are proved rigorously.

- Schenker (2009): $\ell \leq W^8$
- Erdös, Knowles (2011): $\ell \gg W^{7/6}$
- Erdös, Knowles, Yau, Yin (2012): $\ell \gg W^{5/4}$.

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The questions of the order of the localization length closely related to the universality conjecture of the bulk local regime of the random matrix theory.

Global regime

Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the random matrix H_n.

Normalized Counting Measure (NCM):

$$N_n(\triangle) = \sharp\{\lambda_j \in \triangle, j = 1, \ldots, n\}/n, \quad N_n(\mathbb{R}) = 1,$$

where $\triangle$ is an arbitrary interval of the real axis.

It is shown that for many ensembles of random matrices

$$N_n \xrightarrow{w} N.$$

The density of N is called the density of states of the ensemble, the support of N is the spectrum.

Wigner ensembles (in particular GUE, GOE):

$$\rho(\lambda) = (2\pi)^{-1}\sqrt{4-\lambda^2}, \quad \lambda \in (-2, 2).$$

Local regime

The main object of the local regime of the random matrix theory is the k-point correlation function $R_k^{(n)}$, which can be defined as

$$E\left\{\sum_{j_1\neq..\neq j_k}\varphi_k(\lambda_{j_1},\ldots,\lambda_{j_k})\right\}=\int_{\mathbb{R}^k}\varphi_k(\lambda_1,..,\lambda_k)R_k(\lambda_1,..,\lambda_k)d\lambda_1..d\lambda_k,$$

where $\varphi_k:\mathbb{R}^k\to\mathbb{C}$ is bounded, continuous and symmetric in its arguments.

Universality conjecture for the hermitian random matrices in the bulk of the spectrum (Dyson):

$$(n\rho(\lambda_0))^{-k}R_k^{(n)}(\{\lambda_0+\xi_j/n\rho(\lambda_0)\})\overset{n\to\infty}{\longrightarrow}\det\left\{\frac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^k.$$

The correlation function (or mixed moment) of the characteristic polynomials:

$$F_{2k}(\lambda_1, \dots, \lambda_{2k}) = \int \prod_{l=1}^{2k} \det(\lambda_l - H_n) P_n(dH)$$

We are interested in the asymptotic behavior of this function for

$$\lambda_j = \lambda_0 + \frac{\xi_j}{n\rho(\lambda_0)}, \quad j = 1, \dots, 2k, \quad \lambda_0 \in (-2, 2).$$

GUE

Asymptotic behavior of the 2k-point mixed moment for GUE:

$$F_{2k}\left(\Lambda_0+\hat{\xi}/(n\rho(\lambda_0))\right) = C_n\,\frac{\det\left\{\dfrac{\sin\pi(\xi_i-\xi_{j+k})}{\pi(\xi_i-\xi_{j+k})}\right\}_{i,j=1}^{k}}{\triangle(\xi_1,\ldots,\xi_k)\triangle(\xi_{k+1},\ldots,\xi_{2k})}(1+o(1)),$$

where $\triangle(\xi_1,\ldots,\xi_k)$ is the Vandermonde determinant of $\xi_1,\ldots,\xi_k$.

- Similar result for the hermitian matrix model was proved by Brezin, Hikami (2000), Fyodorov, Strahov (2003).
- The same is valid for hermitian Wigner and general sample covariance matrices (Gotze,Kosters (2009-2010) for $k=1$, TS for any k (2010-2011)).

GOE

In the case of real symmetric Gaussian matrices the behavior was found by Brezin, Hikami (2000) for $k = 1$ and by Borodin, Strahov (2006) for any k.

Asymptotic behavior of the 2k-point mixed moment for GOE:

$$F_{2k}\left(\Lambda_0 + \hat{\xi}/(n\rho(\lambda_0))\right) = C_n \frac{\mathrm{Pf}\left\{-\frac{1}{\pi^2}\frac{d}{d\xi_i}\frac{\sin\pi(\xi_i-\xi_j)}{\pi(\xi_i-\xi_j)}\right\}_{i,j=1}^{2k}}{\triangle(\xi_1,\ldots,\xi_{2k})}(1+o(1)),$$

where $\triangle(\xi_1,\ldots,\xi_k)$ is the Vandermonde determinant of $\xi_1,\ldots,\xi_k$. The same is valid for $k = 1$ for hermitian Wigner and general sample covariance matrices (Kosters (2009-2010)).

Main results for 1D RBM:

Let also $D_2 = F_2(\lambda_0, \lambda_0)$.

Theorem (hermitian case)

For 1D Gaussian hermitian random band matrices with $W^2 = n^{1+\theta}$, $0 < \theta < 1$ we have

$$\lim_{n\to\infty} D_2^{-1} F_2\Big(\lambda_0 + \frac{\xi}{2N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{2N\rho(\lambda_0)}\Big) = \frac{\sin(\pi\xi)}{\pi\xi}.$$

Theorem (real symmetric case)

For 1D Gaussian real symmetric random band matrices with $W^2 = n^{1+\theta}$, $0 < \theta < 1$ we have

$$\lim_{n\to\infty} D_2^{-1} F_2\Big(\lambda_0 + \frac{\xi}{2N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{2N\rho(\lambda_0)}\Big) = \frac{\sin(\pi\xi)}{\pi^3\xi^3} - \frac{\cos(\pi\xi)}{\pi^2\xi^2}.$$

Integral representation

$$F_2(\hat{\xi}) = -(2\pi^2)^{-N}\det{}^{-2}J \int \exp\Big\{-\frac{W^2}{2}\sum_{j=-n+1}^{n}\mathrm{Tr}\,(X_j - X_{j-1})^2\Big\}$$

$$\times \exp\Big\{-\frac{1}{2}\sum_{j=-n}^{n}\mathrm{Tr}\,\Big(X_j + \frac{i\Lambda_0}{2} + \frac{i\hat{\xi}}{2N\rho(\lambda_0)}\Big)^2\Big\}\prod_{j=-n}^{n}\det\,\big(X_j - i\Lambda_0/2\big)d\overline{X},$$

where $\{X_j\}$ are hermitian 2×2 matrices,
$dX_j = d\,\mathrm{Re}\,X_{12}\,d\,\mathrm{Im}\,X_{12}\,dX_{11}\,dX_{22}$ and $\hat{\xi} = \xi\sigma_3$.

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$$\times \exp\Big\{ -\frac{1}{2} \sum_{j=-n}^{n} \mathrm{Tr}\Big(X_j + \frac{i\Lambda_0}{2} + \frac{i\hat\xi}{2N\rho(\lambda_0)}\Big)^2 \Big\} \prod_{j=-n}^{n} \det\big(X_j - i\Lambda_0/2\big) d\overline{X},$$

where $\{X_j\}$ are hermitian 2×2 matrices,
$dX_j = d\,\mathrm{Re}\,X_{12}\, d\,\mathrm{Im}\,X_{12}\, dX_{11}\, dX_{22}$ and $\hat\xi = \xi\sigma_3$.
Let us change the variables to $X_j = U_j^* A_j U_j$, where U_j is a unitary matrix and $A_j = \mathrm{diag}\,\{a_j, b_j\}$, $j = -n, \dots, n$. Then dX_j becomes

$$\frac{\pi}{2}(a_j - b_j)^2 da_j\, db_j d\mu(U_j),$$

where $d\mu(U_j)$ is the Haar measure on the unitary group $U(2)$.

Connection to the Heisenberg model

The expected saddle-points: $a_\pm = \pm\sqrt{1-\lambda_0^2/4} = \pm\pi\rho(\lambda_0)$.
Fix $a_j = a_+$, $b_j = a_-$ for each j. Then $X_j = U_j^* A_j U_j = \pi\rho(\lambda_0) U_j^* \sigma_3 U_j$, and the integral representation transforms to

σ-model:

$$\int \exp\Big\{\pi^2\rho(\lambda_0)^2 W^2 \sum_{j=-n+1}^{n} (S_j S_{j-1} - 1) + \frac{i\pi\xi}{2N}\sum_{j=-n}^{n} S_j\sigma_3\Big\} \prod_{j=-n}^{n} dS_j,$$

where $S_j = U_j^*\sigma_3 U_j$. The result states that for $W^2 \gg N$

$$Z_n^{-1}\int e^{\pi^2\rho(\lambda_0)^2 W^2 \sum_{j=-n+1}^{n}(S_j S_{j-1}-1)+\frac{i\pi\xi}{2N}\sum_{j=-n}^{n} S_j\sigma_3} \prod_{j=-n}^{n} dS_j \longrightarrow$$

$$\longrightarrow \int e^{i\pi\xi S_0\sigma_3/2} dS_0 = \frac{\sin(\pi\xi)}{\pi\xi}.$$