Stripes and slabs near the ferromagnetic transition

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Joint work with J. Lebowitz, E. Lieb and R. Seiringer

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Outline

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Periodic patterns

The spontaneous emergence of periodic states is an ubiquitous phenomenon in nature.

Nevertheless, a fundamental understanding of why crystals, or ordered patterns, form is still missing.

In this talk I would like to focus on the phenomenon of formation of periodic arrays of **stripes** or **slabs**, which are observed at low temperatures in a variety of systems, ranging from magnetic films to superconductors, polymer suspensions, twinned martensites, etc.

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Stripes

Fig. 3. Magnetic image of the 308 nm Py film (6 μm scan). The half period is 220 ± 30 nm, slightly smaller than the film thickness.

Competing interactions

The basic mechanism behind stripe formation seems to be the **competition** between a short-range attractive and a long-range repulsive interaction.

The resulting **frustration** induces the system to form mesoscopic islands of a uniform phase, which alternate regularly on the scale of the whole sample.

Striped and slabbed patterns

Theoretically, the understanding of these regular patterns is based on a variational computation of the "best energy" among those of a selected class of periodic states.

Remarkably, in many different situations, the "best state" seems to be striped or slabbed. But why?

One dimension

Motivated by these issues, some years ago we (G-Lebowitz-Lieb) started to investigate the ground state structure of 1D Ising models with competing interactions.

In the presence of n.n. FM, plus long range power-law decaying AF interactions, we proved the non-trivial periodicity of the ground states.

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Our results complemented the few convexity-based proofs of periodic minimizers in one-dimensional non-linear elasticity (Müller, Chen-Oshita).

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In the presence of n.n. FM, plus long range power-law decaying AF interactions, we proved the non-trivial periodicity of the ground states.

Our 1D results have implications in $d = 2, 3$: the best state among those with straight domain walls is periodic and striped.

Generalizations: one and two dimensions

Later on, we generalized our results to:

1. 1D continuum functionals with magnetic field (G-Lebowitz-Lieb): RP + *convexity*;
2. 1D models where the FM interaction is not n.n. (Buttà-Esposito-G-Marra): RP + *coarse graining*;
3. anisotropic 2D system for martensitic phase transitions (G-Müller): RP + *localization bounds*;
4. isotropic 2D magnetic models with in-plane spins (G-Lebowitz-Lieb): FM RP + AF RP.

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Ising models with competing interactions

In many physical contexts, a more natural, though simplified, model for stripe formation in $d \geq 2$ is:

$$H = -J \sum_{\langle \mathbf{x},\mathbf{y} \rangle} (\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - 1) + \sum_{\{\mathbf{x},\mathbf{y}\}} \frac{(\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - 1)}{|\mathbf{x} - \mathbf{y}|^p}$$

Depending on the exponent, the long range interaction can model:

- a Coulomb potential ($p = 1$),
- a dipolar potential ($p = 3$).

More general values of p describe a “generic” antiferromagnetic power law potential.

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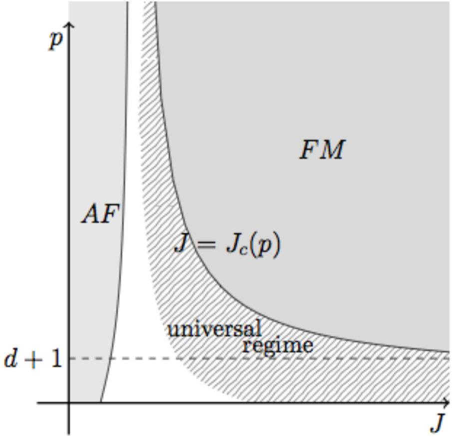
The ground state phase diagram



FM transition at: $$J = J_c(p) = \frac{1}{2} \sum_{\mathbf{y} \in \mathbb{Z}^d} \frac{|y_1|}{|\mathbf{y}|^p}$$

The ground state phase diagram

If $p > 2d$ the model is somewhat simpler to analyze: the optimal stripes energy

$$e_{stripes} \sim -(J_c - J)^{\frac{p-d}{p-d-1}} \ ,$$

is substantially smaller than that of other candidate periodic structures, and we could heuristically identify the elementary excitations of the system (*corners*).

Outline

Main results

Theorem [G-Lieb-Seiringer]. *Let* $d = 2, 3$, $p > 2d$ *and let* $J_c = J_c(p)$ *be the location of the FM transition line. Then:*

$$\lim_{J \to J_c^-} \frac{e_0(J)}{e_S(J)} = 1 \ ,$$

where $e_0(J)$ *is the ground state energy per site, and* $e_S(J)$ *is the minimal energy per site within the class of periodic striped or slabbed configurations.*

Remarks

- The proof comes with explicit error bounds on the remainder. More precisely, we find:

$$1 \geq \frac{e_0(J)}{e_S(J)} \geq 1 - C(J_c - J)^{\frac{p-2d}{(d-1)(p-d-1)}}$$

Remarks

- Our proof also also shows that, if $J \lesssim J_c$, the ground state is striped in a certain sense: namely, if we look at a randomly chosen window, of suitable size $\ell \gg h^*$, we see a striped state with high probability.
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Main steps

The proof consists in a refined lower bound on e_0.
Main steps:

1. Representation of the energy in terms of droplet self-energies and droplet-droplet interactions.
2. Lower bound on self-energy $\gtrsim (J - J_c)|\Gamma| + N_c$.
3. Localization of the droplets' energy functional in boxes of side ℓ (to be optimized over).
4. Key fact: the localized self-energy of droplets with $N_c \geq 1$ is *positive* if $\ell < (J_c - J)^{-1/(d-1)}$.
5. The local configurations without corners can be minimized by using *reflection positivity* and lead to a periodic striped state.

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Thank you!

Summary

- We considered Ising models in $d = 2, 3$ with n.n. FM and power law AF interactions, the decay exponent being $p > 4$ $(d = 2)$ or $p > 6$ $(d = 3)$.
- At a critical J_c, the ground state becomes homogenous. We proved that asymptotically as $J \to J_c^-$ the specific ground state energy approaches the optimal periodic striped energy.

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- The proof comes with explicit error bounds and implies existence of stripes in a suitable sense: most windows of side $\ell \gg h^*$ show striped states, with stripes all of size close to h^*.

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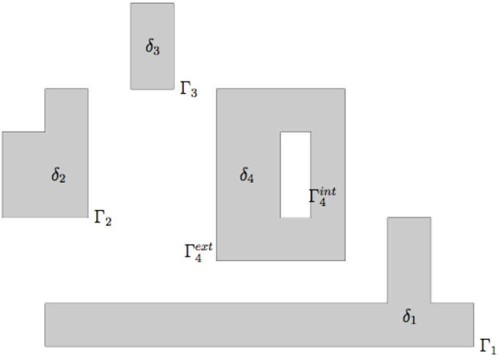
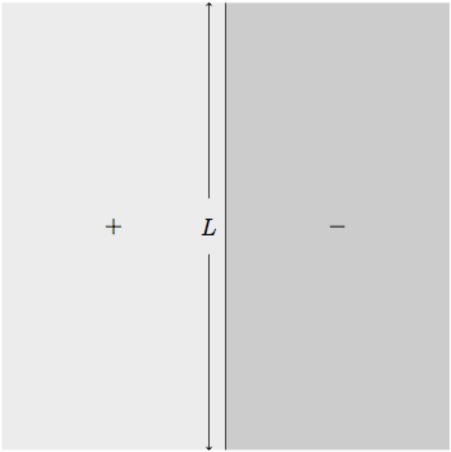
The droplet representation

We choose $+$ boundary conditions and define the droplets δ_i to be the maximal connected regions of negative spins. Their boundaries Γ_i are the usual low-temperature contours of the Ising model.



Energy of a droplet (in $d = 2$)



$$E = (2J - 2J_c)L$$

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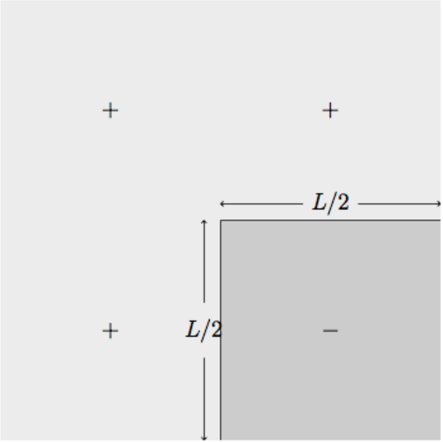

$$E = (2J - 2J_c)L + \kappa_c$$

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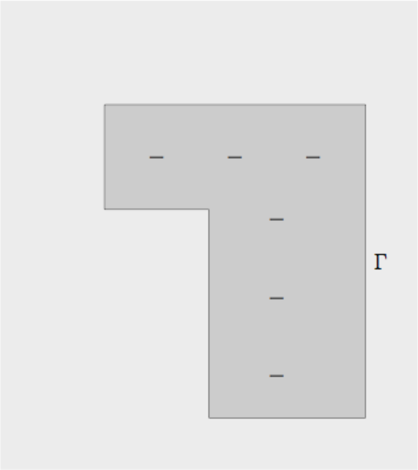

$$E \geq 2(J - J_c)|\Gamma| + \kappa_c N_c$$

Corners

Take home message: corners cost a finite energy! They look like elementary excitations. However: how do we eliminate them by local moves? How do we exclude that their presence does not decrease the interaction energy substantially?

Corners

We localize into boxes of side $h^* \ll \ell \sim \tau^{-1}$. As far as $W(\delta_i, \delta_j)$ is concerned, we just neglect the terms involving droplets belonging to different boxes.

Non trivial part: localization of the self-energy. The resulting expression is positive as soon as $N_c \geq 1$.

We can, therefore, erase all droplets with corners.

At that point we utilize reflection positivity of the long range potential to reflect in the location of the domain walls, so finding our final bound.