Rutgers U, May 6, 2012

Five generalizations of the AKLT model

Bruno Nachtergaele
(UC Davis)

based on joint work with

Mark Fannes, Reinhard Werner, and Sven Bachmann

Dedicated to

Elliott Lieb

who stood at the cradle of so many topics in our beloved field.

[1]Based on work supported by the U.S. National Science Foundation under grants # DMS-0757581, and DMS-1009502.

Outline

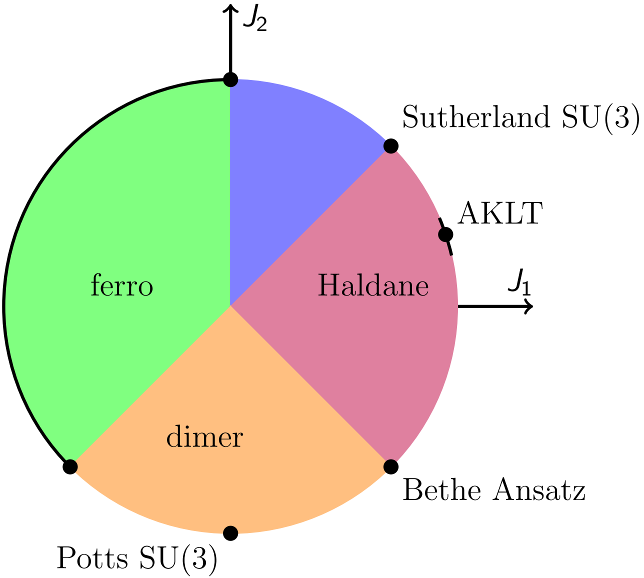
- The AKLT model (Affleck-Kennedy-Lieb-Tasaki 1987-88).
- 'Five' generalizations of the AKLT model
- An algebraic approach to models with frustration-free ground states
- Product vacua with boundary states (PVBS)
- Conclusion and Comments

The AKLT model (Affleck-Kennedy-Lieb-Tasaki, 1987)

Antiferromagnetic spin-1 chain: $[1, L] \subset \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$,

$$H_{[1,L]} = \sum_{x=1}^{L} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L} P^{(2)}_{x,x+1}$$

The ground state space of $H_{[1,L]}$ is 4-dimensional for all $L \geq 2$. In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state (Haldane phase). Exact ground state is "frustration free" (Valence Bond Solid state (VBS), Matrix Product State (MPS), Finitely Correlated State (FCS)).

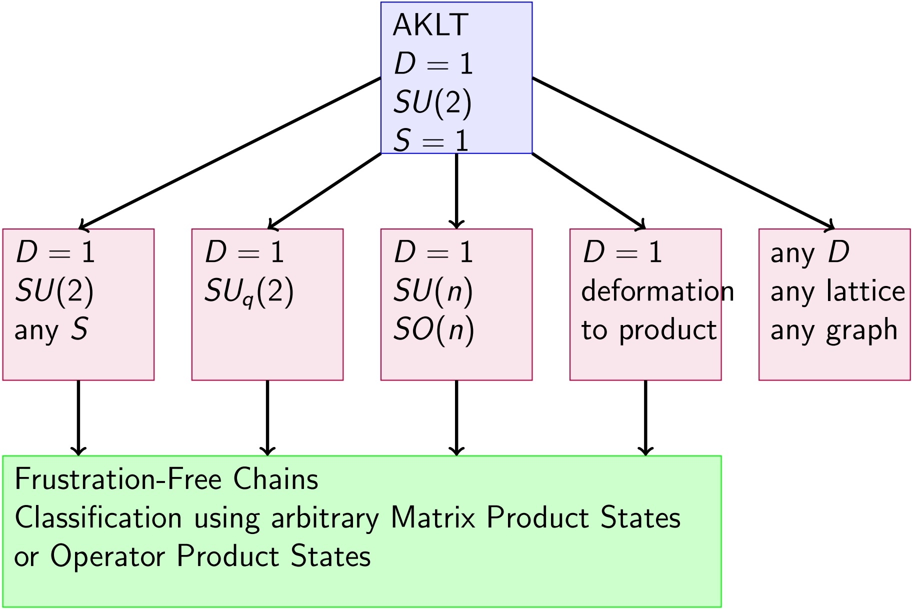


$$H = \sum_x J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

The AKLT model changed the way we look at quantum spin chains and ground states of quantum spin models in general:

- The existence of the Haldane phase
- The role of symmetry
- The development of reliable numerical methods (various Density Matrix Renormalization Group algorithms)
- The nature of ground state correlations, entanglement (e.g., Area Law for entanglement entropy)
- Classification of gapped ground state phases

Many hundreds of authors contributed.


AKLT
$D = 1$
$SU(2)$
$S = 1$
$D = 1$
$SU(2)$
any S
$D = 1$
$SU_q(2)$
$D = 1$
$SU(n)$
$SO(n)$
$D = 1$
deformation
to product
any D
any lattice
any graph
Frustration-Free Chains
Classification using arbitrary Matrix Product States
or Operator Product States

Frustration-free ground states of spin chains

Consider spin chain with for all $x \in \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^d$. A translation invariant nearest neighbor interaction h is a self-adjoint matrix acting on $\mathbb{C}^d \otimes \mathbb{C}^d$, and the Hamiltonian is

$$H_L = \sum_{x=1}^{L-1} h_{x,x+1},$$

We can assume that the smallest eigenvalue of h is 0.
The model is frustration-free if 0 is an eigenvalue for all $L \geq 2$.
Whether the model is frustration-free or not depends on a geometric property of $\ker h = \mathcal{G} \subset \mathbb{C}^d \otimes \mathbb{C}^d$

$$\ker H_{[1,L]} = \bigcap_{x=1}^{L-1} \underbrace{\mathbb{C}^d \otimes \cdots \mathbb{C}^d}_{x-1} \otimes \mathcal{G} \otimes \underbrace{\mathbb{C}^d \otimes \cdots \mathbb{C}^d}_{L-x-1}$$

For which $\mathcal{G}$ is $\ker H_L \neq \{0\}$ for all $L \geq 2$?
In quantum information theory, this is a satisfiability problem.

Operator Product Representation

(with M Fannes and RF Werner, in preparation).

Observation: the existence of 0-eigenvectors of H_L for all finite L is equivalent to the existence of pure states ω of the half-infinite chain with zero expectation of all $h_{x,x+1}, x \geq 1$.

This follows from weak compactness of the set of states and the simple observation that non-negative numbers add up to zero only if they all vanish.

We call such states ω pure zero-energy states.

Zero-energy states are certainly ground states ($h_{x,x+1} \geq 0$); it is a separate question whether they are all the ground states.

Theorem (Fannes-N-Werner (2010))

A pure state ω *is a zero-energy state iff it has an representation in operator product form: there is a Hilbert space* $\mathcal{K}$*, bounded linear operators* $V_1, \ldots, V_d$ *on* $\mathcal{K}$*, and* $\Omega \in \mathcal{K}$*, such that*
$\mathrm{span}\{V_{\alpha_1} \cdots V_{\alpha_n}\Omega \mid n \geq 0, 1 \leq \alpha_1, \ldots, \alpha_n \leq d\} = \mathcal{K}$

$$\omega(|\alpha_1, \ldots, \alpha_n\rangle\langle\beta_1, \ldots, \beta_n|) = \langle\Omega, V^*_{\alpha_1} \cdots V^*_{\alpha_n} V_{\beta_n} \cdots V_{\beta_1}\Omega\rangle$$

and $\mathbb{1}$ *is the only eigenvector with eigenvalue 1 of the operator*

$$\widehat{\mathbb{E}} \in \mathcal{B}(\mathcal{B}(\mathcal{K})) : \quad \widehat{\mathbb{E}}(X) = \sum_{\alpha=1}^{d} V^*_\alpha X V_\alpha$$

and for all $\psi \perp \mathcal{G}$*,* $\psi = \sum_{\alpha,\beta} \psi_{\alpha\beta}|\alpha, \beta\rangle$*, we have the relation*

$$\sum_{\alpha,\beta} \overline{\psi_{\beta\alpha}}\, V_\alpha V_\beta = 0.$$

Matrix Product States (MPS)

If $\mathcal{K}$ is finite-dimensional, say $\dim \mathcal{K} = k$, the theorem is equivalent to the MPS form of the ground state vectors for finite chains: for an arbitrary $k \times k$ matrix B,

$$\psi(B) = \sum_{\alpha_1,\ldots,\alpha_L}^{d} \mathrm{Tr}(BV_{\alpha_L} \cdots V_{\alpha_1}) |\alpha_1, \ldots, \alpha_L\rangle$$

is a ground state of the model.

In the case of the AKLT model we have $k = 2$ and, expressed in the standard basis, the V_α are multiples of the Pauli matrices $\sigma^+, \sigma^3, \sigma^-$.
All one-dimensional generalizations of the AKLT model fit into this framework.

Product Vacua with Boundary States (PVBS)

(joint work with Bachmann, arXiv:1112.4097)

Consider a quantum spin chain with $d = n+1$ states at each site that we interpret as n distinguishable particles labeled $i = 1, \ldots, n$, and an empty state denoted by 0.

The Hamiltonian for a chain of L spins is given by

$$H_L = \sum_{x=1}^{L-1} h_{x,x+1}, \tag{1}$$

where each $h_{x,x+1}$ is a sum of hopping terms normalized to yield and orthogonal projection:

$$h = \sum_{i=1}^{n} |\hat{\phi}_i\rangle\langle\hat{\phi}_i| + \sum_{1\leq i\leq j\leq n} |\hat{\phi}_{ij}\rangle\langle\hat{\phi}_{ij}|,$$

The $\phi_{ij} \in \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ are given by

$$\phi_i = |i,0\rangle - e^{-\theta_{i0}}\lambda_i^{-1}|0,i\rangle\,, \phi_{ij} = |i,j\rangle - e^{-\theta_{ij}}\lambda_i^{-1}\lambda_j|j,i\rangle\,, \phi_{ii} = |i,i\rangle$$

for $i = 1, \ldots, n$ and $i \neq j = 1, \ldots, n$.

The parameters satisfy: $\theta_{ij} \in \mathbb{R}$, $\theta_{ij} = -\theta_{ji}$, and $\lambda_i > 0$, for $0 \leq i, j \leq n$, and $\lambda_0 = 1$.
There exist $n+1$ $2^n \times 2^n$ matrices, $v_0, v_1, \dots, v_n$, satisfying the following commutation relations:

$$\begin{aligned} v_i v_j &= e^{i\theta_{ij}} \lambda_i \lambda_j^{-1} v_j v_i, \quad i \neq j && (2) \\ v_i^2 &= 0, \quad i \neq 0 && (3) \end{aligned}$$

Then, for B an arbitrary $2^n \times 2^n$ matrix,

$$\psi(B) = \sum_{i_1,\dots,i_L=0}^{n} \mathrm{Tr}(B v_{i_L} \cdots v_{i_1}) |i_1, \dots, i_L\rangle \quad (4)$$

is a ground state of the model (MPS vector). In fact, they are all the ground states. E.g., one can pick B such that

$$\psi(B) = \sum_{x=1}^{L} \left(e^{i\theta_{i0}} \lambda_i\right)^x |0, \dots, i, \dots, 0\rangle$$

If we add the assumption that $\lambda_i \neq 1$, for $i = 1, \ldots, n$, we will have n_L particles having $\lambda_i < 1$ that bind to the left edge, and $n_R = n - n_L$ particles with $\lambda_i > 1$, which, when present, bind to the right edge. The bulk ground state is the vacuum state

$$\Omega = |0, \ldots, 0\rangle \, .$$

All other ground states differ from Ω only near the edges. We can prove that the energy of the first excited state is bounded below by a positive constant, independently of the length of the chain. As at most one particle of each type can bind to the edge, any second particle of that type must be in a scattering state. The dispersion relation is

$$\epsilon_i(k) = 1 - \frac{2\lambda_i}{1+\lambda_i^2} \cos(k + \theta_{i0}) \, .$$

We conjecture that the *exact* gap of the infinite chain is

$$\gamma = \min \left\{ \frac{(1-\lambda_i)^2}{1+\lambda_i^2} \,\middle|\, i = 1, \ldots, n \right\} \, .$$

Gapped ground state phases

Two models with interactions $h(0)$ and $h(1)$ are said to belong to the same phase if there is a continuous curve of interactions $h(s), 0 \leq s \leq 1$, interpolating between the two, and such that the spectral gap of

$$H_L(s) = \sum_{x=1}^{L-1} h_{x,x+1}(s)$$

is bounded below by a constant $\gamma > 0$, for all s and L.

The AKLT model belongs to the same phase as the PVBS models with $n_L = n_R = 1$. The 4 ground states of the a finite AKLT chains are usually described in terms of a spin $1/2$ particle attached to the two ends of the chain.

Denote the two particle states by $-$ and $+$. For $s \in [0, s_0]$ where $\sin(s_0) = \sqrt{2/3}$, the following 4 vectors span the ground state space of two neighboring spins, $\mathcal{G}(s)$, of the the interpolating models as a function of s:

$$
\begin{aligned}
\psi^0(s) &= \mu(s)\sin(s)\left[\lambda(s)^2|-,+\rangle + |+,-\rangle\right] \\
&\quad - \cos^2(s)(1+\lambda(s)^4)|0,0\rangle \\
\psi^{0-}(s) &= -\lambda(s)|0,-\rangle + |-,0\rangle \\
\psi^{0+}(s) &= -\lambda(s)|+,0\rangle + |0,+\rangle \\
\psi^{-+}(s) &= |-,+\rangle - \lambda(s)^2|+,-\rangle\,,
\end{aligned}
$$

$\lambda(s)$ is a smooth function such that $\lambda(s_0) = 1$, $0 < \lambda(s) < 1$, for all $s < s_0$, and $\mu(s) = (1 - \lambda(s)^2\cos^2(s))^{1/2}$. The corresponding nearest neighbor interaction, $h(s)$, is taken to be the projection onto the orthogonal complement of this 4-dimensional space.

$H_L(s_0)$ is the AKLT Hamiltonian and that $H_L(0)$ is the PVBS model with $n_L = n_R = 1$, the coefficients $\lambda_- = \lambda(0)$ and $\lambda_+ = \lambda(0)^{-1}$, and all the phases $\theta_{ij} = \pi$.
The path of interactions is smooth as the four ground state vectors are smooth, remain orthogonal to each other and of finite norm for all s, and the spectral gap does not close.
Hence, the AKLT model is in the same gapped quantum phase as the PVBS model with $n_L = n_R = 1$.

The sets of ground states of these models are equivalent for the finite, half-infinite and infinite chains, where they are isomorphic to a pair of qubits, a single qubit, and a unique pure state, respectively.

Conclusion and Comments

- 25 years ago the AKLT model changed the world (of quantum spin systems)
- There are interesting models (e.g., XXZ Heisenberg ferromagnet) with $\dim \mathcal{K} = \infty$.
- The PVBS Hamiltonians are just toy models, but we conjecture that a generalization of this class describes a complete classification of gapped ground state phases in one dimension.
- If one imposes a local symmetry, a representation of this symmetry on $\mathcal{K}$ enters the classification problem.
- We are close to a comprehensive picture in one dimension, but in two (and more) dimensions many questions remain open.