Hydrodynamic limit of interacting particle systems: the zero-range process

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Joint works w/. Mischler, Bodineau-Lebowitz-Villani, Marahrens
The zero-range process

- Introduced by [Spitzer1970] as a model system for interacting random walks
- Particles on a lattice $\Lambda_N$ hop randomly to other neighboring sites
- Hopping rate depends on the number of particles at the departure site (zero-range interactions)
- Boundary conditions: reservoirs, periodic conditions...
The periodic case

- Periodic spatial geometry: $u \in \mathbb{T}^d$
- At the discrete level: $x, y, z \in \Lambda_N = \mathbb{T}^d_N = \{1, \ldots, N\}^d$
- Microscopic configuration phase space: $\eta \in X_N = \mathbb{N}^{\mathbb{T}^d_N}$
- Stochastic “trajectories” for $\eta$ which preserve the total normalized mass $\frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta(x)$
- Law $f_t^N \in \mathcal{P}(X_N)$ depending on time
The zero-range process: master equation

\[ \frac{\mathrm{d}}{\mathrm{d}t} \langle f_t^N, \varphi^N \rangle = \langle f_t^N, G^N \varphi^N \rangle \]

\[ G^N \varphi^N(\eta) := N^2 \sum_{x,y \in \mathbb{T}_N^d} g(\eta(x)) \left[ \varphi^N(\eta^{x,y}) - \varphi^N(\eta) \right] \]

\[ \eta^{x,y}(z) = \begin{cases} 
\eta(x) - 1 & \text{for } z = x \\
\eta(y) + 1 & \text{for } z = y \\
\eta(z) & \text{otherwise}
\end{cases} \]

- Rate function \( g \) which satisfies
  - \( g(0) = 0 \) and \( g(k) > 0 \) for all \( k > 0 \)
- Factor \( N^2 \): diffusive (parabolic) scaling
The limit evolution system: nonlinear diffusion

- **Hydrodynamic limit:** $N \to +\infty$

- **Empirical measure**
  $$\mu_N^{\eta} = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N} \in \mathcal{M}_+(\mathbb{T}^d)$$

- Does $\mu_N^{\eta}$, where $\eta$ has law $f_t^N$, approaches a deterministic profile $f_t$ with a macroscopic evolution equation?

  $$\mathbb{P}_{f_t^N} \left( \left\{ \left| \langle \mu^{\eta}_N, \varphi \rangle - \langle f_t, \varphi \rangle \right| > \varepsilon \right\} \right) \xrightarrow[N \to \infty]{} 0$$

- **Nonlinear diffusion equation for $f_t$:**
  $$\partial_t f = \Delta(\sigma(f)), \quad f_t(u) \geq 0, \quad u \in \mathbb{T}^d$$

- **Remark:** Defining solutions requires more regularity than measures $\mathcal{M}_+(\mathbb{T}^d)$, e.g. $L^\infty(\mathbb{T}^d)$
Invariant measure structure

- **Particle system:**
  Invariant product measure $\nu^N_\phi(\eta) = \otimes \nu_\phi(\eta_i)$ with
  $$\nu^N_\phi(\{\eta(x) = k\}) = \frac{1}{Z(\phi)} \frac{\phi^k}{g(k)!} \quad \text{with} \quad Z(\phi) = \sum_{k \geq 0} \frac{\phi^k}{g(k)!}$$

- **Notation:** $g(k)! = g(k)g(k-1)\cdots g(1)$ and $g(0)! = 1$

- **Limit equation:**
  Stationary solution $f_\infty = \text{constant}$

- **Nonlinearity functional $\sigma$ prescribed by**
  $$\mathbb{E}_{\nu^N_{\sigma(\rho)}} \left[ \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \right] = \frac{1}{Z(\sigma(\rho))} \sum_{k \geq 0} \frac{k \sigma(\rho)^k}{g(k)!} = \rho$$

- $\sigma(0) = 0$ and increasing
Consider $\Omega$ bounded regular, $\sigma \in C^2$ increasing and
\[
\partial_t f = \Delta \sigma(f), \quad u \in \Omega, \quad f|_{\partial\Omega}(u) = f_b(u).
\]

Then for any $\Phi \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ convex with $\Phi(1) = \Phi'(1) = 0$:
\[
\frac{d}{dt} H_{\Phi}(f_t|f_\infty) := \int_{\Omega} \left( \int_{f_\infty(u)}^{f_t(u)} \Phi' \left( \frac{\sigma(v)}{\sigma(f_\infty(u))} \right) \, dv \right) \, du
\]
\[
= - \int_{\Omega} \Phi''(h) |\nabla h|^2 \sigma(f_\infty(u)) \, du \leq 0
\]

with $h := \frac{\sigma(f_t(u))}{\sigma(f_\infty(u))}$. 

**Theorem (Bodineau-Lebowitz-CM-Villani)**
Idea of the strategy

- Example: In the case corresponding to the Boltzmann relative entropy \( \Phi(z) = z \ln z - z + 1 \) one finds

\[
H(f_t|f_\infty) := \int_\Omega \left( \int_{f_\infty(x)}^{f_t(u)} \ln \left( \frac{\sigma(v)}{\sigma(f_\infty(u))} \right) \, dv \right) \, du
\]

different from the usual relative entropy due to \( \sigma \)

- Heuristic: computation of the large deviation function for the zero-range process with reservoirs in order to guess the relative entropy structure for the limit equation

- Key point: the invariant measure is still a product measure with a varying density, leading to explicit calculations

- Proof: Use that \( h \) satisfies \( h = 1 \) at the boundary and use of \( \Phi(1) = \Phi'(1) = 0 \) to kill the boundary terms
The hydrodynamic limit: framework

Assumptions:

▶ $g(0) = 0$, $g(k) > 0$ for all $k > 0$

▶ $g(k + 1) - g(k) \leq g^* < \infty$ for all $k > 0$

▶ $g(k) - g(j) \geq \delta$ for some $k_0 > 0$ and $\delta > 0$, and any $k \geq j + k_0$

Evolution systems:

$$\frac{d}{dt} \langle f_t^N, \varphi^N \rangle = \langle f_t^N, G^N \varphi^N \rangle \text{ on } \mathcal{P}(X_N)$$

$$\frac{\partial f_t}{\partial t} = \Delta \sigma(f_t) \text{ on } \mathcal{X} \subset \mathcal{M}_+(\mathbb{T}^d)$$

Question: $\mu^N_\eta \sim f_t$ as $N \to \infty$ where $\eta$ has law $f^N_t$?
Assume on the initial data $f_0 \in L^\infty(\mathbb{T}^d)$

$$H \left( f_0^N | \nu_{\sigma(\rho)}^N \right) = \int_\Omega \ln \left( \frac{df_0^N}{d\nu_{\sigma(\rho)}^N} \right) df_0^N \lesssim N^d$$

$$\left\langle f_0^N, \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)^2 \right\rangle \lesssim 1$$

Then there is propagation in time of the deterministic limit:

$$P_{f_0^N} \left( \left\{ \left| \langle \mu_{\eta}^N, \varphi \rangle - \langle f_0, \varphi \rangle \right| > \varepsilon \right\} \right) \xrightarrow{N \to \infty} 0$$

implies for later times $t > 0$

$$P_{f_t^N} \left( \left\{ \left| \langle \mu_{\eta}^N, \varphi \rangle - \langle f_t, \varphi \rangle \right| > \varepsilon \right\} \right) \xrightarrow{N \to \infty} 0$$
The hydrodynamic limit: some existing results (II)

[Yau1991] If furthermore $f \in C^3(\mathbb{T}^d)$ (smooth solution at the limit) and

$$\frac{1}{N^d} H \left( f_0^N \mid \nu_0^N \right) \xrightarrow{N \to \infty} 0$$

for the local equilibrium product measure

$$\nu_f^N := \prod_{x \in \mathbb{T}_N^d} \frac{\sigma(f(x/N))^{\eta(x)}}{Z(\sigma(f(x/N)))! g(\eta(x))!},$$

then at later times $t > 0$

$$\frac{1}{N^d} H \left( f_t^N \mid \nu_t^N \right) \xrightarrow{N \to \infty} 0$$

Questions and motivations

- **Quantitative rate of convergence?** not in GLV, almost in GPVW, Yau’s relative entropy methods can be made explicit, but rate $O(e^{\lambda t})$ with $\lambda > 0$ large, and for smooth solutions

- However both the many-particle and limit systems are **dissipative**, hence ergodicity and relaxation should win over stochastic fluctuations at the level of the laws

\[
\begin{align*}
\lim_{N \to \infty} f_t^N & \in \mathcal{P}(X_N) \quad \xrightarrow{N \to \infty} \quad f_t \in L^\infty(\mathbb{T}^d) \\
\lim_{t \to \infty} f_t^N & \xrightarrow{N \to \infty} \nu_{\sigma(\rho)}^N \in \mathcal{P}(X_N) \quad \xrightarrow{N \to \infty} \quad f_{\infty}
\end{align*}
\]

- **Goals:**
  - Fully quantitative rate
  - Does not use regularity of the limit solution
  - Uniform in time
  - In entropic form...
The main result

Theorem (Marahrens-CM, 2012)

\( f_t \in L^\infty \) the solution to the nonlinear diffusion equation and

\[
\frac{1}{N^d} H \left( f_0^N \big| \nu^N_{\sigma(\rho)} \right) \lesssim 1 \quad \text{and} \quad \left\langle f_0^N, \left( \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)^4 \right) \right\rangle \lesssim 1
\]

Consider \( F \in C^2_b(\mathbb{R}) \) and \( \varphi \in C^3(\mathbb{T}^d) \) then

\[
\sup_{t \geq 0} \left| \left\langle f_t^N, F \left( \langle \mu^N_\eta, \varphi \rangle \right) \right\rangle - F \left( \langle f_t, \varphi \rangle \right) \right| \leq r_{HL}(\varepsilon, N)
\]

\[
+ C_{F,\varphi} \left\langle f_0^N, \left\| \mu^N_\eta,\varepsilon - f_0 \right\|_{H^{-1}} \right\rangle
\]

for some quantitative polynomial rate \( r_{HL} \)

Consequence: rate of convergence \( O(N^{-\alpha}) \) uniform in time
An inspirative previous work on Kac’s model

- Abstract ideas developed in another work w/. Mischler on Kac’s program in kinetic theory:

  - Many-particle collision jump process on the velocities (“Kac’s walk”) in $\mathbb{S}^{N-1}(\sqrt{N})$ with jump rates depending on the velocities (hard spheres)

    \[
    \partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle
    \]

    \[
    (G^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^{N} |v_i - v_j| \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) \left[ \varphi^{*}_{ij} - \varphi \right] d\sigma
    \]

- Mean-field limit towards nonlinear (spatially homogeneous) Boltzmann equation when propagation of chaos

  \[
  \partial_t f = Q(f)
  \]

  \[
  Q(f)(v) := \int_{v\in\mathbb{R}^d} \int_{\sigma\in\mathbb{S}^{d-1}} \left( f(v')f(v') - f(v)f(v*) \right) |v - v*| d\sigma
  \]
Kac’s program in kinetic theory

- Estimate on the spectral gap in $L^2(S^{N-1}([\sqrt{N}]))$: [Carlen-Carvalho-Loss2003] (cf. also [Janvresse], [Maslen])
- However not sufficient for answering main motivation of Kac: to connect the asymptotic behavior and entropies of the many-particle and limit systems in the chaotic limit

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**Theorem (Mischler-CM, 2011)**

**Uniform in time chaos** (for any number of marginals $1 \leq \ell \leq N$)

$$W_1\left( \prod_{\ell} f_t^N, f_t^{\otimes \ell} \right)_{t \geq 0} \leq \alpha(N) \to 0$$

**Entropic chaos:**

$$\frac{H(f_t^N)}{N} \xrightarrow{N \to \infty} H(f)$$

**Unif. in $N$ relaxation times:**

$$\sup_{N \geq 1} \frac{W_1(f_t^N, \gamma^N)}{N} \leq \beta(t) \to 0$$
Abstract strategy

- View the many-particle system as a perturbation of the limit nonlinear PDE
- Consistency-stability estimates on semigroups
- Requires an appropriate functional framework for comparing generators (consistency) and establishing stability estimates
- The latters correspond to statistical stability estimates on the flow of the limit nonlinear PDE

\[
\begin{align*}
X_N & \xrightarrow{\text{Kolmogorov}} \mathcal{P}(X_N) & \xleftarrow{\text{duality}} & C_b(X_N) \\
\mathcal{X} \subset \mathcal{M}_+(\mathbb{T}^d) & \xrightarrow{\text{"Liouville"}} P(\mathcal{X}) & \xleftarrow{\text{duality}} & C_b(\mathcal{X})
\end{align*}
\]

with \((\pi_N \Phi)(\eta) := \Phi(\mu^N_\eta) = (\mu^N_\cdot)_* \Phi\)
Evolution $N$-particle semigroups

- Stochastic process $(\eta^N_t)$ on $X_N$
- Corresponding linear semigroup $\mathcal{F}^N_t$ on $P(X_N)$:

$$\partial_t f^N = A^N f^N, \quad f^N \in P(X_N),$$

Forward Kolmogorov equation or “Master equation”

- Dual linear semigroup $T^N_t$ of $\mathcal{F}^N_t$:

$$\forall f^N \in P(X_N), \varphi^N \in C_b(X_N), \quad \langle f^N, T^N_t(\varphi^N) \rangle := \langle \mathcal{F}^N_t(f^N), \varphi^N \rangle$$

- Semigroup of the observables:

$$\partial_t \varphi^N = G^N(\varphi^N), \quad \varphi^N \in C_b(X_N)$$
Evolution limit semigroups

- Nonlinear semigroup $F^\infty_t$ on $\mathcal{X}$ solution to
  \[ \partial_t f_t = Q(f_t) = \Delta \sigma(f), \quad f_{|t=0} = f_0 \]

- Pullback linear semigroup $T^\infty_t$ on $C_b(\mathcal{X})$:
  \[ \forall f \in \mathcal{X}, \Phi \in C_b(\mathcal{X}), \quad T^\infty_t[\Phi](f) := \Phi(F^\infty_t(f)) \]
  solution to the linear evolution equation on $C_b(P(E))$:
  \[ \partial_t \Phi = G^\infty(\Phi) \quad \text{with generator} \quad G^\infty \]

- Comparison of semigroups $T^N_t$ and $T^\infty_t$

- Additional difficulty: $\mu^N_\eta \not\in \mathcal{X} = L^\infty(\mathbb{T}^d)$, hence we introduce the mollified empirical measure $\mu^{N,\varepsilon}_\eta = \chi_\varepsilon \ast \mu^N_\eta$, $\varepsilon > 0$ and the corresponding projection $\pi_{N,\varepsilon}$
Given a nonlinear ODE $\dot{Y} = F(Y)$ on $\mathbb{R}^d$, one can define (at least formally) the linear Liouville transport PDE

$$\partial_t \rho + \nabla_v \cdot (F \rho) = 0,$$

where $\rho_t(v) = Y^*_t(\rho_0) = \rho_0 \circ Y_{-t}$ (characteristics)

Dual viewpoint: for $\phi_0$ function defined on $\mathbb{R}^d$, evolution $\phi_t(v) = \phi_0(Y_t(v)) = (Y_t)_*\phi_0 = \phi_0 \circ Y_t$ solution to the linear PDE

$$\partial_t \phi - F \cdot \nabla_v \phi = 0,$$
Go “one level above” and replace $\mathbb{R}^d$ by $\mathcal{X}$:
The infinite dimensional “ODE” $f' = Q(f)$ on $\mathcal{X}$ yields first the abstract transport equation

$$\partial_t \pi + \nabla \cdot (Q(f) \pi) = 0, \quad d\pi(t, \cdot) \in P(\mathcal{X})$$

and second the abstract dual equation

$$\partial_t \Phi - Q(f) \cdot \nabla \Phi = 0, \quad \Phi(t, \cdot) \in C_b(\mathcal{X}).$$

Provide intuition and formal formula for the generator “$(G^\infty \Phi)(f) = Q(f) \cdot \nabla \Phi(f)$”: but requires to define correctly the objects...
Well-posedness for the limit equation in $L^\infty(\mathbb{T}^d)$

Propagation of $H^k$ regularity

Lipschitz stability: $\|F_t^\infty f_2 - F_t^\infty f_1\|_{H^{-1}} \leq \|f_2 - f_1\|_{H^{-1}}$

Higher-order stability:

$$\|F_t^\infty f_2 - F_t^\infty f_1 - D F_t^\infty(f_1)(f_2 - f_1)\|_{H^{-1}} \leq C(f)\|f_2 - f_1\|_{H^{-1}}^{1+\theta}$$

$D F_t^\infty(f) \in \mathcal{L}(\mathcal{X})$: solution $h_t := D F_t^\infty(f_1)(f_2 - f_1) \in \mathcal{X}$ to

$$\partial_t h_t = \Delta \left( \sigma'(f_1(t)) h \right) \quad \text{such that} \quad h_0 = (f_2 - f_1)|_{t=0}$$

$T_t^\infty \psi : H \rightarrow \mathbb{R}$ is $C^{1+\theta}(\mathcal{X})$ for the $H^{-1}$-norm and

$$D \left[ T_t^\infty \Phi \right](f) \cdot h = D \Phi(F_t^\infty(f)) \cdot [DF_t^\infty[f](h)], \quad \Phi \in C^1(\mathcal{X})$$

$G^\infty[\Phi](f) = F'(\langle f, \varphi \rangle) \langle \sigma(f), \Delta \varphi \rangle$ for $\Phi(f) = F(\langle f, \varphi \rangle)$
Consistency and convergence (I)

\[
\left| \langle f^N_t, F(\langle \mu^N_\eta, \varphi \rangle_{L^2}) \rangle - F(\langle f_t, \varphi \rangle_{L^2}) \right| = \left| \langle F^N f^N_0, \Phi(\mu^N_\eta) - \Phi(F^\infty f_0) \rangle \right|
\]

\[
= \left| \langle f^N_0, T^N_t \Phi(\mu^N_\eta) - [T^\infty_t \Phi](f_0) \rangle \right|
\]

\[
\leq \left| \langle f^N_0, T^N_t \Phi(\mu^N_\eta, \epsilon) - [T^\infty_t \Phi](\mu^N_\eta, \epsilon) \rangle \right|
+ \left| \langle f^N_0, [T^\infty_t \Phi](\mu^N_\eta, \epsilon) - [T^\infty_t \Phi](f_0) \rangle \right|
+ \left| \langle f^N_0, T^N_t \Phi(\mu^N_\eta) - T^N_t \Phi(\mu^N_\eta, \epsilon) \rangle \right|
\]

\[=: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3\]

Error terms:

- **\(\mathcal{T}_2\):** initial data approximation error propagated by \(T^\infty_t\) (Lipschitz stability)
- **\(\mathcal{T}_3\):** \(\epsilon\)-error due to the mollification at the level of the particle system
- **\(\mathcal{T}_1\):** core of the proof - comparison of the two semigroups
- Linear growth in time of error + interpolation with the exponential relaxation of the limit equation: uniform in time estimates
Consistency and convergence (II): the $\mathcal{T}_1$ estimate

\[
\frac{d}{ds} (T_s^N \pi_{N,\varepsilon} T_t^\infty \Phi(\eta)) = T_s^N G^N \pi_{N,\varepsilon} T_t^\infty \Phi(\eta) - T_s^N \pi_{N,\varepsilon} G^\infty T_t^\infty \Phi(\eta)
\]

hence

\[
\mathcal{T}_1 \leq \left| \int_0^t \left\langle f_{t-s}^N, (G^N \pi_{N,\varepsilon} - \pi_{N,\varepsilon} G^\infty) T_s^\infty \psi \right\rangle ds \right|
\]

- First-order Taylor expansion of $T_t^\infty$:
  \[
  T_t^\infty \psi(\mu_{N,\varepsilon}^{x,y}) - T_t^\infty \psi(\mu_{\varepsilon}^{N,\eta}) - DT_t^\infty \psi(\mu_{\varepsilon}^{N,\eta})(\mu_{N,\varepsilon}^{x,y} - \mu_{N,\varepsilon}^{x})
  \]
  (remainder term controlled by high-order statistical stability)

- Replacing the discrete Laplacian by the continuous one
  (propagation of regularity on $D \mathcal{F}_t^\infty (\mu_{\varepsilon}^{N,\eta})^*$)

- Quantitative version of replacement lemma
  [Kipnis-Landim1999] with the help of LSI

\[
\frac{1}{T} \int_0^T \int_{\mathbb{T}^d} \left\langle f_t^N, \left( g \circ \eta^{(\varepsilon)}(uN) - \sigma(\eta^{(\varepsilon)}(uN)) \right) \right\rangle du \, dt \leq r_{RL}(\varepsilon, N)
\]