# Hydrodynamic limit of interacting particle systems: the zero-range process

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Joint works w/. *Mischler, Bodineau-Lebowitz-Villani, Marahrens*

# The zero-range process

- Introduced by [Spitzer1970] as a model system for interacting random walks
- Particles on a lattice $\Lambda_N$ hop randomly to other neighboring sites
- Hopping rate depends on the number of particles at the departure site (zero-range interactions)
- Boundary conditions: reservoirs, periodic conditions...

# The periodic case

- Periodic spatial geometry: $u \in \mathbb{T}^d$
- At the discrete level: $x, y, z \in \Lambda_N = \mathbb{T}_N^d = \{1, \ldots, N\}^d$
- Microscopic configuration phase space: $\eta \in X_N = \mathbb{N}^{\mathbb{T}_N^d}$
- Stochastic "trajectories" for $\eta$ which preserve the total normalized mass $\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)$
- Law $f_t^N \in \mathcal{P}(X_N)$ depending on time

# The zero-range process: master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f_t^N, \varphi^N\rangle = \langle f_t^N, G^N\varphi^N\rangle$$

$$G^N\varphi^N(\eta) := N^2 \sum_{x,y\in\mathbb{T}_N^d} g(\eta(x)) \left[\varphi^N\left(\eta^{x,y}\right) - \varphi^N(\eta)\right]$$

- $\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 \text{ for } z = x \\ \eta(y) + 1 \text{ for } z = y \\ \eta(z) \text{ otherwise} \end{cases}$
- Rate function $g$ which satisfies
- $g(0) = 0$ and $g(k) > 0$ for all $k > 0$
- Factor $N^2$: diffusive (parabolic) scaling

# The limit evolution system: nonlinear diffusion

- Hydrodynamic limit: $N \to +\infty$
- Empirical measure $\mu_\eta^N = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N} \in \mathcal{M}_+(\mathbb{T}^d)$
- Does $\mu_\eta^N$, where $\eta$ has law $f_t^N$, approaches a deterministic profile $f_t$ with a macroscopic evolution equation?

$$\mathbb{P}_{f_t^N}\left(\left\{\left|\langle \mu_\eta^N, \varphi\rangle - \langle f_t, \varphi\rangle\right| > \varepsilon\right\}\right) \xrightarrow[N\to\infty]{} 0$$

- Nonlinear diffusion equation for $f_t$:

$$\partial_t f = \Delta(\sigma(f)), \quad f_t(u) \geq 0, \quad u \in \mathbb{T}^d$$

- Remark: Defining solutions requires more regularity than measures $\mathcal{M}_+(\mathbb{T}^d)$, e.g. $L^\infty(\mathbb{T}^d)$

# Invariant measure structure

- Particle system:

  Invariant product measure $\nu_\phi^N(\eta) = \otimes \nu_\phi(\eta_i)$ with

  $$\nu_\phi^N(\{\eta(x) = k\}) = \frac{1}{Z(\phi)} \frac{\phi^k}{g(k)!} \quad \text{with} \quad Z(\phi) = \sum_{k \geq 0} \frac{\phi^k}{g(k)!}$$

- Notation: $g(k)! = g(k)g(k-1)\cdots g(1)$ and $g(0)! = 1$
- Limit equation:

  Stationary solution $f_\infty =$ constant

- Nonlinearity functional $\sigma$ prescribed by

$$\mathbb{E}_{\nu^N_{\sigma(\rho)}}\left[\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\eta(x)\right] = \frac{1}{Z(\sigma(\rho))}\sum_{k\geq 0}\frac{k\sigma(\rho)^k}{g(k)!} = \rho$$

- $\sigma(0) = 0$ and increasing

# The relative entropy structure with Dirichlet conditions

**Theorem (Bodineau-Lebowitz-CM-Villani)**

Consider $\Omega$ bounded regular, $\sigma \in C^2$ increasing and

$$\partial_t f = \Delta \sigma(f), \quad u \in \Omega, \; f_{|\partial\Omega}(u) = f_b(u).$$

Then for any $\Phi \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ convex with $\Phi(1) = \Phi'(1) = 0$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H_\Phi\left(f_t | f_\infty\right) &:= \int_\Omega \left( \int_{f_\infty(x)}^{f_t(u)} \Phi'\left( \frac{\sigma(v)}{\sigma(f_\infty(u))} \right) \mathrm{d}v \right) \mathrm{d}u \\ &= - \int_\Omega \Phi''(h) \left| \nabla h \right|^2 \sigma(f_\infty(u)) \, \mathrm{d}u \leq 0 \end{aligned}$$

with $h := \dfrac{\sigma(f_t(u))}{\sigma(f_\infty(u))}$.

# Idea of the strategy

- Example: In the case corresponding to the Boltzmann relative entropy $\Phi(z) = z \ln z - z + 1$ one finds

$$H\left(f_t | f_\infty\right) := \int_\Omega \left( \int_{f_\infty(x)}^{f_t(u)} \ln\left( \frac{\sigma(v)}{\sigma(f_\infty(u))} \right) \mathrm{d}v \right) \mathrm{d}u$$

  different from the usual relative entropy due to $\sigma$
- Heuristic: computation of the large deviation function for the zero-range process with reservoirs in order to guess the relative entropy structure for the limit equation
- Key point: the invariant measure is still a product measure with a varying density, leading to explicit calculations
- Proof: Use that $h$ satisfies $h = 1$ at the boundary and use of $\Phi(1) = \Phi'(1) = 0$ to kill the boundary terms

# The hydrodynamic limit: framework

Assumptions:

- $g(0) = 0$, $g(k) > 0$ for all $k > 0$
- $g(k+1) - g(k) \leq g^* < \infty$ for all $k > 0$
- $g(k) - g(j) \geq \delta$ for some $k_0 > 0$ and $\delta > 0$, and any $k \geq j + k_0$

Evolution systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f_t^N, \varphi^N \rangle = \langle f_t^N, G^N \varphi^N \rangle \quad \text{on} \quad \mathcal{P}(X_N)$$

$$\frac{\partial f_t}{\partial t} = \Delta \sigma(f_t) \quad \text{on} \quad \mathcal{X} \subset \mathcal{M}_+(\mathbb{T}^d)$$

Question: $\mu_\eta^N \sim f_t$ as $N \to \infty$ where $\eta$ has law $f_t^N$?

# The hydrodynamic limit: some existing results (I)

[Guo-Papanicolaou-Varadhan1988]

Assume on the initial data $f_0 \in L^\infty(\mathbb{T}^d)$

$$H\left(f_0^N \middle| \nu^N_{\sigma(\rho)}\right) = \int_\Omega \ln\left(\frac{\mathrm{d}f_0^N}{\mathrm{d}\nu^N_{\sigma(\rho)}}\right) \mathrm{d}f_0^N \lesssim N^d$$

$$\left\langle f_0^N, \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta(x)^2 \right\rangle \lesssim 1$$

Then there is propagation in time of the deterministic limit:

$$\mathbb{P}_{f_0^N}\left(\left\{\left|\langle \mu^N_\eta, \varphi\rangle - \langle f_0, \varphi\rangle\right| > \varepsilon\right\}\right) \xrightarrow[N\to\infty]{} 0$$

implies for later times $t > 0$

$$\mathbb{P}_{f_t^N}\left(\left\{\left|\langle \mu^N_\eta, \varphi\rangle - \langle f_t, \varphi\rangle\right| > \varepsilon\right\}\right) \xrightarrow[N\to\infty]{} 0$$

# The hydrodynamic limit: some existing results (II)

[Yau1991]

If futhermore $f \in C^3(\mathbb{T}^d)$ (smooth solution at the limit) and

$$\frac{1}{N^d} H\left(f_0^N | \nu^N_{f_0(\cdot)}\right) \xrightarrow[N\to\infty]{} 0$$

for the local equilibrium product measure

$\nu^N_{f(\cdot)} := \prod_{x \in \mathbb{T}^d_N} \frac{\sigma(f(x/N))^{\eta(x)}}{Z(\sigma(f(x/N)))g(\eta(x))!}$, then at later times $t > 0$

$$\frac{1}{N^d} H\left(f_t^N | \nu^N_{f_t(\cdot)}\right) \xrightarrow[N\to\infty]{} 0$$

[Grunewald-Otto-Villani-Westdinckenberg2009] on a related model (Gunzburg-Landau with Kawasaki dynamics): another coarse-graining method with (almost...) explicit estimates

# Questions and motivations

- Quantitative rate of convergence? not in GLV, almost in GPVW, Yau's relative entropy methods can be made explicit, but rate $O(e^{\lambda t})$ with $\lambda > 0$ large, and for smooth solutions
- However both the many-particle and limit systems are dissipative, hence ergodicity and relaxation should win over stochastic fluctuations at the level of the laws

$$
\begin{tikzcd}
f_t^N \in \mathcal{P}(X_N) \arrow[r, "N\to\infty"] \arrow[d, "t\to\infty"'] & f_t \in L^\infty(\mathbb{T}^d) \arrow[d, "t\to\infty"] \\
\nu^N_{\sigma(\rho)} \in \mathcal{P}(X_N) \arrow[r, "N\to\infty"] & f_\infty
\end{tikzcd}
$$

- Goals:
  - Fully quantitative rate
  - Does not use regularity of the limit solution
  - Uniform in time
  - In entropic form...

# The main result

**Theorem (Marahrens-CM, 2012)**

$f_t \in L^\infty$ the solution to the nonlinear diffusion equation and

$$\frac{1}{N^d} H\left(f_0^N | \nu^N_{\sigma(\rho)}\right) \lesssim 1 \quad \text{and} \quad \left\langle f_0^N, \left(\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)^4\right)\right\rangle \lesssim 1$$

Consider $F \in C_b^2(\mathbb{R})$ and $\varphi \in C^3(\mathbb{T}^d)$ then

$$\sup_{t \geq 0} \left| \left\langle f_t^N, F\left(\langle \mu_\eta^N, \varphi \rangle\right) \right\rangle - F\left(\langle f_t, \varphi \rangle\right) \right| \leq r_{HL}(\varepsilon, N) + C_{F,\varphi} \left\langle f_0^N, \left\| \mu_\eta^{N,\varepsilon} - f_0 \right\|_{H^{-1}} \right\rangle$$

for some quantitative polynomial rate $r_{HL}$

Consequence: rate of convergence $O(N^{-\alpha})$ uniform in time

# An inspirative previous work on Kac's model

- Abstract ideas developped in another work w/. Mischler on Kac's program in kinetic theory:
- Many-particle collision jump process on the velocities ("Kac's walk") in $\mathbb{S}^{N-1}(\sqrt{N})$) with jump rates depending on the velocities (hard spheres)

$$\partial_t \langle f_t^N, \varphi \rangle = \langle f_t^N, G^N \varphi \rangle$$

$$(G^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^{N} |v_i - v_j| \int_{\mathbb{S}^{d-1}} b(\cos\theta_{ij}) \, \left[\varphi_{ij}^* - \varphi\right] \, d\sigma$$

- Mean-field limit towards nonlinear (spatially homogeneous) Boltzmann equation when propagation of chaos

$$\partial_t f = Q(f)$$

$$Q(f)(v) := \int_{v_* \in \mathbb{R}^d} \int_{\sigma \in \mathbb{S}^{d-1}} \Big( f(v_*')f(v') - f(v)f(v_*) \Big) \, |v - v_*|$$

# Kac's program in kinetic theory

- Estimate on the spectral gap in $L^2(\mathbb{S}^{N-1}(\sqrt{N}))$: [Carlen-Carvalho-Loss2003] (cf. also [Janvresse], [Maslen])
- However not sufficient for answering main motivation of Kac: to connect the asymptotic behavior and entropies of the many-particle and limit systems in the chaotic limit

**Theorem (Mischler-CM, 2011)**

Uniform in time chaos (for any number of marginals $1 \leq \ell \leq N$)

$$\sup_{t\geq 0} \frac{W_1\left(\Pi_\ell f_t^N, f_t^{\otimes \ell}\right)}{\ell} \leq \alpha(N) \to 0$$

Entropic chaos: $\dfrac{H\left(f_t^N\right)}{N} \xrightarrow{N\to\infty} H(f)$

Unif. in $N$ relaxation times: $\displaystyle\sup_{N\geq 1} \frac{W_1\left(f_t^N, \gamma^N\right)}{N} \leq \beta(t) \to 0$

# Abstract strategy

- View the many-particle system as a perturbation of the limit nonlinear PDE
- Consistency-stability estimates on semigroups
- Requires an appropriate functional framework for comparing generators (consistency) and establishing stability estimates
- The latters correspond to statistical stability estimates on the flow of the limit nonlinear PDE

$$
\begin{tikzcd}
X_N \arrow[rr, "\text{Kolmogorov}"] \arrow[d, "\mu_\eta^N"'] & & \mathcal{P}(X_N) \arrow[rr, leftrightarrow, "\text{duality}"] \arrow[d, "\pi_N^*"'] & & C_b(X_N) \\
\mathcal{X} \subset \mathcal{M}_+(\mathbb{T}^d) \arrow[rr, "\text{"Liouville"}"] & & P(\mathcal{X}) \arrow[rr, leftrightarrow, "\text{duality}"] & & \boxed{C_b(\mathcal{X})} \arrow[u, "\pi_N"']
\end{tikzcd}
$$

with $(\pi_N \Phi)(\eta) := \Phi(\mu_\eta^N) = (\mu_\cdot^N)_* \Phi$

# Evolution $N$-particle semigroups

- Stochastic process $(\eta_t^N)$ on $X_N$
- Corresponding linear semigroup $\mathcal{F}_t^N$ on $P(X_N)$:

$$\partial_t f^N = A^N f^N, \qquad f^N \in P(X_N),$$

  Forward Kolmogorov equation or "Master equation"
- *Dual* linear semigroup $T_t^N$ of $\mathcal{F}_t^N$:

$$\forall\, f^N \in P(X_N),\ \varphi^N \in C_b(X_N), \qquad \left\langle f^N, T_t^N(\varphi^N) \right\rangle := \left\langle \mathcal{F}_t^N(f^N), \varphi^N \right\rangle$$

- Semigroup of the observables:

$$\partial_t \varphi^N = G^N(\varphi^N), \quad \varphi^N \in C_b(X_N)$$

# Evolution limit semigroups

- Nonlinear semigroup $\mathcal{F}_t^\infty$ on $\mathcal{X}$ solution to

$$\partial_t f_t = Q(f_t) = \Delta\sigma(f), \quad f_{|t=0} = f_0$$

- Pullback linear semigroup $T_t^\infty$ on $C_b(\mathcal{X})$:

$$\forall\, f \in \mathcal{X},\ \Phi \in C_b(\mathcal{X}), \quad T_t^\infty[\Phi](f) := \Phi\left(\mathcal{F}_t^\infty(f)\right)$$

  solution to the *linear* evolution equation on $C_b(P(E))$:

$$\partial_t \Phi = G^\infty(\Phi) \quad \text{with generator} \quad G^\infty$$

- Comparison of semigroups $T_t^N$ and $T_t^\infty$
- Additional difficulty: $\mu_\eta^N \notin \mathcal{X} = L^\infty(\mathbb{T}^d)$, hence we introduce the mollified empirical measure $\mu_\eta^{N,\varepsilon} = \chi_\varepsilon * \mu_\eta^N$, $\varepsilon > 0$ and the corresponding projection $\pi_{N,\varepsilon}$

# Interpretation of the pullback semigroup (I)

- Given a nonlinear ODE $\dot{Y} = F(Y)$ on $\mathbb{R}^d$, one can define (at least formally) the linear Liouville transport PDE

$$\partial_t \rho + \nabla_v \cdot (F\,\rho) = 0,$$

  where $\rho_t(v) = Y_t^*(\rho_0) = \rho_0 \circ Y_{-t}$ (characteristics)
- Dual viewpoint: for $\phi_0$ function defined on $\mathbb{R}^d$, evolution $\phi_t(v) = \phi_0(Y_t(v)) = (Y_t)_* \phi_0 = \phi_0 \circ Y_t$ solution to the linear PDE

$$\partial_t \phi - F \cdot \nabla_v \phi = 0,$$

# Interpretation of the pullback semigroup (II)

- Go "one level above" and replace $\mathbb{R}^d$ by $\mathcal{X}$:
  The infinite dimensional "ODE" $f' = Q(f)$ on $\mathcal{X}$ yields first the abstract transport equation

  $$\partial_t \pi + \nabla \cdot (Q(f)\,\pi) = 0, \qquad \mathrm{d}\pi(t,\cdot) \in P(\mathcal{X})$$

  and second the abstract dual equation

  $$\partial_t \Phi - Q(f) \cdot \nabla \Phi = 0, \qquad \Phi(t,\cdot) \in C_b(\mathcal{X}).$$

- Provide intuition and formal formula for the generator "$(G^\infty \Phi)(f) = Q(f) \cdot \nabla \Phi(f)$": but requires to define correctly the objects...

# Statistical stability analysis

- Well-posedness for the limit equation in $L^\infty(\mathbb{T}^d)$
- Propagation of $H^k$ regularity
- Lipschitz stability: $\|\mathcal{F}_t^\infty f_2 - \mathcal{F}_t^\infty f_1\|_{H^{-1}} \leq \|f_2 - f_1\|_{H^{-1}}$
- Higher-order stability:

$$\|\mathcal{F}_t^\infty f_2 - \mathcal{F}_t^\infty f_1 - D\mathcal{F}_t^\infty(f_1)(f_2 - f_1)\|_{H^{-1}} \leq C(f)\|f_2 - f_1\|_{H^{-1}}^{1+\theta}$$

- $D\mathcal{F}_t^\infty(f) \in \mathcal{L}(\mathcal{X})$: solution $h_t := D\mathcal{F}_t^\infty(f_1)(f_2 - f_1) \in \mathcal{X}$ to

$$\partial_t h_t = \Delta\left(\sigma'(f_1(t))h\right) \quad \text{such that} \quad h_0 = (f_2 - f_1)_{|t=0}$$

- $T_t^\infty \psi : H \to \mathbb{R}$ is $C^{1+\theta}(\mathcal{X})$ for the $H^{-1}$-norm and

$$D\left[T_t^\infty \Phi\right](f) \cdot h = D\Phi(\mathcal{F}_t^\infty(f)) \cdot [D\mathcal{F}_t^\infty[f](h)], \quad \Phi \in C^1(\mathcal{X})$$

- $G^\infty[\Phi](f) = F'\left(\langle f, \varphi\rangle\right)\langle \sigma(f), \Delta\varphi\rangle$ for $\Phi(f) = F(\langle f, \varphi\rangle)$

# Consistency and convergence (I)

$$\begin{aligned}
\left|\left\langle f_t^N, F\left(\langle\mu_\eta^N,\varphi\rangle_{L^2}\right)\right\rangle - F\left(\langle f_t,\varphi\rangle_{L^2}\right)\right| &= \left|\left\langle \mathcal{F}_t^N f_0^N, \Phi\left(\mu_\eta^N\right) - \Phi\left(\mathcal{F}_t^\infty f_0\right)\right\rangle\right| \\
&= \left|\left\langle f_0^N, T_t^N[\Phi\left(\mu_\eta^N\right)] - [T_t^\infty\Phi]\left(f_0\right)\right\rangle\right| \\
&\leq \left|\left\langle f_0^N, T_t^N[\Phi\left(\mu_\eta^{N,\epsilon}\right)] - [T_t^\infty\Phi]\left(\mu_\eta^{N,\epsilon}\right)\right\rangle\right| \\
&\quad + \left|\left\langle f_0^N, [T_t^\infty\Phi]\left(\mu_\eta^{N,\epsilon}\right) - [T_t^\infty\Phi]\left(f_0\right)\right\rangle\right| \\
&\quad + \left|\left\langle f_0^N, T_t^N[\Phi\left(\mu_\eta^N\right)] - T_t^N[\Phi\left(\mu_\eta^{N,\epsilon}\right)]\right\rangle\right| =: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3
\end{aligned}$$

Error terms:

- $\mathcal{T}_2$: initial data approximation error propagated by $T_t^\infty$ (Lipschitz stability)
- $\mathcal{T}_3$: $\varepsilon$-error due to the mollification at the level of the particle system
- $\mathcal{T}_1$: core of the proof - comparision of the two semigroups
- Linear growth in time of error + interpolation with the exponential relaxation of the limit equation: uniform in time estimates

# Consistency and convergence (II): the $\mathcal{T}_1$ estimate

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T_s^N \pi_{N,\varepsilon} T_{t-s}^\infty \Phi(\eta)\right) = T_s^N G^N \pi_{N,\varepsilon} T_{t-s}^\infty \Phi(\eta) - T_s^N \pi_{N,\varepsilon} G^\infty T_{t-s}^\infty \Phi(\eta)$$

hence

$$\mathcal{T}_1 \leq \left| \int_0^t \left\langle f_{t-s}^N, \left(G^N \pi_{N,\varepsilon} - \pi_{N,\varepsilon} G^\infty\right) T_s^\infty \psi \right\rangle \mathrm{d}s \right|$$

- First-order Taylor expansion of $T_t^\infty$:
  $T_t^\infty \psi(\mu_{\eta^{x,y}}^{N,\epsilon}) - T_t^\infty \psi(\mu_\eta^{N,\epsilon}) - DT_t^\infty \psi(\mu_\eta^{N,\epsilon})(\mu_{\eta^{x,y}}^{N,\epsilon} - \mu_\eta^{N,\epsilon})$
  (remainder term controlled by high-order statistical stability)
- Replacing the discrete Laplacian by the continuous one (propagation of regularity on $D\mathcal{F}_t^\infty(\mu_\eta^{N,\varepsilon})^*$)
- Quantitative version of replacement lemma [Kipnis-Landim1999] with the help of LSI

$$\frac{1}{T}\int_0^T \int_{\mathbb{T}^d} \left\langle f_t^N, \left|(g\circ\eta)^{(\varepsilon)}(uN) - \sigma(\eta^{(\varepsilon)}(uN))\right|^2 \right\rangle \mathrm{d}u\,\mathrm{d}t \leq r_{\mathrm{RL}}(\epsilon, N)$$