Solving a four dimensional gauge theory using integrability

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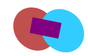
Based on: arXiv:1012.3982 (review on Integrability in gauge theory)
arXiv:1203.1913 (paper on the quark anti-quark potential)

Large N gauge theories and strings

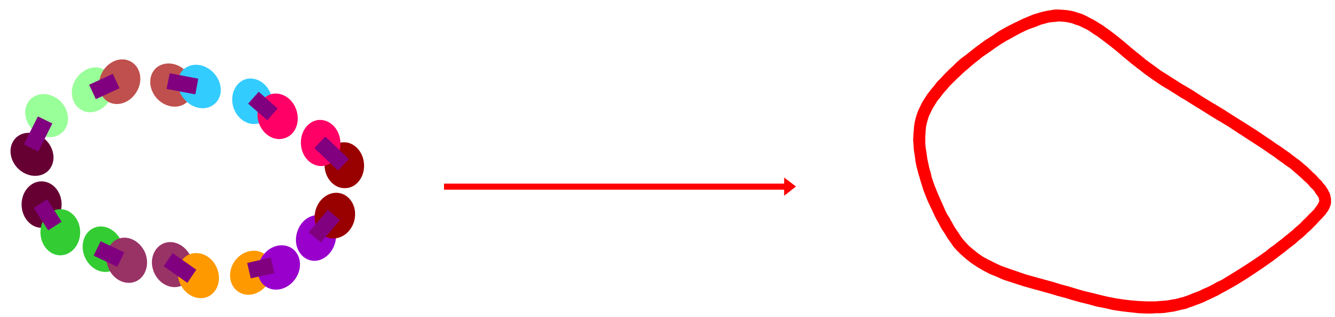
Gluon: colors



Take N colors instead of 3, SU(N)

t' Hooft '74

Large N limit



g^2N = effective interaction strength.
Keep this fixed when N → infinity

String coupling ~ 1/N

Closed strings → glueballs

Gauge theory = String theory

Most supersymmetric Quantum Chromodynamics	=	String theory on $AdS_5 \times S^5$

J.M. '97

N colors

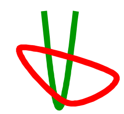
N = magnetic flux through S^5

Radius of curvature

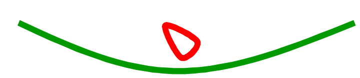
$$R_{S^5} = R_{AdS_5} = \left(g_{YM}^2 \ N \right)^{1/4} l_s$$

Duality:

$g^2 N$ is small $\rightarrow$ gauge theory is easy – string side is hard



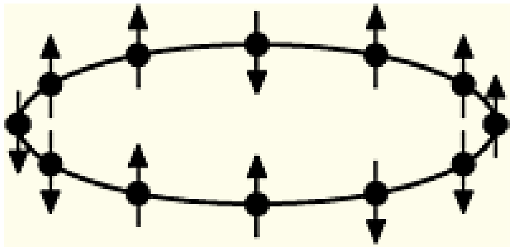
$g^2 N$ is large $\rightarrow$ gravity is easy – gauge theory is hard



Strings made with gluons become fundamental strings.

Strings from gluons

- Large N → Unbreakable chains
- Spin chains where ``spins'' = states of gluons, includes momentum and spin…



- When the coupling is weak, gluons move independently = weakly coupled spins.
- As the coupling increases the ``spins'' start interacting and they start moving collectively.
- This collective motion and behavior is more easily understood in terms of strings moving in a five dimensional space. (10 dimensions → superstring theory)

Integrability

- Integrability= large set of symmetries that makes it possible to find exact solutions.

- The usual Heisemberg spin chain is integrable and can be solved by Bethe ansatz.

Bethe 1931

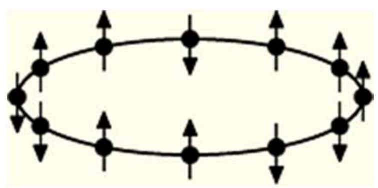
- Many 1 +1 dimensional relativistic models are integrable and can be solved by the bootstrap method.

Zamolodchikov
Zamolodchikov 1978

Integrable chains from gluons

- Special gauge theories give integrable chains

- Maximally supersymmetric, four dimensional gauge theory. Conformal gauge theory, coupling λ.
- Gauge field + 6 scalar fields + 4 fermions, all in the adjoint of the gauge group.



How to solve a spin chain

1- Choose a simple vacuum (i.e. all spin down)

2- Find the simplest excitations of this vacuum (magnons).

3- Find their two particle S-matrices.

4- Write the Bethe equations. (Integrability $\rightarrow$ factorized scattering).

How to solve a 1+1 relativistic theory

1- Choose a simple vacuum.

2- Find the simplest excitations of this vacuum.

3- Find their two particle asymptotic S-matrices.

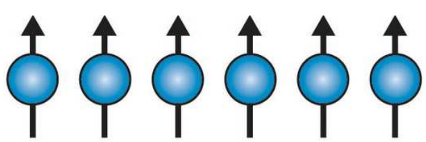
4- Use TBA trick to find the spectrum on a circle.

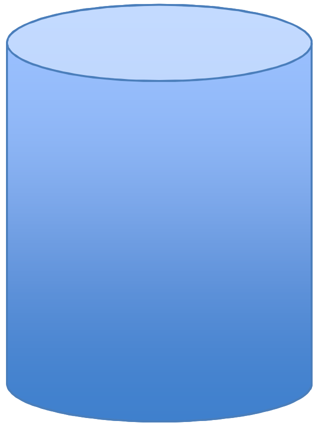
Zamolodchikov

Simple vacuum for the chain

- Select a complex scalar field Z .

- Consider a state of the theory on S^3 x R with minimum energy for Z.



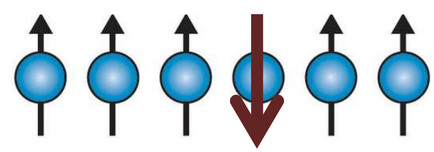


- State created by the operator $\mathrm{Tr}[Z^L]$

Simplest excitations

Make the Z quantum move slightly on the sphere.

Or change one Z quantum by another scalar or fermion.



$$Tr[\partial_\mu Z Z^L] \ , \qquad Tr[\psi Z^L]$$

These excitations can move along the chain

$$\overrightarrow{p}$$

$$Tr[ZZ\cdots ZZ\partial_\mu ZZZZ....]$$

Dispersion relation:

$$\epsilon(p) = \sqrt{1+\lambda\sin^2\frac{p}{2}}$$

Fixed by supersymmetry

All other excitations?

- Just add the simplest many times!!

The 2 particle S-matrix

- Matrix structure = fixed by symmetries.

Beisert 2005

- Overall phase = fixed by solving a crossing equation.

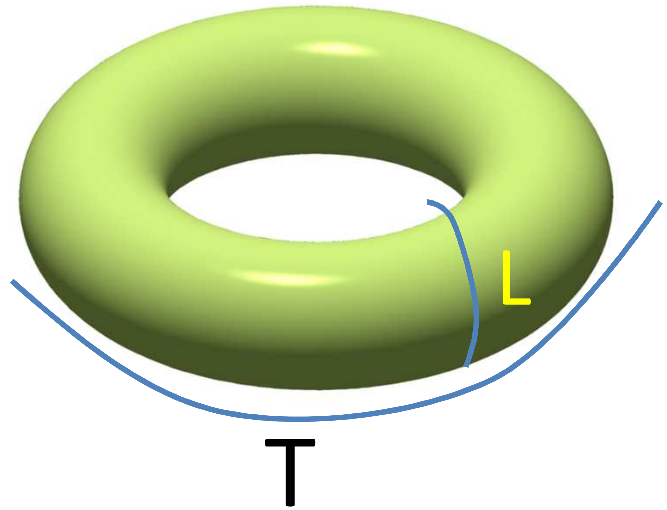
Beisert Janik
Hernandez Lopez
Eden Staudacher
2006.

- Very long chain → Bethe equations.

Beisert Staudacher

Short strings

- Consider a closed string propagating over a long time T



TBA trick:
Zamolodchikov

- View it as a very long string of length T, propagating over eucildean time L = Thermodynamic configuration.
- In our case the physical theory and the mirror theory are different.

$$e^{-TE_0} \sim Tr[e^{-TH_L}] = Z = Tr[e^{-LH_T}]$$

Yang-Yang

- Solve it by the Thermodynamic Bethe Ansatz = sum over all the solutions of the mirror Bethe equations over a long circle T weighed with the Boltzman factor.

$$\log Y_A = -L\epsilon_A(p) + K_{AB} * \log(1+Y_B)$$

$$\mathcal{E} = -\frac{1}{2\pi}\int dp_A \log(1+Y_A)$$

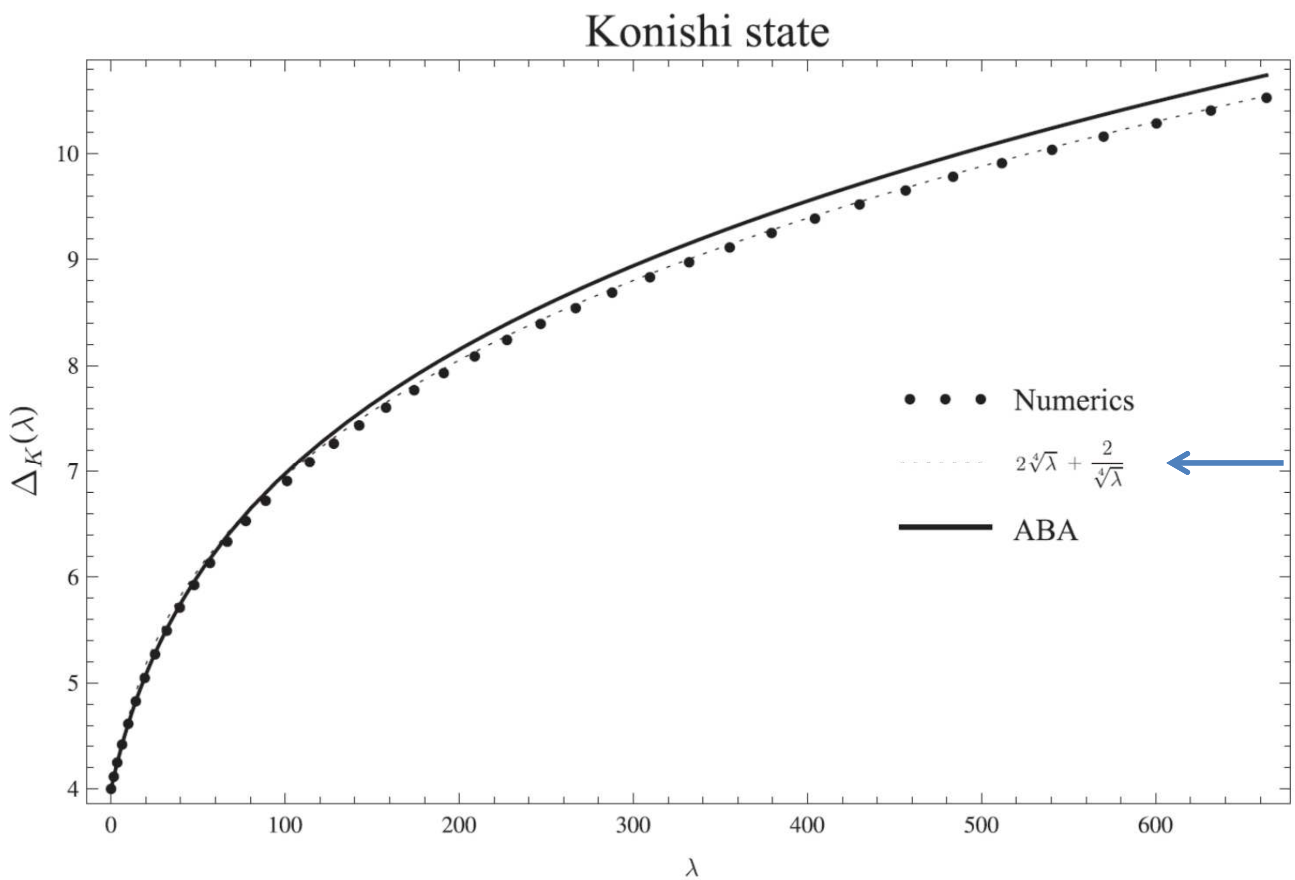
Arutyunov Frolov
Gromov Kazakov Vieira
Bombardelli, Fioravanti, Tateo 2009

Y_A= (densities of particles)/(densities of holes) as a function of momentum

A runs over all particles and bound states of particles in the fully diagonalized nested bethe ansatz

Conclusions

- We can find the energy of any state of the chain of gluons for any coupling.
- Exact solutions for critical exponents of a four dimensional CFT.
- Results interpolate between perturbative gauge theory and strings on $AdS_5 \times S^5$.

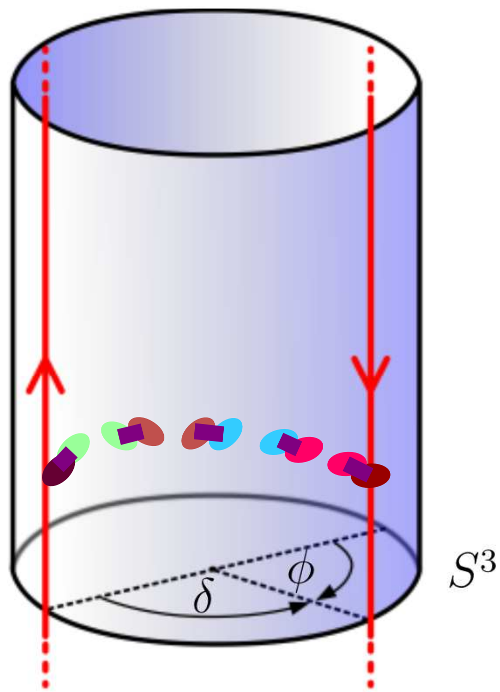


String theory result at
Large λ . Computed
from ten dimensional string
Theory in $AdS_5 \times S^5$.

Perturbation theory
Feynman diagrams
Small coupling

Gromov, Kazakov Vieira

Quark – Anti-quark potential

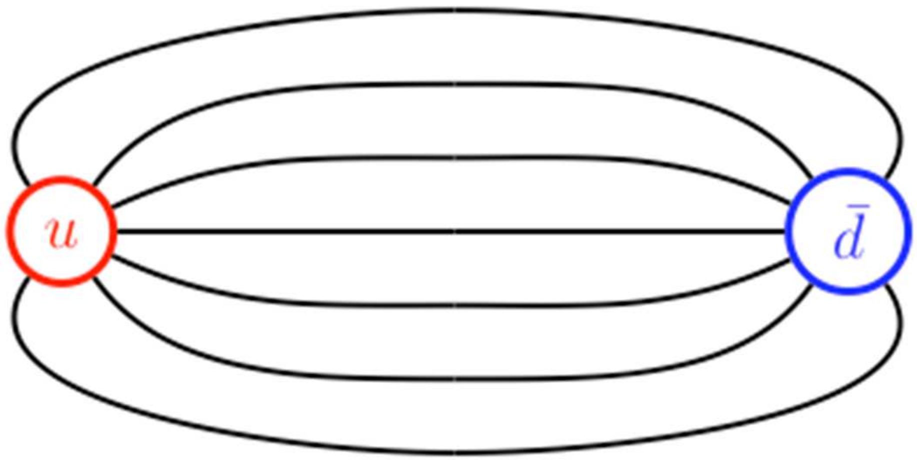


Correa Sever
J. M. 2012
Drukker

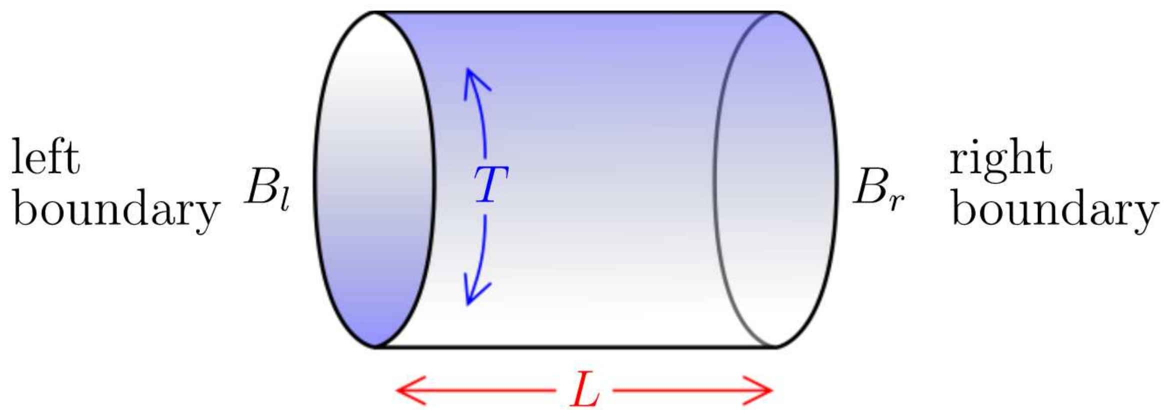
Simple: Spin chain with boundaries.

Motivations

- Flux lines going between quark and anti-quark → become a string. Understand this string and its excitations.



Boundary TBA trick: LeClair, Mussardo, Saleur, Skorik



Closed string between two boundary states.

$$Z^{\text{open}}_{B_l,B_r} = \mathrm{Tr}_{\text{open}}[e^{-TH^{\text{open}}_{B_l,B_r}}] = \langle B_l | e^{-LH_{\text{closed}}} | B_r \rangle \,,$$

Final Answer

Write integral equations for a set of functions $Y_A(u)$:

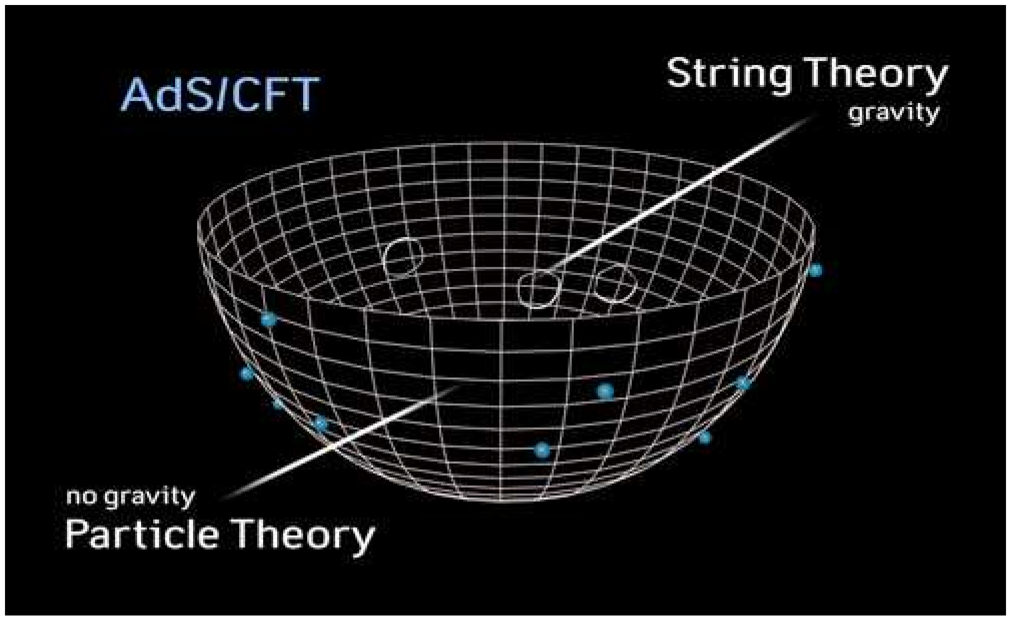
$$\log Y_A = \log r_{bdy} + (\text{angle}) + K_{AB} * \log(1 + Y_B)$$

Compute the potential as:

$$V = -\frac{1}{2\pi} \sum_A \int_0^\infty dq_A \log(1 + Y_A) \,.$$

Conclusions

- There are very special and interesting “spin” chains that arise in planar gauge theories.
- They beautifully show the transition between free gluons and strings in AdS .
- One can calculate all critical exponents, or anomalous dimensions of a interacting four dimensional theory.
- Also a similar example in d=3. It involves a Chern Simons matter theory = Massless non-abelian anyons.


AdS/CFT
String Theory
gravity
no gravity
Particle Theory

Final Equations

$$\log \frac{Y_{1,1}}{\mathbf{Y}_{1,1}} = K_{m-1} \star \log \frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}} \frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + R^{(01)}_{1a} \star \log(1+Y_{a,0}) \qquad (62)$$

$$\log \frac{\overline{Y}_{2,2}}{\overline{\mathbf{Y}}_{2,2}} = K_{m-1} \star \log \frac{1+\overline{Y}_{1,m}}{1+\overline{\mathbf{Y}}_{1,m}} \frac{1+\mathbf{Y}_{m,1}}{1+Y_{m,1}} + B^{(01)}_{1a} \star \log(1+Y_{a,0}) \qquad (63)$$

$$\log \frac{\overline{Y}_{1,s}}{\overline{\mathbf{Y}}_{1,s}} = -K_{s-1,t-1} \star \log \frac{1+\overline{Y}_{1,t}}{1+\overline{\mathbf{Y}}_{1,t}} - K_{s-1} \,\hat{\star} \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}} \qquad (64)$$

$$\log \frac{Y_{a,1}}{\mathbf{Y}_{a,1}} = -K_{a-1,b-1} \star \log \frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} - K_{a-1} \,\hat{\star} \log \frac{1+Y_{1,1}}{1+\overline{Y}_{2,2}} + \left[R^{(01)}_{ab} + B^{(01)}_{a-2,b}\right] \star \log(1+Y_{b,0}) \qquad (65)$$

$$\log \frac{Y_{a,0}}{\mathbf{Y}_{a,0}} = \left[2S_{ab} - R^{(11)}_{ab} + B^{(11)}_{ab}\right] \star \log(1+Y_{b,0}) + 2\left[R^{(10)}_{ab} + B^{(10)}_{a,b-2}\right] \star_{\mathrm{sym}} \log \frac{1+Y_{b,1}}{1+\mathbf{Y}_{b,1}} + 2R^{(10)}_{a1} \,\hat{\star}_{\mathrm{sym}} \log \frac{1+Y_{1,1}}{1+\mathbf{Y}_{1,1}} - 2B^{(10)}_{a1} \,\hat{\star}_{\mathrm{sym}} \log \frac{1+\overline{Y}_{2,2}}{1+\overline{\mathbf{Y}}_{2,2}} \qquad (66)$$

$$\mathbf{Y}_{1,1} = -\frac{\cos\theta}{\cos\phi}\,, \qquad \overline{\mathbf{Y}}_{1,s} = \frac{\sin^2\theta}{\sin[(s+1)\theta]\sin[(s-1)\theta]}$$

$$\overline{\mathbf{Y}}_{2,2} = -\frac{\cos\theta}{\cos\phi}\,, \qquad \mathbf{Y}_{a,1} = \frac{\sin^2\phi}{\sin[(a+1)\phi]\sin[(a-1)\phi]}$$

$$\mathbf{Y}_{a,0} = 4\frac{e^{i\chi(z^{[+a]})+i\chi(1/z^{[-a]})}}{e^{i\chi(z^{[-a]})+i\chi(1/z^{[+a]})}} \left(\frac{z^{[-a]}}{z^{[+a]}}\right)^{2L+2} (\cos\phi-\cos\theta)^2 \frac{\sin^2 a\,\phi}{\sin^2\phi}\,.$$

Method

How did Coulomb do it ?

$$\longrightarrow \quad V = -\frac{\lambda}{r}$$

We will follow a less indirect route, but still indirect..

Integrability in N=4 SYM

- N=4 is integrable in the planar limit.

Minahan Zarembo
Bena Polchinski Roiban
Beisert, etc…

- Large number of symmetries→ How do we use them ?

- There is a well developed method that puts these symmetries to work and has lead to exact results.

- It essentially amounts to choosing light-cone gauge for the string in AdS, and then solving the worldsheet theory by the bootstrap method.

- First we consider a particular SO(2) in SO(6) and consider states carrying charge L under SO(2) .
  Operator $Z = \phi_5 + i\, \phi_6$

- States with the lowest dimension carrying this charge → Lightcone ground state for the string

Infinite chain or string

- L= ∞
- Ground state: chain of ZZ….ZZZ..ZZZ

p $\longrightarrow$

- Impurities that propagate on the state ZZ…ZWZ…ZZZ
- Symmetries $\widetilde{su}(2|2)^2$

Beisert

- Elementary worldsheet excitations 4 x 4 under this symmetry.

- Fix dispersion relation

- Fix the 2 → 2 S matrix. Matrix structure by symmetries + overall phase by crossing.

Beisert Janik
Hernandez Lopez
Eden Staudacher

- Solves the problem completely for an infinite (or very long) string.

Beisert Staudacher equations

Back to our case

- Where is the large L ?
- Nowhere ? Just introduce it and then remove it !

$$Z^L$$

x

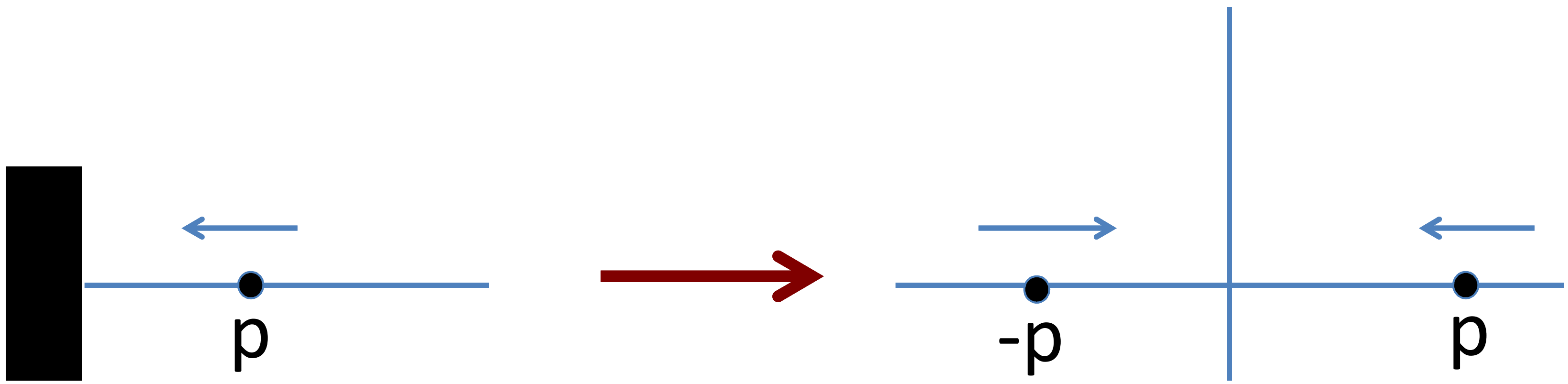
- Operators on a Wilson line: For large L = same chain but with two boundaries.

Drukker Kawamoto

- Need to find the boundary reflection matrix.
- Constrained by the symmetries preserved by the boundary = Wilson line

$$\widetilde{su}(2|2)^2 \cap OSp(4^*|4) = \widetilde{su}(2|2)_D$$

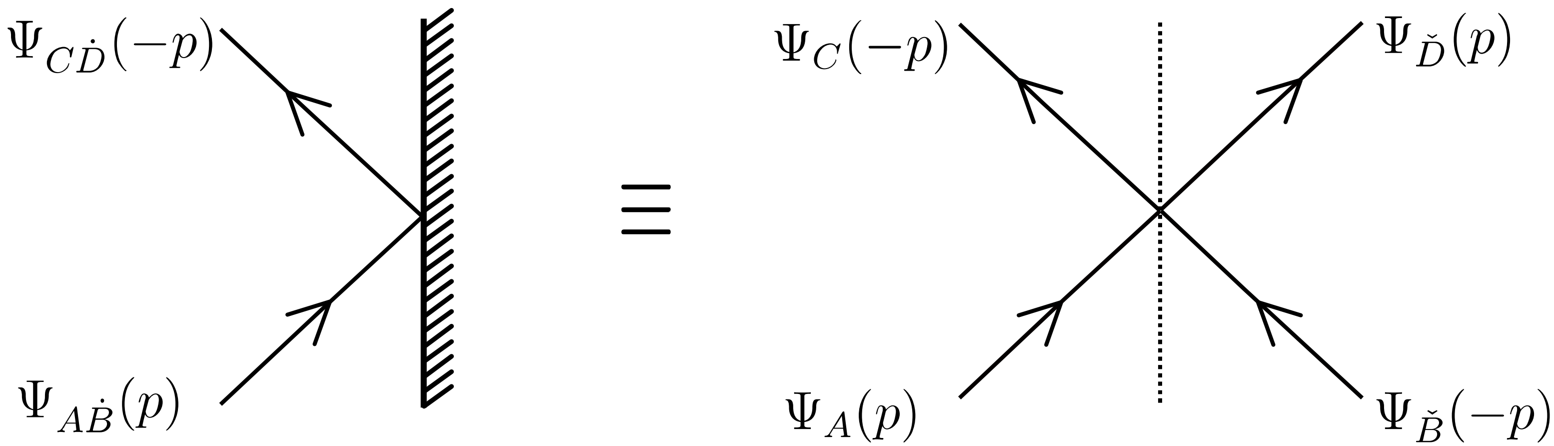
- Bulk magnon = pair of magnons of $\widetilde{su}(2|2)_D$



Half line with

$SU(2|2)^2$ in bulk

full line, single $SU(2|2)_D$



Reflection matrix = bulk matrix for one SU(2|2) factor up to an overall phase

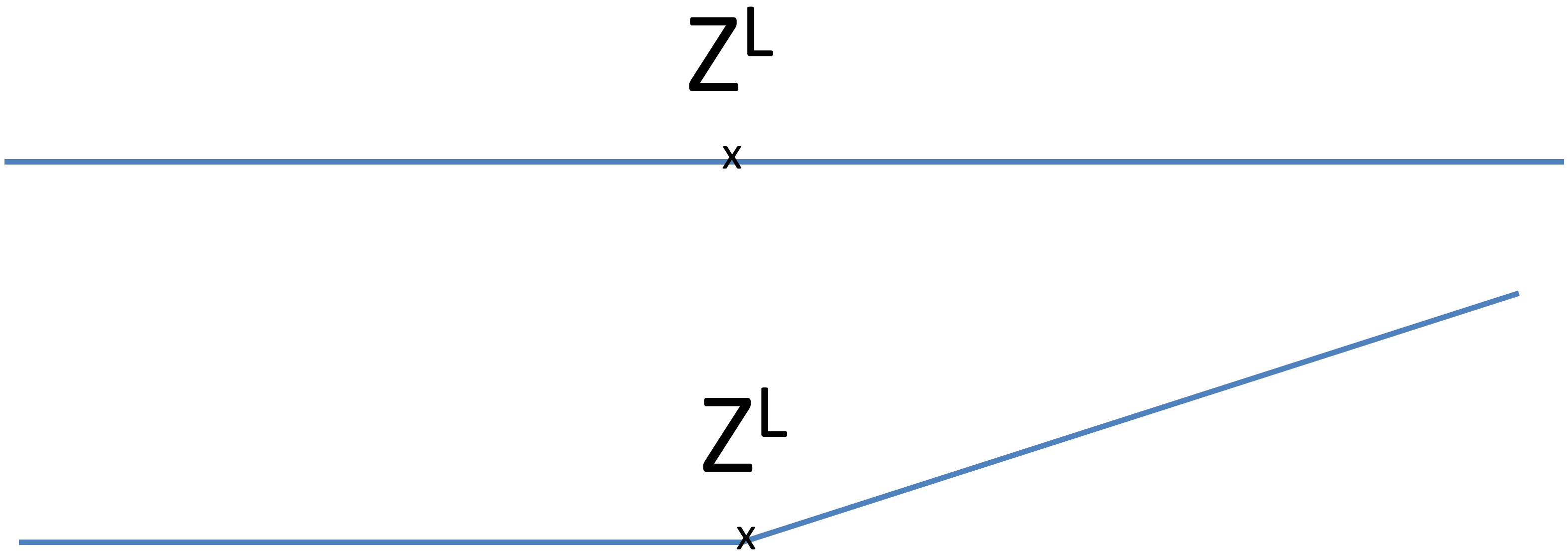
$$R = S(p, -p)\sigma_B$$

σ_B Determined via a crossing equation.

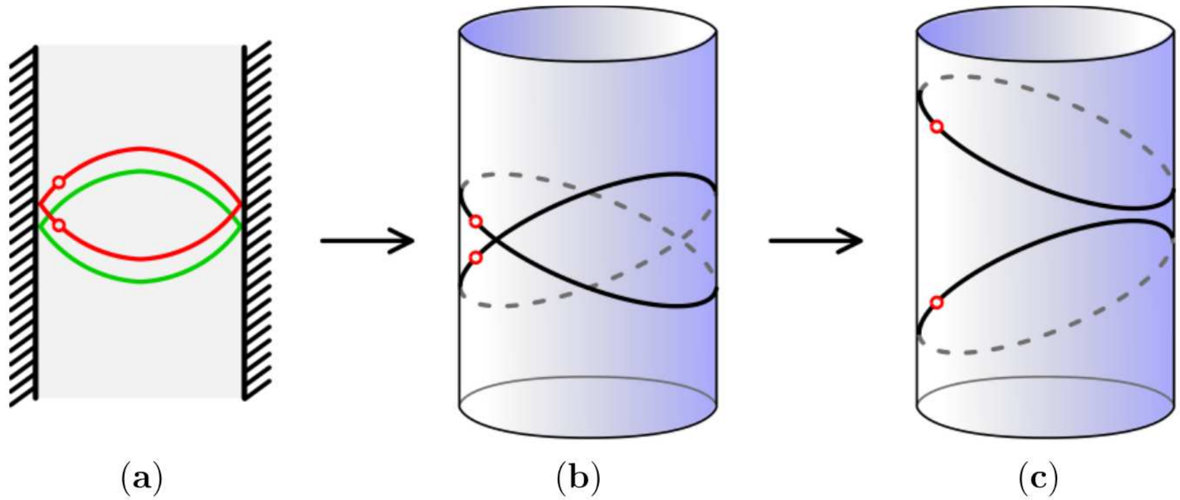
We used the method given by Volin, Vieira

This is the hardest step, and the one that is not controlled by a symmetry.
In involves a certain degree of guesswork. We checked it in various limits.

- We solve the problem for large L: add the two boundaries



Rotate one boundary → Introduce extra phases in the reflection matrix.



Bulk magnons emanate from the boundary with amplitude given by the (analytic continued) reflection matrix.

Using the bulk crossing equation we can untangle the lines and cancel the bulk S matrix factors.

We are left with a pair of lines on the cylinder. This looks like a thermodynamic computation restricted to pairs with p and –p .

Then we essentially get the same as in the bulk for a thermodynamic computation with

$$\beta = 2L$$

And a constraint on the particle densities:

$$Y_{a,s}(u) = Y_{a,-s}(-u)$$