Overcoming critical slowing-down:

Where do we stand
23 years after Swendsen and Wang?

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Critical slowing-down (a quick review)

- Near a critical point, spins have *long-range* correlations
- But ... traditional MC algorithms make only *local* updates

  $\Longrightarrow$ expect autocorrelation time τ to *diverge* at critical point
  (critical slowing-down)
- Heuristic random-walk argument suggests $\tau \sim \xi^{\approx 2}$
  (i.e. **dynamic critical exponent** $z \approx 2$)
- Truth is not far from this (e.g. $z \approx 2.13$ for 2D Ising)
- Rayleigh quotient with trial vector $\mathcal{M}^2$ gives rigorous lower bound
  $z \geq \gamma/\nu$ (i.e. $\tau \geq \text{const} \times \chi$)

- How to speed up the slow (long-distance) modes?
- Need *collective-mode* (nonlocal) updating
- But ... need to offer the system the collective moves *that it wants*

- **Example:** When the long-distance modes are *spin waves*,
  multi-grid Monte Carlo (MGMC) works fairly well
  (CSD is reduced from $z \approx 2$ to $z \approx 0.5$ – 0.7)
- This applies to *continuous-spin models* like nonlinear σ-models
- But what about discrete-spin models like Ising and Potts???

Enter Swendsen and Wang (1987)

- Collective-mode algorithm for q-state Potts ferromagnet

$$H(\sigma) \;=\; \sum_{\langle ij \rangle} J_{ij}\,(1 - \delta_{\sigma_i,\sigma_j})$$

  $(\sigma_i = 1, 2, \ldots, q;\;\; J_{ij} \geq 0)$

- Based on Fortuin–Kasteleyn (1969) representation (reinterpreted)

- Partition function $Z \;=\; \displaystyle\sum_{\{\sigma\}} \exp\left[\sum_{\langle ij \rangle} J_{ij}\,(\delta_{\sigma_i,\sigma_j} - 1)\right]$

$$= \sum_{\{\sigma\}} \prod_{\langle ij \rangle} [(1 - p_{ij}) \;+\; p_{ij}\delta_{\sigma_i,\sigma_j}]$$

  where $p_{ij} = 1 - \exp(-J_{ij})$

- Now insert identity $a + b \;=\; \displaystyle\sum_{n=0}^{1} [a\delta_{n,0} \;+\; b\delta_{n,1}]$ on each bond $\langle ij \rangle$

$$\Longrightarrow \quad Z \;=\; \sum_{\{\sigma\}} \sum_{\{n\}} \prod_{\langle ij \rangle} [(1 - p_{ij})\,\delta_{n_{ij},0} \;+\; p_{ij}\delta_{n_{ij},1}\delta_{\sigma_i,\sigma_j}]$$

  with **auxiliary variables** $n_{ij} = 0, 1$

  (**joint model** of Potts spins σ and bond occupation variables n)

- Sum over σ at fixed n:

$$Z \;=\; \sum_{\{n\}} \left(\prod_{\langle ij \rangle:\; n_{ij}=1} p_{ij} \right) \left(\prod_{\langle ij \rangle:\; n_{ij}=0} (1 - p_{ij}) \right) q^{\mathcal{C}(n)}$$

  where $\mathcal{C}(n) = \#$ connected components ("clusters")
  (Fortuin–Kasteleyn random-cluster model)
  Remark: q can now take arbitrary *positive real* values

Recapitulating Fortuin–Kasteleyn ...

- $Z_{\text{Potts}} = Z_{\text{joint}} = Z_{\text{RC}}$
- Marginal distribution of joint model on the Potts variables σ (integrating out the n) gives the Potts model
- Marginal distribution of joint model on the bond variables n (integrating out the σ) gives the random-cluster model
- Conditional distribution of n given σ is as follows:
  Independently for each bond $\langle ij \rangle$, set $n_{ij} = 0$ if $\sigma_i \neq \sigma_j$,
  and set $n_{ij} = 0, 1$ with probability $1 - p_{ij}, p_{ij}$ if $\sigma_i = \sigma_j$
- Conditional distribution of σ given n is as follows:
  Independently for each connected component, set all spins σ_i to the same value, chosen uniformly at random from $\{1, 2, \ldots, q\}$

The Swendsen–Wang algorithm:

- Alternately apply the two conditional distributions!
- Each half-step can be carried out in time of order V
- Local move in one set of variables can have nonlocal effects in the other $\implies$ it's a collective-mode algorithm!
  (in which the collective modes are *chosen by the system*)

Dynamic critical behavior of Swendsen–Wang (empirical)

- 2D Ising model at criticality (SW versus Metropolis):

L	χ	Swendsen–Wang $\tau_{\text{int},\mathcal{E}}$	Metropolis $\tau_{\text{exp},\mathcal{M}}$
4	12.183 ± 0.007	2.027 ± 0.010	
8	41.396 ± 0.008	2.590 ± 0.004	
16	139.584 ± 0.039	3.253 ± 0.008	285.6 ± 4.3
32	470.022 ± 0.140	4.012 ± 0.011	1258 ± 28
64	1581.319 ± 0.378	4.892 ± 0.011	5380 ± 140
128	5320.644 ± 1.680	5.875 ± 0.018	23950 ± 480
256	17899.581 ± 5.846	6.928 ± 0.022	
512	60184.698 ± 18.670	8.107 ± 0.025	

$\Longrightarrow \tau_{\text{SW}} \sim L^{\approx 0.22}$ (or maybe $\sim \log^2 L$) versus $\tau_{\text{Metropolis}} \sim L^{\approx 2.13}$

- For q-state Potts model in dimension d:

	Estimates of z_{SW}			
	$q = 1$	$q = 2$	$q = 3$	$q = 4$
$d = 1$	0	0	0	0
$d = 2$	0	0.222 ± 0.007	0.514 ± 0.006	$1\ (\times \log^{??})$
$d = 3$	0	0.46 ± 0.03	—	—
$d = 4$	0	$1\ (\times \log^{??})$	—	—

Warning: Error bars shown here are statistical errors only. Systematic errors due to corrections to scaling may be much larger.

Dynamic critical behavior of Swendsen–Wang: What do we *understand*?

(Very little, alas.)

- $q = 1 \implies \tau = 0 \implies z_{\text{SW}} = 0$
- $d = 1$ (or when lattice is a *tree*) $\implies$ bonds are independent, $\tau_{\text{exp}} \to -1/\log(1 - 1/q) < \infty$ as $\beta \to +\infty \implies z_{\text{SW}} = 0$
- Non-rigorous (but probably rigorizable) argument gives $z_{\text{SW}} = 1$ for Ising model on complete graph (Curie–Weiss mean-field model) $\implies$ *suggests* $z_{\text{SW}} = 1$ also for Ising in $d \geq 4$
- Rayleigh quotient with trial vector $\mathcal{N}$ (bond density) gives rigorous **lower bound** (Li–A.S. 1989)

$$\tau_{\text{exp}} \gtrsim \tau_{\text{int},\mathcal{E}} \geq \tau_{\text{int},\mathcal{N}} \geq \text{const} \times C_H$$

  and hence $z_{\text{SW}} \geq \alpha/\nu$ (**slowness of energy-like modes**)
- Empirically this bound is close to sharp in $d = 2$ (it might even be sharp modulo a logarithm)
- But it is *very far from sharp* for Ising in $d = 3, 4$ $\implies$ there is a slow mode we don't understand!

Open problems concerning this unknown slow mode:

- Find Rayleigh trial vector giving different lower bound on z_{SW}
- Find heuristic argument predicting z_{SW} (even roughly) in terms of static exponents
- Ossola–A.S. 2004 conjectured $z_{\text{SW}} \geq \beta/\nu$ and maybe even $z_{\text{SW}} = \max(\alpha/\nu, \beta/\nu)$

It's hard to understand SW dynamics based on 5 nontrivial data points!
($d = 2$, $q = 2, 3, 4$ and $d = 3, 4$, $q = 2$)

Enter Chayes and Machta (1998)

- Collective-mode algorithm for FK random-cluster model with $q \geq 1$ (q need not be an integer!)
- Reduces to a variant of SW when q is an integer
- Reverses SW auxiliary-variable idea:
  - SW starts from Potts spin model, introduces auxiliary variables $n_{ij} = 0, 1$ living on bonds
  - CM starts from random-cluster model, introduces auxiliary variables $\sigma_i = 1, 2, \ldots, k$ living on sites (k integer $\leq q$)
- Provides natural interpolation of SW dynamics to noninteger q (albeit only for $q \geq 1$)
- Can study whole range $1 \leq q \leq q_c(\mathcal{L})$ as a function of *continuous* q [here $q_c(\mathcal{L})$ = maximum q for which the phase transition is second-order on the lattice $\mathcal{L}$]

Chayes–Machta algorithm as reinterpreted by Deng *et al.* 2007

- Consider *generalized random-cluster model*

$$Z = \sum_{\{n\}} \left(\prod_{\langle ij \rangle :\ n_{ij}=1} p_{ij} \right) \left(\prod_{\langle ij \rangle :\ n_{ij}=0} (1-p_{ij}) \right) \left(\prod_{i=1}^{k} W(H_i) \right)$$

  where $H_1, \ldots, H_k$ are the clusters and $\{W(H)\}$ are nonnegative weights
  (reduces to ordinary RC model if $W(H) = q$ for all H)

- Fix an integer $m \geq 1$

- Decompose each weight $W(H)$ into m nonnegative pieces,
  any way we like: $W(H) = \sum_{\alpha=1}^{m} W_\alpha(H)$

- First half-step of generalized CM algorithm:
  Given bond configuration n, choose independently for each cluster H_i
  a "color" $\alpha \in \{1, \ldots, m\}$ with probabilities $W_\alpha(H_i)/W(H_i)$;
  assign this color to all sites in H_i.

- **Fact:** Subgraph consisting of sites colored α is a generalized RC model
  with weights $\{W_\alpha(H)\}$

- Now update these generalized RC models with any valid MC algorithm
  - One valid update is "do nothing" (**inactive** colors)
  - Must include at least one nontrivial update
  - Idea: Have at least one α for which $\{W_\alpha(H)\}$ is easy to simulate
  - Example: If $W(H) = q$ for all H, with $q \geq 1$, can take
    $W_\alpha(H) = 1$ for one or more colors α (**active** colors):
    independent bond percolation is trivial to update!
  - If q is integer and we make all colors active, recover standard SW

Dynamic critical behavior of S–W–C–M (empirical)

Dimension $d = 2$			
q	$z_{\mathrm{int},\mathcal{E}'}$	α/ν	β/ν
1.00	0	-0.5000	**0.1042**
1.25	0	-0.3553	**0.1112**
1.50	0	-0.2266	**0.1168**
1.75	0.06(1)	-0.1093	**0.1213**
2.00	0.14(1)	0 (log)	0.1250
2.25	0.24(1)	0.1036	0.1280
2.50	0.31(1)	0.2036	0.1303
2.75	0.40(2)	0.3017	0.1321
3.00	0.49(1)	0.4000	0.1333
3.25	0.57(1)	0.5013	0.1339
3.50	0.69(1)	0.6101	0.1338
3.75	0.78(1)	0.7376	0.1324
4.00	0.93(2)	1.0000	0.1250

Dimension $d = 3$			
q	$z_{\mathrm{int},\mathcal{E}'}$	α/ν	β/ν
1	0	$-0.710(2)$	**0.4774(1)**
1.5	0.13(1)	$-0.32(4)$	**0.500(4)**
1.8	0.29(1)	$-0.15(5)$	**0.5117(6)**
2	0.46(3)	0.174(1)	0.5184(1)
2.2	0.76(1)	0.50(4)	0.508(4)

- Ossola–A.S. conjecture $z \geq \beta/\nu$ is definitely **false** (see **boldface**)
- z_{SWCM} very close to α/ν in $d = 2$ for all q
- $z_{\mathrm{SWCM}} \gg \alpha/\nu$ in $d = 3$ for all q — but we still don't understand *why*!

Sweeny (1983) local algorithm for the random-cluster model

- Single-bond-update dynamics for FK random-cluster model with $q > 0$
  (q need not be an integer, nor need it be ≥ 1!)
- Choose a bond b at random, erase its current occupation state, and give it a new occupation state according to the conditional distribution of the RC model with the other bonds held fixed
- In other words: Occupy it with probability $v/(1+v)$ [resp. $v/(q+v)$] in case the endpoints of b are (resp. are not) already connected by a path of occupied bonds not using b [here $v = e^J - 1 = p/(1-p)$]
- Requires nonlocal connectivity check at each *bond* update
  $\Longrightarrow$ nontrivial computational issues (dynamic connectivity algorithms)
- Rayleigh quotient with trial vector $\mathcal{N}$ (bond density) gives rigorous **lower bound**

  $$\tau_{\text{exp}} \gtrsim \tau_{\text{int},\mathcal{N}} \geq \text{const} \times C_H$$

  and hence $z \geq \alpha/\nu$ (where time is measured in "sweeps")
- Analogous to $z \geq \gamma/\nu$ for single-spin-update algorithms
  ... but C_H diverges much more slowly than χ
  $\Longrightarrow$ *this* local algorithm may not be so bad after all!
- **Numerical results in $d = 2, 3$:**
  - $z_{\text{exp}} \approx z_{\text{int},\mathcal{N}}$ only slightly larger than lower bound α/ν, and conceivably *equal* to it
  - Exponents slightly *smaller* than those found for Chayes–Machta–Swendsen–Wang!

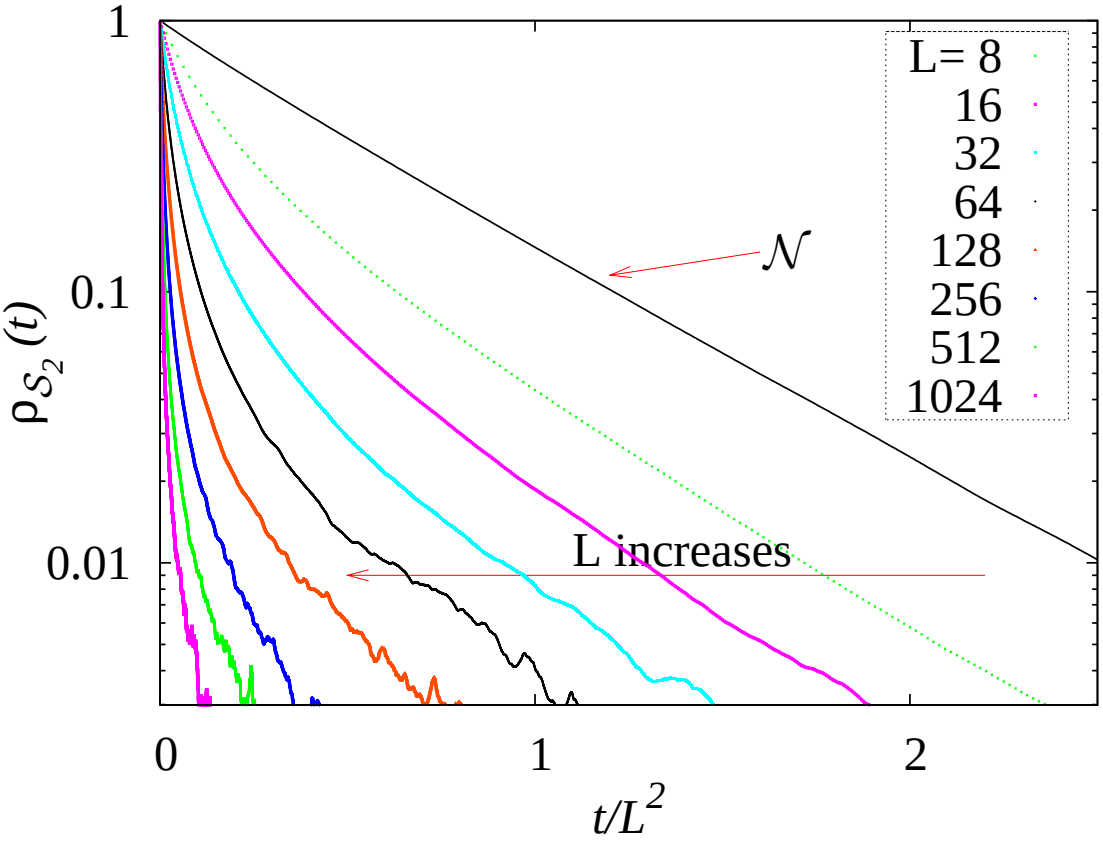
  $\Longrightarrow$ Sweeny algorithm extremely efficient (despite being local) *provided that* computational difficulties can be overcome

Surprise in the Sweeny algorithm: Critical *speeding-up*!

- Critical slowing-down (well known):
  Slowest mode exhibits $\tau_{\text{exp}} \sim \xi^{z_{\text{exp}}}$ with $z_{\text{exp}} > 0$
- Critical speeding-up (quite unexpected):
  Some observables $\mathcal{O}$ exhibit strong decorrelation
  on time scales much *less* than one sweep
  $\Longrightarrow$ can even have $z_{\text{int},\mathcal{O}} < 0$!
- **Example:** $\mathcal{O} = \mathcal{S}_2 = \sum |\mathcal{C}|^2$ in the Sweeny algorithm
  - FK clusters are fractals
  - A large cluster can sometimes be broken into two large pieces by one or a few bond deletions
  - Two large clusters can sometimes be joined by one or a few bond insertions
  - Not surprising that a "global" observable like $\mathcal{S}_2$ could exhibit significant decorrelation in a time much *less* than one sweep, i.e. of order L^w "hits" with $w < d$
    (We will later make this reasoning quantitative)
  - Crucial that we update bonds randomly rather than by sweeps

Critical speeding-up for $\mathcal{O} = \mathcal{S}_2$ in the Sweeny dynamics

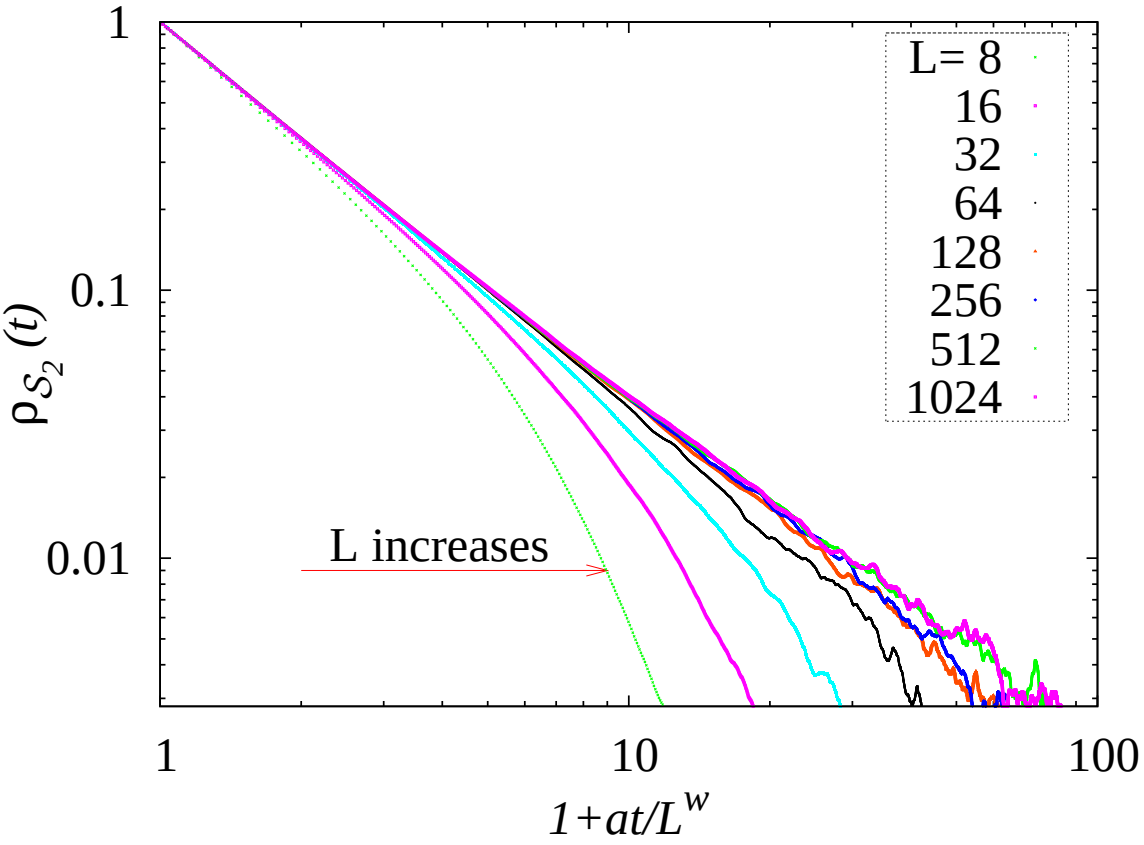
- $d = 2$ random-cluster model at criticality (here $q = 0.2$)
- Time t measured in "hits" of a single bond
- Autocorrelation function $\rho_{\mathcal{S}_2}(t)$ plotted versus t/L^2 for varying L
- Autocorrelation function $\rho_{\mathcal{N}}(t)$ shown for comparison (here all values of L collapse onto a single curve)



- Two time scales:
  - Fast decay in time much *less* than a single sweep (i.e. of order L^w for some $w < d$)
  - Ultimate exponential decay $e^{-t/\tau_{\text{exp}}}$ with $\tau_{\text{exp}} \sim L^{d+z_{\text{exp}}}$

Same data plotted versus rescaled time

- Plot $\rho_{\mathcal{S}_2}(t)$ versus t/L^w
- Adjust the exponent w until all the points fall on a scaling curve $\rho_{\mathcal{S}_2}(t) = f(t/L^w)$ in the limit $L \to \infty$
- E.g. we find $w \approx 0.99$ for $q = 0.2$
- Scaling function f is very close to $f(x) = (1 + ax)^{-r}$



Sweeny algorithm: Critical exponents versus q

Dimension $d = 2$							
q	z_{exp}	α/ν	w	r	$z_{\text{int},\mathcal{S}_2}$	d_{red}	d_{F}
0.0005	0	-1.9576	0.77	4.83	-1.23	1.2376	1.9965
0.005	0	-1.8679	0.79	4.18	-1.21	1.2111	1.9891
0.05	0	-1.6005	0.88	2.84	-1.12	1.1299	1.9679
0.2	0	-1.2467	0.99	1.42	-1.01	1.0168	1.9417
0.5	0	-0.8778	1.11	0.80	-0.71	0.8904	1.9172
1.0	0	-0.5000	1.26	0.43	-0.32	0.7500	1.8958
1.5	0	-0.2266	1.36	0.25	-0.16	0.6398	1.8832
2.0	0 (log)	0 (log)	1.49	0.15	-0.08	0.5417	1.8750
2.5	0.26(1)	0.2036	1.64	0.10	0.20	0.4474	1.8697
3.0	0.45(1)	0.4000	1.84	0.06	0.41	0.3500	1.8667
3.5	0.636(2)	0.6101	2.04	0.04	0.61	0.2375	1.8662

Dimension $d = 3$							
q	z_{exp}	α/ν	w	r	$z_{\text{int},\mathcal{S}_2}$	d_{red}	d_{F}
0	0	$-1.44(5)$	1.52	1.04	-1.48	?	2.5838(5)
1	0	$-0.713(1)$	1.87	0.32	-0.36	1.1437(6)	2.5219(2)
2	0.35(1)	0.174(2)	2.55	0.08	0.29	0.757(2)	2.4816(1)

- Note that $z_{\text{int},\mathcal{S}_2} < 0$ for $q \lesssim 2$!
- Critical speeding-up is strongest for small q
- This is when clusters are most fragile,
  i.e. red-bond fractal dimension d_{red} is largest
  (Red bonds are those whose removal disconnects the infinite cluster into two infinite pieces)

Scaling argument predicting critical speeding-up

- *Conjecture* that decorrelation of $\mathcal{S}_2$ is caused principally by hitting a few (order 1) red bonds $\Longrightarrow$ predicts $w = d - d_{\text{red}}$
- Prediction verified when $z_{\text{exp}} = 0$
- Deviations (not yet understood) occur when $z_{\text{exp}} > 0$
- We lack (at present) any theory for the other exponent r

Summary of Sweeny algorithm

- Despite being *local*, Sweeny algorithm is unexpectedly efficient
- For $0 < q < 1$ it is the *only* known algorithm
- For $q \lesssim 2$ efficiency is enhanced by strong critical speeding-up
- Even for larger q, it is a potential competitor to Chayes–Machta if efficient connectivity-checking algorithms can be found

Let's step back and take stock ...

- Local updates for spin models lead to critical slowing-down
- A general idea for overcoming critical slowing-down:
  - Replace the underlying spins by an alternate representation, obtained from the original model by algebraic transformation
  - Local moves in the new variables may have nonlocal effects in the original variables
- Swendsen–Wang is of this kind (uses joint Potts–FK representation)
- Even Sweeny is of this kind (local move in FK bond representation has nonlocal effects when reinterpreted for spins)

Can we invent other algorithms of this type?

- One well-known alternate representation of spin models is the *high-temperature expansion*
- Consider high-temperature graphs as a statistical-mechanical model in their own right [works when weights are nonnegative]
- But not so obvious how to update "vacuum" graphs alone
- Clever idea (Prokof'ev and Svistunov 2001):

  Update "vacuum" *and* "two-point-function" graphs together
  $\Longrightarrow$ **worm algorithm**

  (see also Jerrum and Sinclair 1993 for a closely related idea)

High-temperature expansion of ferromagnetic Ising model (a quick review)

- Consider ferromagnetic Ising model on graph $G = (V, E)$ with nearest-neighbor coupling $J > 0$ and zero magnetic field
- High-temperature graphs are subsets $A \subseteq E$ of "occupied bonds"
- Write ∂A for the set of vertices that touch an *odd* number of bonds of A (**boundary mod 2**)
- Vacuum graphs are $\mathcal{S}_\varnothing = \{A\colon\ \partial A = \varnothing\}$
- Two-point-function graphs are $\mathcal{S}_{x,y} = \{A\colon\ \partial A = \{x, y\}\}$ (Use convention $\mathcal{S}_{x,x} = \mathcal{S}_\varnothing$)
- High-temperature expansion then states that

$$Z = \sum_{A \in \mathcal{S}_\varnothing} w^{|A|}$$

$$ZG(x, y) = \sum_{A \in \mathcal{S}_{x,y}} w^{|A|}$$

  where $G(x, y) = \langle \sigma_x \sigma_y \rangle$ and $w = \tanh J$

- This expansion can be interpreted *probabilistically* if $w > 0$

Worm algorithm as reinterpreted by Deng *et al.* 2007

- Configuration space $\mathcal{S}$ of worm algorithm consists of ordered triplets (A, x, y) with $x, y \in V$ and $A \in \mathcal{S}_{x,y}$
- Weight of configuration (A, x, y) is $w^{|A|}$
- Elementary move of worm algorithm:
  - Pick uniformly at random one of the two "endpoints" (say, x) and one of the edges emanating from x (say, $e = xx'$)
  - Propose to move from the current configuration (A, x, y) to the new configuration $(A \triangle e, x', y)$, where $\triangle$ denotes symmetric difference
    (i.e. delete the bond e if it is present, or insert it if it is absent)
  - Accept or reject this move according to Metropolis or heat-bath criterion
- Optional additional move: Whenever we reach $x = y$, move endpoints from (x, x) to randomly chosen (x', x')
- Some observables of interest in this unusual ensemble:
  - Number of occupied bonds $\mathcal{N} = |A|$
  - Short-distance observable $\mathcal{D}_{\mathbf{0}} = \delta_{x-y,\mathbf{0}}$ measuring returns to $x = y$
    $\Longrightarrow$ estimate susceptibility by $\langle \mathcal{D}_{\mathbf{0}} \rangle = 1/\chi$
  - Low-momentum observable $\mathcal{F}_{\mathbf{p}} = e^{i\mathbf{p}\cdot(x-y)}$ for $|\mathbf{p}| = 2\pi/L$
    $\Longrightarrow$ estimate correlation length using $\langle \mathcal{F}_{\mathbf{p}} \rangle = \widetilde{G}(\mathbf{p})/\chi$

Dynamic critical behavior of worm algorithm (theory)

- Measure time t in "hits" of a single bond
  (but natural unit of time is one "sweep" $= L^d$ hits)

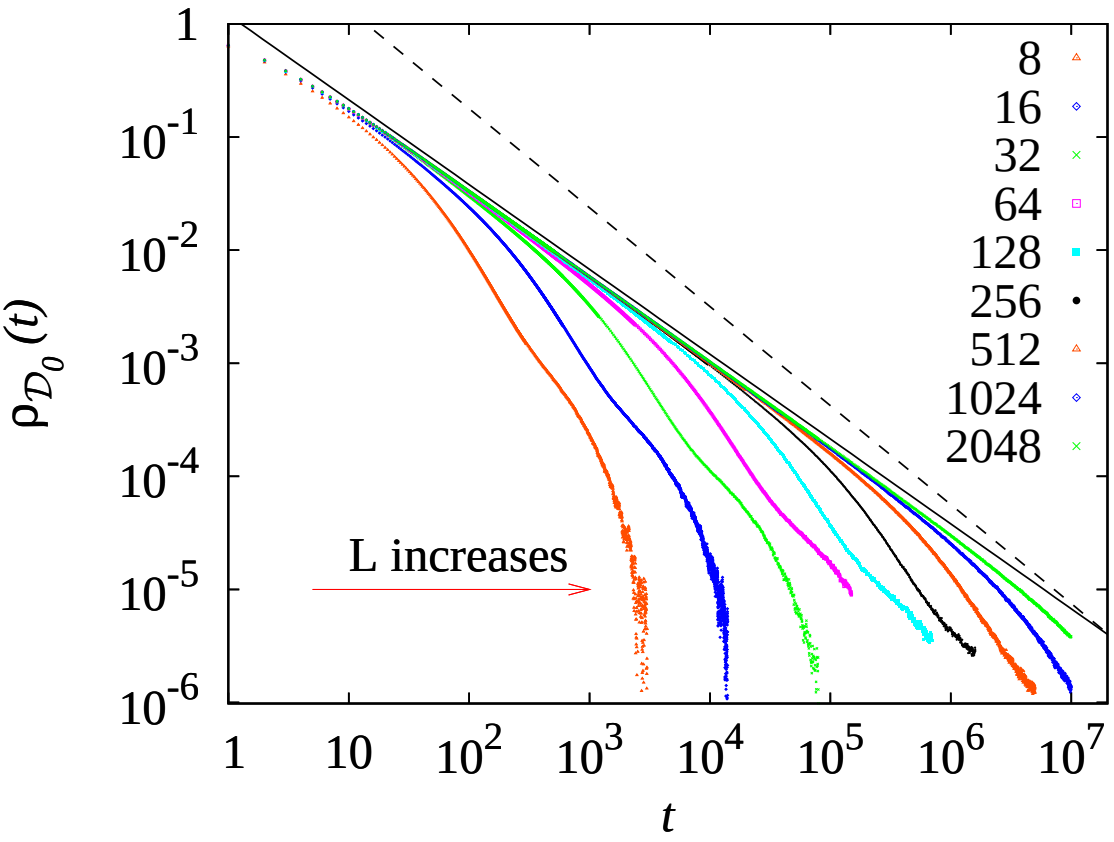
- Bond density $\mathcal{N}$: Rayleigh–Ritz argument gives
  $\tau_{\text{int},\mathcal{N}} \geq \text{const} \times L^{d+\alpha/\nu}$ in "hits", i.e. $z_{\text{int},\mathcal{N}} \geq \alpha/\nu$
  [this will be slowest mode]

- Short-distance observable $\mathcal{D}_{\mathbf{0}} = \delta_{x-y,\mathbf{0}}$:
  - Assume *perfect equilibration of bonds* A for the given endpoints x, y
  - Then $\mathbf{z} = x - y$ performs a random walk with drift, with equilibrium distribution $G(\mathbf{z})/\chi$
  - Fokker–Planck analysis predicts $\rho_{\mathcal{D}_{\mathbf{0}}}(t) \sim t^{-(1-\eta/2)}$ in the limit $L \to \infty$
  - Actual decay might be slower because bonds are not perfectly equilibrated.

- But $\mathcal{D}_{\mathbf{0}}$ estimates χ via a "rare" event,
  i.e. binomial random variable with probability $1/\chi$
  So $\sim \chi$ samples needed to get relative variance of order 1
  ("statistical inefficiency due to large static variance")

Dynamic critical behavior of worm algorithm (empirical)

- $d = 2$ Ising model at criticality
- Bond density $\mathcal{N}$: Decay of $\rho_{\mathcal{N}}(t)$ is nearly pure exponential, fit $\tau_{\mathrm{int},\mathcal{N}} \sim L^{d+z}$ yields $z_{\mathrm{exp}} = z_{\mathrm{int},\mathcal{N}} \approx 0.34 \quad (> \alpha/\nu)$
- Short-distance observable $\mathcal{D}_{\mathbf{0}}$ shows significant decorrelation on time scale of order 1 (in "hits"):



- Short-time behavior is $\rho_{\mathcal{D}_{\mathbf{0}}}(t) \sim t^{-s}$ with $s \approx 0.75$ (perfect equilibration of bonds predicted $s = 1 - \eta/2 = 7/8$)

Dynamic critical behavior of worm algorithm (continued)

- Worm algorithm exhibits two-time-scale behavior (like Sweeny):
  - Initial decay $\rho_{\mathcal{D}_\mathbf{0}}(t) \sim t^{-s}$ on time scale of order 1
  - Ultimate exponential decay $e^{-t/\tau_{\text{exp}}}$ with $\tau_{\text{exp}} \sim L^{d+z_{\text{exp}}}$
- Crossover scaling Ansatz yields $z_{\text{int},\mathcal{D}_\mathbf{0}} = -sd + (1-s)z_{\text{exp}}$
- With $z_{\text{exp}} \approx 0.34$ and $s \approx 0.75$, this gives $z_{\text{int},\mathcal{D}_\mathbf{0}} \approx -1.42$
- Behavior in $d = 3$ is similar, but now:
  - $z_{\text{exp}} = z_{\text{int},\mathcal{N}}$ appears to be *equal* to $\alpha/\nu \approx 0.174$
  - Short-time behavior is $\rho_{\mathcal{D}_\mathbf{0}}(t) \sim t^{-s}$ with $s \approx 0.66$ (perfect equilibration of bonds predicted $s = 1-\eta/2 \approx 0.98$)
  - Crossover scaling Ansatz yields $z_{\text{int},\mathcal{D}_\mathbf{0}} \approx -1.92$
- Practical efficiency for estimating χ:
  - Multiply $\tau_{\text{int},\mathcal{D}_\mathbf{0}} \sim L^{z_{\text{int},\mathcal{D}_\mathbf{0}}}$ by factor $\chi \sim L^{\gamma/\nu}$ due to static variance
  - Yields "effective dynamic critical exponent" $z_{\text{eff},\mathcal{D}_\mathbf{0}} = z_{\text{int},\mathcal{D}_\mathbf{0}} + \gamma/\nu$
  - We find $z_{\text{eff},\mathcal{D}_\mathbf{0}} \approx 0.33$ in $d = 2$, $z_{\text{eff},\mathcal{D}_\mathbf{0}} \approx 0.04$ in $d = 3$
  - Compare to Swendsen–Wang: $z_{\text{SW}} \approx 0.14$–$0.22$ in $d = 2$, $z_{\text{SW}} \approx 0.46$
  - Worm is slightly worse than SW in $d = 2$, significantly better in $d = 3$
- Results for estimating ξ using $\mathcal{F}_\mathbf{p}$ are similar

Dynamic critical behavior of worm algorithm (continued)

- **Conclusion:**
  - Worm algorithm is best currently available algorithm for estimating χ and ξ in 3D Ising
  - In practice, worm outperforms SW when $L \gtrsim 32$, at a rate growing like $L^{\approx 0.32}$
- Two-time-scale (or even *three*-time-scale) dynamics needs further elucidation: Can we *explain* the dynamic critical exponents?
- "Worm" idea is a general principle that is widely applicable

- **Recent developments:**
  - Wolff 2009 combines variant of worm algorithm with Aizenman random-current identity to study connected 4-point function
  - Zhang–Garoni–Deng 2009 devise worm algorithm for fully-packed loop model on hexagonal lattice ($\Longleftrightarrow$ $T = 0$ triangular-lattice Ising antiferromagnet)