Animal Gaits

and

Symmetries of Periodic Solutions

Statistical Mechanics

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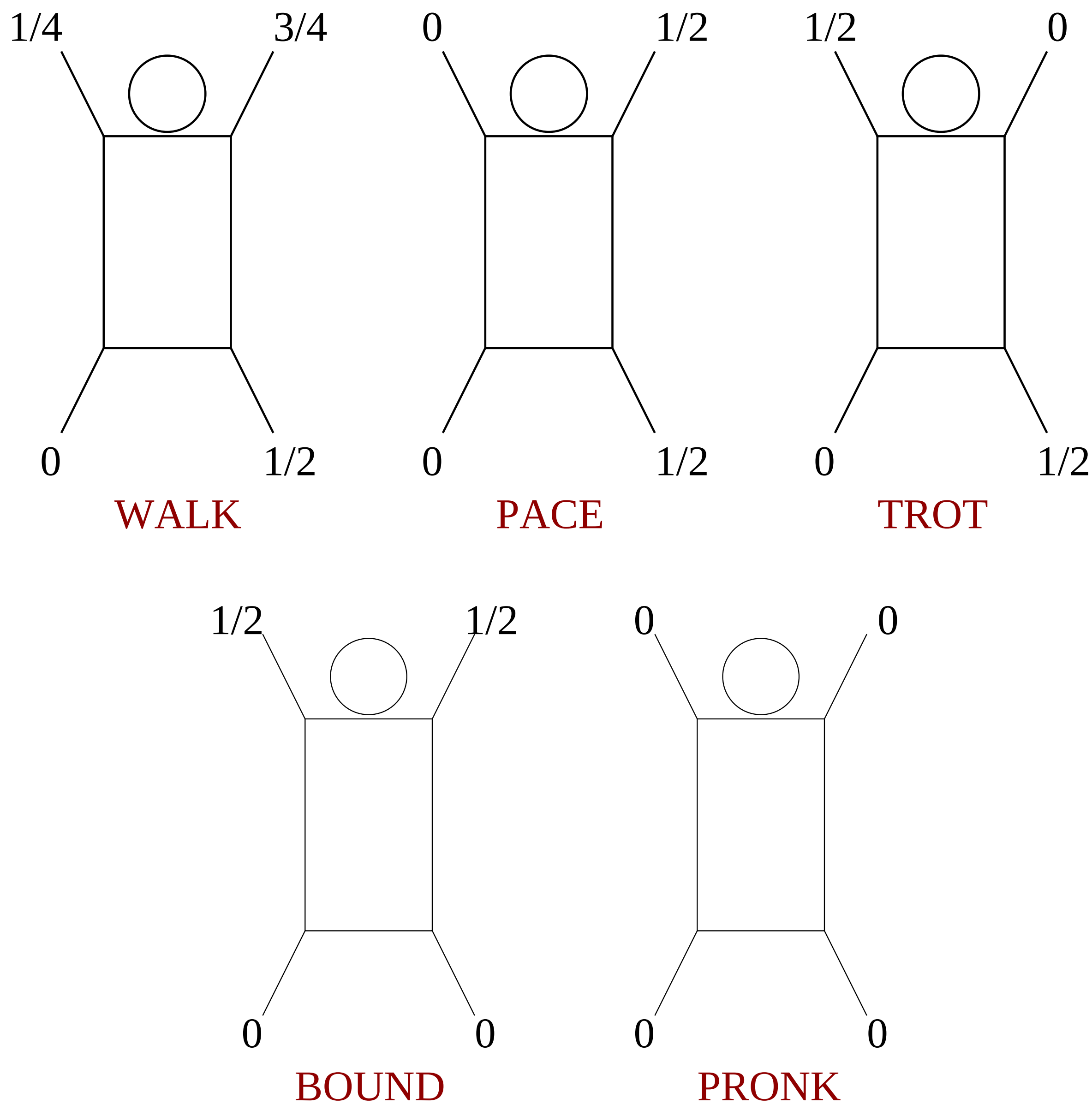
Standard Gait Phases



Gait Symmetries

Gait	Spatio-temporal symmetries
Trot	(Left/Right, $\frac{1}{2}$) and (Front/Back, $\frac{1}{2}$)
Pace	(Left/Right, $\frac{1}{2}$) and (Front/Back, 0)
Walk	(Figure Eight, $\frac{1}{4}$)

- **Walk, trot, pace are different gaits**

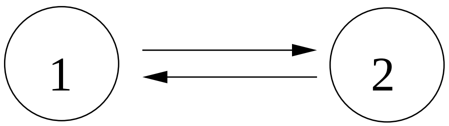
Collins and Stewart (1993)

Central Pattern Generators (CPG)

Gaits modeled mechanically and/or electrically — we discuss electrical system

- **Assumption**: In nervous system is network of neurons that produces gait rhythms
- Hodgkin - Huxley: **Neuron** modeled by system of differential equations
- **CPG** = network of coupled identical systems
- Design simplest network to produce walk, trot, and pace

Two Identical Cells



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1)\end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$

- Robust time-periodic solutions:
  - in phase oscillation

$$x_2(t) = x_1(t)$$

Note: $x_1 = x_2$ is flow-invariant subspace

  - half-period out-of-phase oscillation

$$x_2(t) = x_1(t + \frac{T}{2})$$

Spatio-Temporal Symmetries

- A symmetry $\dot{x} = F(x)$ is a linear map γ where

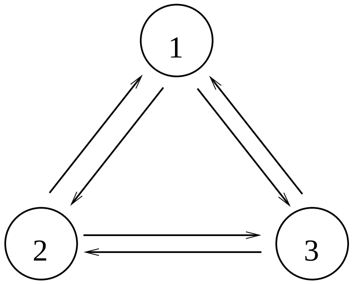
  γ(sol'n) = sol'n $\quad\Longleftrightarrow\quad F(\gamma x) = \gamma F(x)$

- Let $x(t)$ be a time-periodic solution
  - $K = \{\gamma \in \Gamma : \gamma x(t) = x(t)\}$ **space symmetries**
  - $H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}$ **spatiotemporal symm's**

- Facts:
  - $\gamma \in H \Longrightarrow \theta \in \mathbf{S}^1$ such that $\gamma x(t) = x(t+\theta)$
  - H/K is cyclic since
    $\gamma \mapsto \theta$ is a homomorphism with kernel K

- Example: $H = \mathbf{Z}_2(1\ 2); K = 1; \theta = \frac{T}{2}$

Three-Cell Bidirectional Ring: $\Gamma = \mathbf{S}_3$



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_3, x_1) \qquad f(x_2, x_1, x_3) = f(x_2, x_3, x_1) \\ \dot{x}_3 &= f(x_3, x_1, x_2) \end{aligned}$$

- Discrete rotating waves: $H = \mathbf{Z}_3$, $K = \mathbf{1}$

$$x_2(t) = x_1\left(t + \tfrac{T}{3}\right) \quad \text{and} \quad x_3(t) = x_2\left(t + \tfrac{T}{3}\right)$$

In-phase periodic solutions: $H = \mathbf{Z}_2(1\ 3) = K$

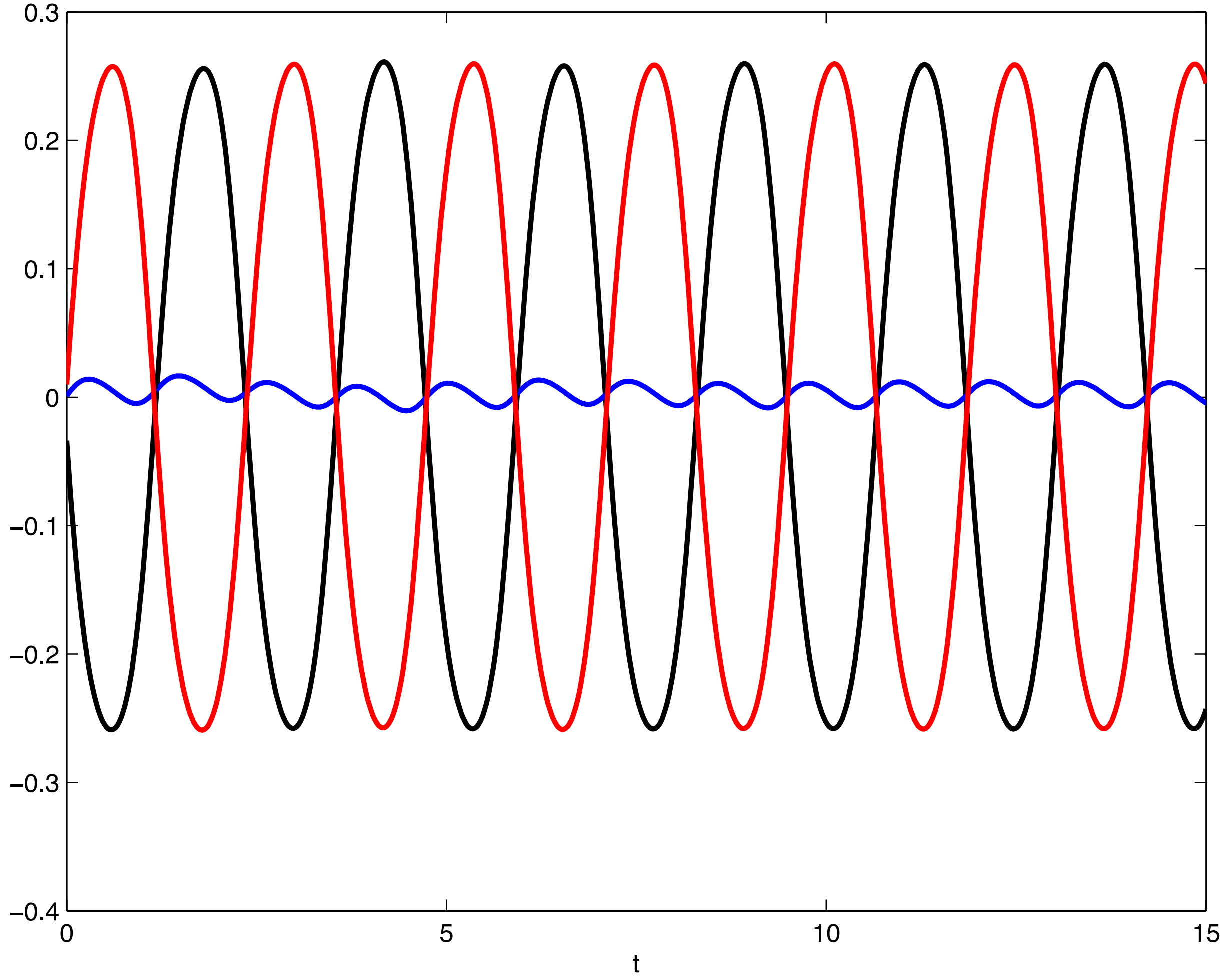
$$x_3(t) = x_1(t)$$

Out-of-phase periodic solutions: $H = \mathbf{Z}_2(1\ 3)$, $K = \mathbf{1}$

$$x_3(t) = x_1\left(t + \tfrac{T}{2}\right) \quad \text{and} \quad x_2(t) = x_2\left(t + \tfrac{T}{2}\right)$$

G. and Stewart (1986)

Out of Phase



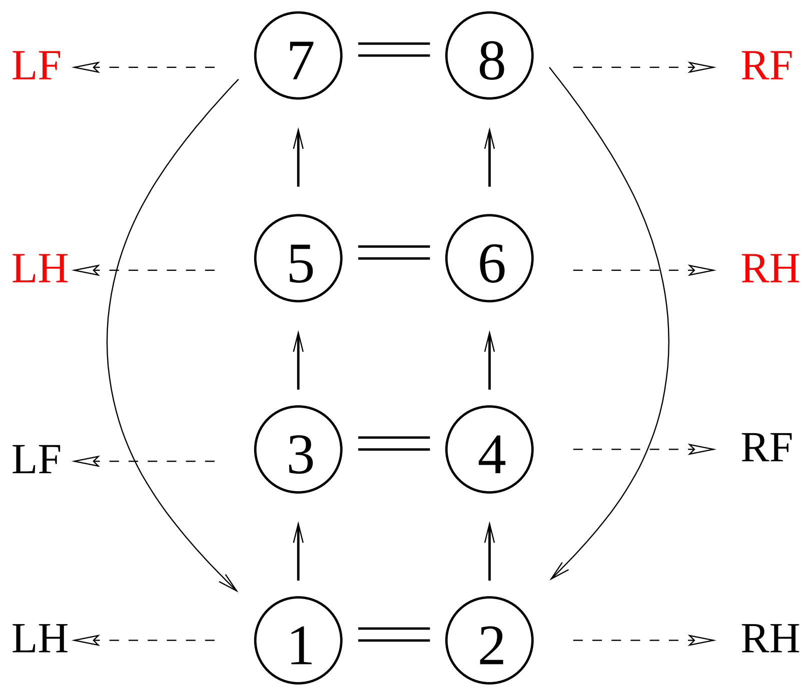
Central Pattern Generators (CPG)

- **Use gait symmetries to construct coupled network**

  1) walk $\Longrightarrow$ four-cycle ω in symmetry group

  2) pace or trot $\Longrightarrow$ transposition κ in symmetry group

- Simplest network has $\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$ symmetry



G., Stewart, Buono, and Collins (1999); Buono and G. (2001)

Primary Gaits: $H = \mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$

K	Phase Diagram	Gait
$\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	pronk
$\mathbf{Z}_4(\omega)$	$\begin{matrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix}$	pace
$\mathbf{Z}_4(\kappa\omega)$	$\begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{matrix}$	trot
$\mathbf{Z}_2(\kappa) \times \mathbf{Z}_2(\omega^2)$	$\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{matrix}$	bound
$\mathbf{Z}_2(\kappa\omega^2)$	$\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{matrix}$	walk
$\mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$	jump

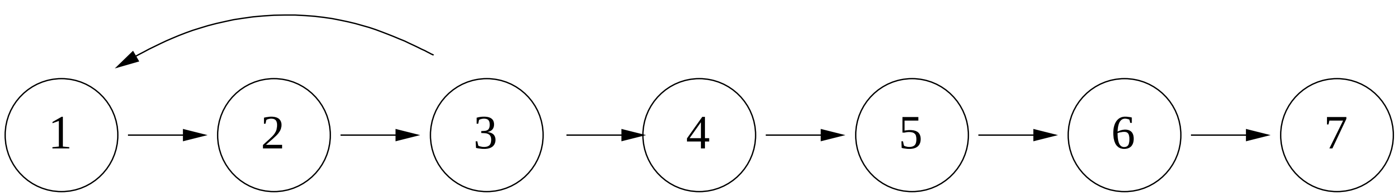
Synchrony Subspaces

- A polydiagonal is a subspace

$$\Delta = \{x : x_c = x_d \quad \text{for some subset of cells}\}$$

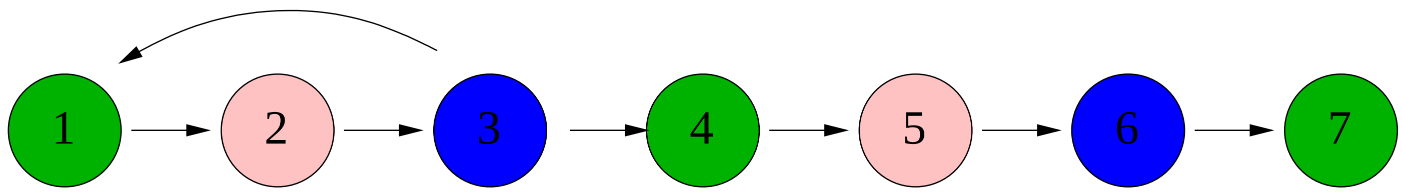
- A **synchrony subspace** is a flow-invariant polydiagonal

- **Chain with Back Coupling**



$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}(\mathbf{x}_1, \mathbf{x}_3) & \dot{\mathbf{x}}_2 &= \mathbf{f}(\mathbf{x}_2, \mathbf{x}_1) & \dot{\mathbf{x}}_3 &= \mathbf{f}(\mathbf{x}_3, \mathbf{x}_2) \\ \dot{\mathbf{x}}_4 &= \mathbf{f}(\mathbf{x}_4, \mathbf{x}_3) & \dot{\mathbf{x}}_5 &= \mathbf{f}(\mathbf{x}_5, \mathbf{x}_4) & \dot{\mathbf{x}}_6 &= \mathbf{f}(\mathbf{x}_6, \mathbf{x}_5) \\ \dot{\mathbf{x}}_7 &= \mathbf{f}(\mathbf{x}_7, \mathbf{x}_6) \end{aligned}$$

Chain with Back Coupling

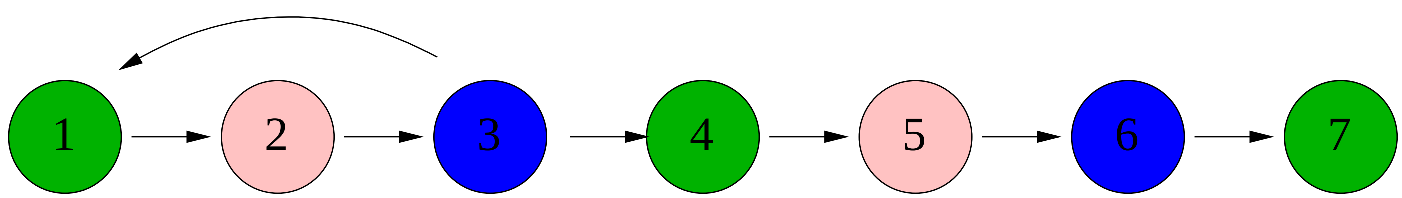


$$\dot{x}_1 = f(x_1, x_3) \qquad \dot{x}_2 = f(x_2, x_1) \qquad \dot{x}_3 = f(x_3, x_2)$$
$$\dot{x}_4 = f(x_4, x_3) \qquad \dot{x}_5 = f(x_5, x_4) \qquad \dot{x}_6 = f(x_6, x_5)$$
$$\dot{x}_7 = f(x_7, x_6)$$

- $Y = \{x : x_1 = x_4 = x_7;\ x_2 = x_5;\ x_3 = x_6\}$ is flow-invariant
- Y is a synchrony subspace

Balanced Coloring

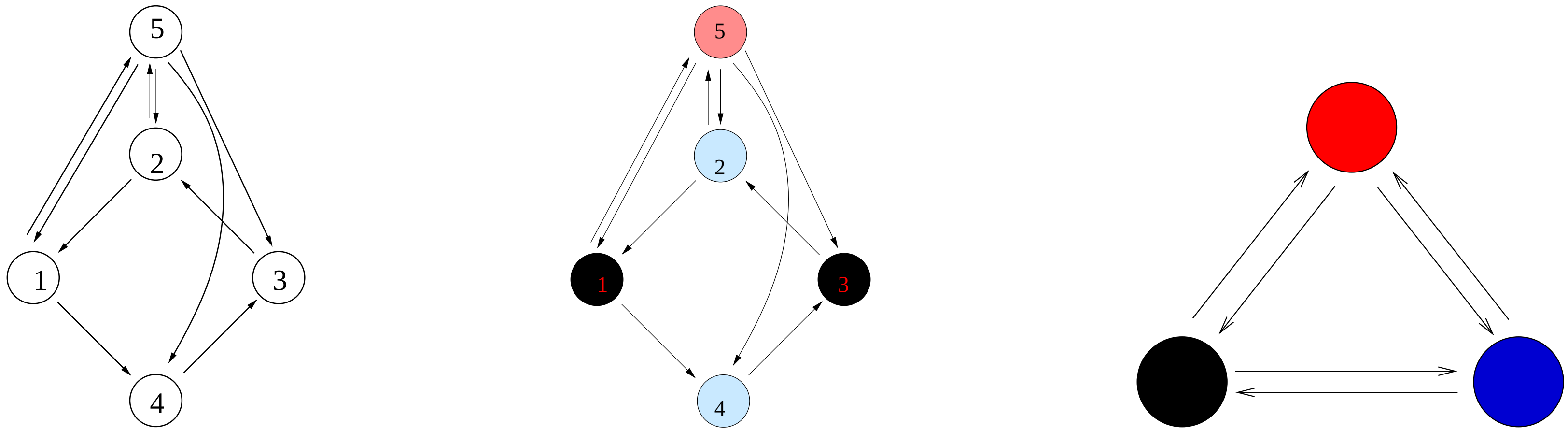
- Let Δ be a polydiagonal
- Color equivalent cells the same color
  if cell coord's in Δ are equal
- Coloring is **balanced** if all cells with same color receive
  equal number of inputs from cells of a given color



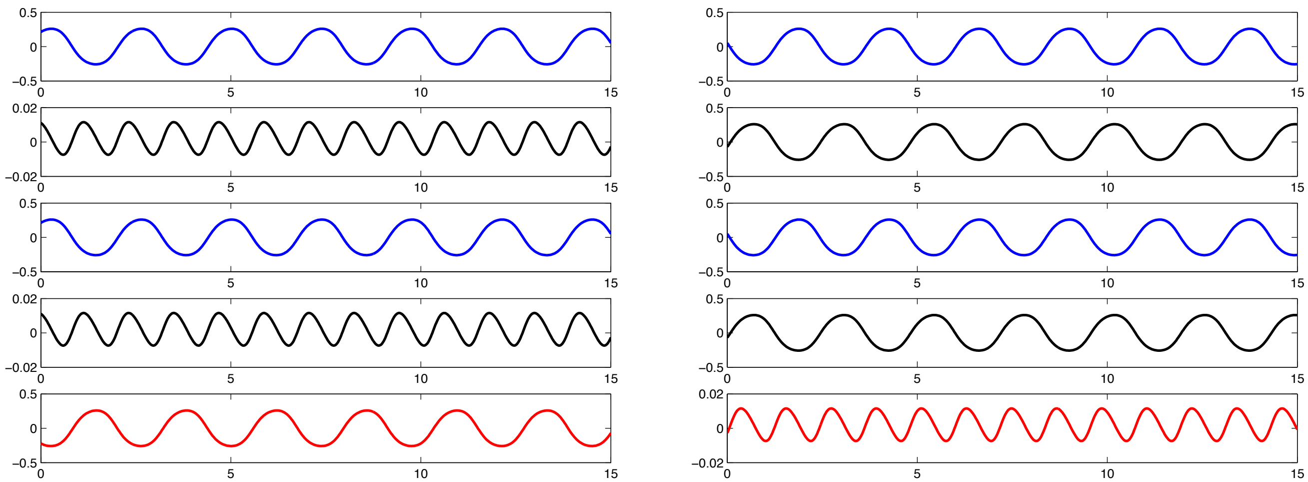
- **Theorem**: **synchrony subspace $\Longleftrightarrow$ balanced**

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

Asym Network; Symmetric Quotient



- Quotient is bidirectional 3-cell ring with D_3 symmetry



- **Rigid phase shift; no symmetry**

Phase-Shift Synchrony

- $Z(t) = (z_1(t), \ldots, z_N(t))$ is stable T-periodic solution
- **Phase-shift synchrony** between nodes i, j

$$z_i(t) = z_j(t + \theta T) \text{ where } 0 \leq \theta < 1$$

- Phase-shift synchrony is **rigid** if perturbing system leads to periodic state with same phase-shift θ
- $i = j \Longrightarrow$ multirhythms
- **Theorem:** Transitive network: nonzero rigid phase-shift occurs only when phase-shift forced by symmetry on quotient network

Stewart and Parker (2008, 2009); G., Romano and Wang (2010); Aldis (2010)

How to Find Quotient Network

- $\Delta_Z = \{x : x_c = x_d \text{ if } z_c(t) = z_d(t)\}$
- Δ_Z is **rigid** if periodic state near Z in perturbed system always has same polydiagonal
- **Theorem**: Δ_Z rigid implies coloring associated to Δ_Z is balanced
- Restrict admissible system to Δ_Z
  On quotient network $Z(t)$ has no zero rigid phase-shifts

G., Romano and Wang (2010); Aldis (2010)

Theorem of Stewart & Parker

Given periodic solution $Z(t)$ on path connected network with a **nonzero** rigid phase-shift. Assume $Z(t)$

- has no **zero** phase-shifts
- is fully oscillatory
- satisfies the rigid phase conjecture

Then there exists a network symmetry that generates the rigid phase-shifts

Stewart and Parker (2008)

Idea of Proof: Def'n of Symmetry

- Choose a node c

  Let $\theta > 0$ be smallest phase-shift s.t. $z_d(t) = z_c(t + \theta T)$

  Define $g(c) = d$. Note

  - Fully oscillatory implies smallest θ exists
  - No zero phase-shifts implies d is unique

- Rigid phase conjecture implies g is **symmetry** of network

Pattern of Synchrony

Let G be a path connected network

- (Q, σ) is **pattern of synchrony** if Q is quotient network and $\sigma : Q \to Q$ is permutation symmetry

- A T-periodic solution $Z(t)$ to a G-admissible system **has pattern of synchrony** (Q, σ) if
  - $\{Z(t)\} \subset \Delta_Q$
  - $\sigma Z(t) = Z\left(t + \frac{T}{m}\right)$ where m is order of σ

- If $Z(t)$ has pattern of synchrony (Q, σ), then $z_c(t) = z_d(t)$ when nodes c and d have the same color

Pattern of Synchrony (2)

- $\sigma = \sigma_1 \cdots \sigma_s$ is product of cycles of orders $m_1, \ldots, m_s$
- Let $\sigma_j = (c_1 \cdots c_{m_j})$. Let $Y(t)$ be the projection of $Z(t)$ to quotient network Q. Then $\sigma Y(t) = Y(t + \frac{T}{m})$ implies

$$\begin{aligned} y_{c_2}(t) &= y_{c_1}(t + \frac{T}{m}) \\ y_{c_3}(t) &= y_{c_2}(t + \frac{T}{m}) \\ &\vdots \\ y_{c_{m_j}}(t) &= y_{c_{m_{j-1}}}(t + \frac{T}{m}) \\ y_{c_1}(t) &= y_{c_{m_j}}(t + \frac{T}{m}) \end{aligned}$$

- So $y_{c_1}(t) = y_{c_1}(t + \frac{m_j}{m} T)$ and y_{c_j} has period $T_j = \frac{m_j}{m} T$
- Cycles of different lengths in σ imply multirhythms

Thanks

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