Spectral renormalization group

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- Landau free energy expansion and
  "Field theoretic" RG a la Wilson
- Graph Laplacian, eigenvalue distribution, eigenvectors
- Integration and rescaling on deterministic hierarchical graphs
- Random graphs: Gaussian theory, replicas
- Critical behaviour

Landau free energy expansion in the order parameter

$$H = \int_\Omega d\mathbf{x}\{\frac{1}{2}\left[r_o\psi^2(\mathbf{x}) - \psi(\mathbf{x})\nabla^2\psi\right] + u_o\psi^4(\mathbf{x})\}$$

$$H_0 = \frac{1}{2}\int_0^\Lambda \frac{k^{d-1}dk}{(2\pi)^d}[r_o + k^2]\widehat{\psi}^*(\mathbf{k})\widehat{\psi}(\mathbf{k})$$

$$H_{\text{int}} = u_0\int_0^\Lambda d\mathbf{k}_1 \ldots d\mathbf{k}_4\widehat{\psi}(\mathbf{k}_1)\ldots\widehat{\psi}(\mathbf{k}_4)\delta(\sum_{i=1}^4 \mathbf{k}_i)$$

where $\widehat{\psi}(\mathbf{k}) = \text{F.T.}\psi(\mathbf{x})$

Integrate out the small wavelength (large k) fluctuations

$$Z = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{0 \leq |\mathbf{k}| \leq \Lambda/b} d\widehat{\psi}_{\mathbf{k}}^{\text{lower}} \prod_{\Lambda/b < |\mathbf{k}| \leq \Lambda} d\widehat{\psi}_{\mathbf{k}}^{\text{upper}} e^{-H}$$

The density of states $\sim k^{d-1}$ is scale free!

Rescale to restore original Hamiltonian $\rightarrow$ renormalized coupling constants r and u

Try similar scheme on networks?

The Graph Laplacian

- The analogue of $(-1\times)$ the Laplacian on arbitrary networks is given by
$$L = D - A$$
- A is the adjacency matrix
$D_{ij} = \delta_{ij} d_i$
d_i is the degree of the ith node; $i = 1, \ldots, N$

The normalized graph Laplacian is defined to be

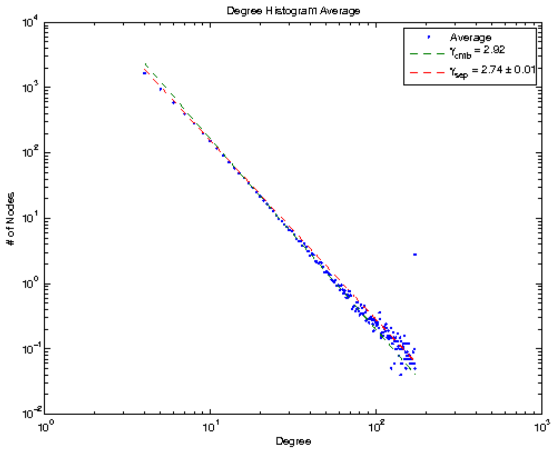
$$\tilde{L} = D^{-1} L = I - D^{-1} A$$

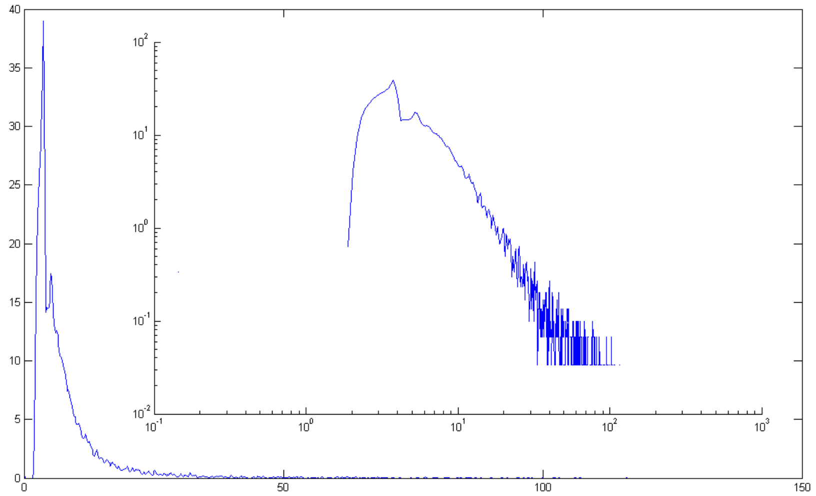
where I is the identity matrix.

Properties of the eigenvectors and eigenvalues

- $\sum_j L_{ij} = 0$
- Eigenvectors $L\,\mathbf{u}_\lambda = \lambda \mathbf{u}_\lambda$ orthagonal $\mathbf{u}_\lambda^\dagger \mathbf{u}_{\lambda'} = \delta_{\lambda,\lambda'}$
- $\lambda_0 = 0$, $\lambda_1 > 0$ (connected NW), $0 < \lambda_1 \leq \lambda_2 \ldots \lambda_N \equiv \Lambda$
- $u_{\lambda_0}(i) =$ const.
- $\sum_i u_\lambda(i) = 0$ for $\lambda \neq 0$
- In general, $u_{\lambda_k}(i) u_{\lambda_l}(i) \ldots u_{\lambda_m}(i) \neq u_{\lambda_n}(i)$ for some n.
- Eigenvectors of the normalized Laplacian $\mathbf{v}_\mu^\dagger D \mathbf{v}_{\mu'} = \delta_{\mu,\mu'}$
- Eigenvalues $\mu_0 = 0$, $0 < \mu_1 \leq \mu_2 \ldots \mu_N \leq 2$.

Degree distribution and Laplacian eigenvalue spectra for a "scale free" network



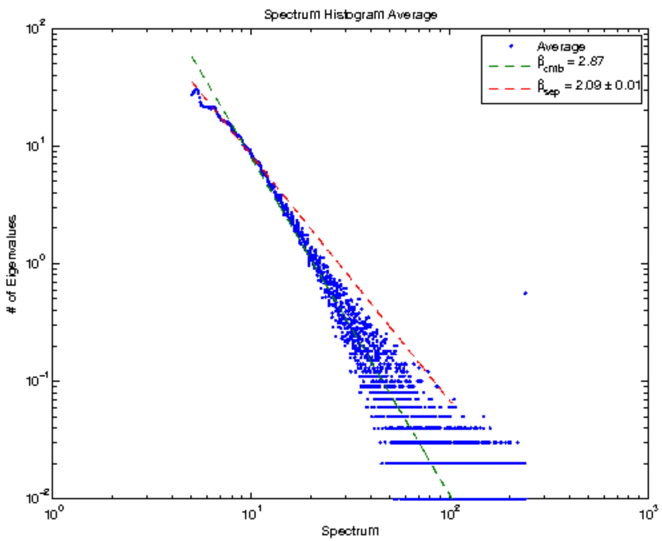


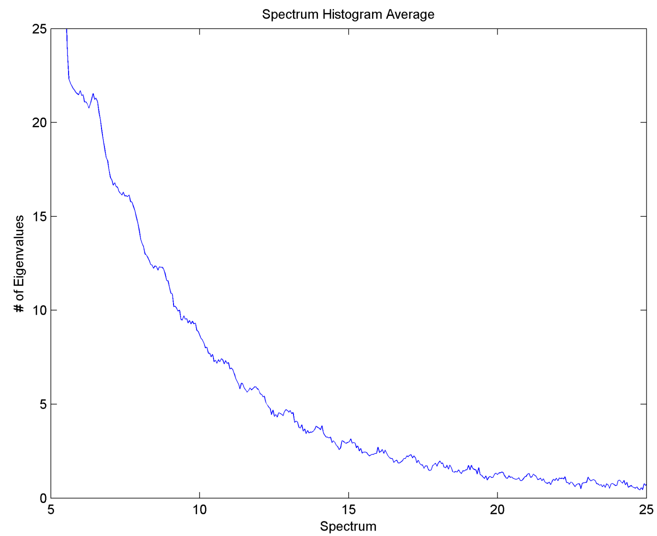
Preferential attachment(BA) model

$m0 = 5$, $m = 5$ $N = 5000$, 100 realizations.

The Laplace spectrum is not scale free; exponential crosses over to power law

Tail end of the Laplace spectrum





a) superposition, 100 realizations b) averaged to display fine structure.

Self similar curve

$$f(z) \sim f_0(z) + \sum_{n=0}^{N} \frac{1}{a^n} \sum_{m}^{M} g[b^n(z - m/b^n)]$$

Expand the order parameter in terms of the eigen-vectors of the Laplacian

Consider just a Gaussian model

$$H_0 = \frac{1}{2}\sum_{i}^{N}\psi(i)[r_o + L]\psi(i) = \frac{1}{2}\int_0^{\Lambda}\rho(\lambda)d\lambda\widehat{\psi}(\lambda)[r_o+\lambda]\widehat{\psi}(\lambda)$$

- $\widehat{\psi}(\lambda) = \frac{1}{N}\sum_i^N \psi(i)u_\lambda(i)$
- $\rho(\lambda) = \sum_i^N \delta(\lambda - \lambda_i)$

Naive renormalization

Gaussian theory

Integrate out the coefficients $\widehat{\psi}_\lambda$ for $\lambda > \Lambda/s$

Assume $\rho(s\lambda) \sim s^{-\beta}\rho(\lambda)$
β depends on the degree distribution

Fix the renormalization factors by requiring the coefficient of the λ term in the Hamiltonian to remain constant

$r = sr_0$ giving $\nu' = 1$ for any β - correct within Gaussian or MF theory.

f.p. again Gaussian to first order in u_0.

$\rho(\lambda)$ is not scale free
“rescaling” is not trivial!

- Introduce replicas and average over the realizations of the stochastic network
- Consider hierarchical networks obtained by successive decorations for which the spectrum can be computed iteratively.

Averaging out over replicas

$$Z^n = \int_{-\infty}^{\infty} \prod_{\alpha=1}^{n} \prod_{\lambda} d\widehat{\psi}_{\lambda}^{(\alpha)} e^{-\sum_\alpha H_0^\alpha}$$

$$\sum_\alpha H_0^\alpha = \frac{1}{2} \int_0^\Lambda \sum_i^N \sum_\alpha^n \widehat{\psi}_z^{(\alpha)} (r_0 + z) \widehat{\psi}_z^{(\alpha)} \delta(z - \lambda_i)$$

Assuming that a limiting distribution $\rho(\lambda)$ exists, and approximating the distribution over different realizations of the spectrum by $\mathcal{P}(\{\lambda_i\}) \prod_i d\lambda_i = \prod_i p(\lambda_i) d\lambda_i$

$$\langle Z^n \rangle = \int_0^\infty \prod_i p(\lambda_i) d\lambda_i Z^n$$

Cumulant expansion for effective Hamiltonian

Evaluating the expectation value by means of a cumulant expansion up to second order gives, for the effective Hamiltonian

$$\int_0^{\Lambda_{\mathrm{max}}} dz\rho(z)\,[r_0 + z]\,\sum_{\alpha}\,[\widehat{\psi}^{(\alpha)}(z)]^2\;+$$

$$\int_0^{\Lambda_{\mathrm{max}}} dz\rho(z)z^2\sum_{\alpha,\beta}[\widehat{\psi}^{(\alpha)}(z)]^2[\widehat{\psi}^{(\beta)}(z)]^2$$

Note, however that there is no small parameter in which to expand the fourth order term.

Restoring the integrated out part of the spectrum

$\lambda' = T_s(\lambda)$, a nonlinear transformation parametrized by the scale factor s calls for an initial λ-dependent temperature like coupling constant, $r(\lambda)$.

Under graph decoration, **Matrix extension transformation** yields all the new eigenvalues obtained in terms of the existing ones

The spectrum of the *normalized* Laplacian is given by the preimages of the *decimation* transformation $\lambda = R(\lambda')$, and converges to the Julia set of R^{-1} as $N \to \infty$.

See:

- N. Bajorin, T. Chen, ..A. Teplyaev, *J. Phys.*A (2008)
- Z. Zhang et al., *PRE* **80**, (2009)

Conclusions?

The Julia set is chaotic. Thus there is no smooth way in which to rescale existing eigenvalues to restore the integrated-out ones.

The λ-dependent temperature a way to understand the critical regions with power law behaviour of correlations obtained by Real Space RG on scale free networks.