Stochastic Domination for Ising models and other processes

Tom Liggett and J.S.

Stochastic Domination:

Given $X := \{X_i\}_{i\in I}$ and $Y := \{Y_i\}_{i\in I}$ (defined on different probability spaces), we write

$$X \preceq Y$$

if there are random variables $\{(\tilde{X}_i, \tilde{Y}_i)\}_{i\in I}$ such that

- $\{X_i\}_{i\in I} = \{\tilde{X}_i\}_{i\in I}$ (in distribution)

- $\{Y_i\}_{i\in I} = \{\tilde{Y}_i\}_{i\in I}$ (in distribution)

  and

- $P(\tilde{X}_i \le \tilde{Y}_i \ \forall i) = 1.$

If the random variables take values in $\{0,1\}$ and $A \subseteq \{0,1\}^I$ is an **upset***, then $X \preceq Y$ easily implies

$$(1) \quad P(X \in A) \leq P(Y \in A).$$

Theorem (Strassen): If (1) holds for all up sets A, then $X \preceq Y$.

* A is an **upset** if $x \in A$, $y \geq x$ implies $y \in A$.

A particular case of the easy implication is that if $X := \{X_i\}_{i \in I}$ is i.i.d. "with density ρ", then $X \preceq Y$ implies that

$$(**)\ \ P(Y_i = 0\ \ \forall i \in I) \leq (1-\rho)^{|I|}.$$

The reverse implication is true under some conditions.

Definition: $Y := \{Y_i\}_{i \in I}$ satisfies the FKG lattice condition if, when we condition on all but 2 of the variables, these 2 are positively correlated.

Theorem 1. (Liggett & S.): Let $X := \{X_i\}_{i \in I}$ be i.i.d. "with density ρ" and $Y := \{Y_i\}_{i \in I}$ be permutation invariant and satisfy the FKG lattice condition. Then $(**)$ implies that $X \preceq Y$.

Of course $X \preceq Y$ also implies

$$\text{(!!)} \;\; P(Y_i = 1 \;\; \forall i \in I) \geq \rho^{|I|}$$

but this is not sufficient for $X \preceq Y$ (even if permutation invariant and FKG).

Let

$$(Y_1, Y_2) = \begin{cases} (1,1) & \text{with probability } 1/2, \\ (0,0) & \text{with probability } 1/2. \end{cases}$$

Then Y dominates X if and only if $p \leq 1 - \sqrt{1/2}$ but (!!) holds if and if $p \leq 1/2$.

Definition: We let ν_ρ denote a product measure "with density ρ" on $\{0,1\}^I$ and given any process $Y = \{Y_i\}_{i \in I}$, we let

$$\rho_{\text{max}}(Y) := \sup\{\rho : \nu_\rho \preceq Y\}.$$

A variation of the previous result (explained later) will have implications for the Ising model.

Quick review of the Ising model

Definition: A process $Y := \{Y_i\}_{i \in Z^2}$ is an Ising model with parameter J if for all i

$$P(Y_i = 1 \mid \{Y_j\}_{j \neq i}) = \begin{cases} \frac{e^{4J}}{e^{4J}+e^{-4J}} & \text{if 4 } 1's \\ \frac{e^{2J}}{e^{2J}+e^{-2J}} & \text{if 3 } 1's \\ \frac{1}{2} & \text{if 2 } 1's \\ \frac{e^{-2J}}{e^{2J}+e^{-2J}} & \text{if 1 } 1 \\ \frac{e^{-4J}}{e^{4J}+e^{-4J}} & \text{if 0 } 1's \end{cases}$$

Phase transition For J large, there is more than 1 Ising model! In this case, there is a stochastically smallest μ^- and a stochastically largest μ^+ (obtained by choosing appropriate boundary conditions and taking limits).

Theorem 2. (Liggett & S.): For all J,

$$\rho_{\max}(\mu^-) = \rho_{\max}(\mu^+).$$

Theorem 3. (Liggett & S.): $\rho_{\max}(\mu^+)$ is strictly decreasing in J.

Ising models on Trees One can define Ising models on trees analogously and one also has μ^- and μ^+.

The situation on trees is completely difference.

Theorem 4. (Liggett & S.): For all J satisfying $\mu^- \neq \mu^+$,

$$\rho_{\max}(\mu^-) < \rho_{\max}(\mu^+).$$

We will see later than Theorem 3 is also false on trees.

The results for trees are obtained by explicit computations using the exact known "recursion" type structure of the Ising model on a tree together with an analysis of certain fixed points.

Question: Does the above dichotomy generalize to an amenable/nonamenable dichotomy?

Definition: A graph is amenable if there are subsets whose boundary/volume ratios are arbitrary small (like boxes in Z^d). Otherwise, it is nonamenable (like homogeneous trees).

Another difference is the following and concerns the relationship between the plus states as J varies.

Theorem 5. ("Liggett & S."): If $J_1 \neq J_2$, then on Z^d, $\mu^{J_1,+}$ and $\mu^{J_2,+}$ are not stochastically ordered.

Theorem 6. (Liggett & S.): Consider the Ising model on T and let J_c be the critical value for J.
(i). If $J_c < J_1 < J_2$, then $\mu^{J_2,+}$ dominates $\mu^{J_1,+}$.
(ii). For every $\rho < 1$, there exists J such that $\mu^{+,J}$ dominates ν_ρ.

Question: Does the above dichotomy generalize to an amenable/nonamenable dichotomy?

Theorem 2 uses certain basic properties of the Ising model together with the following theorem.

Theorem 7. (Liggett & S.): Let μ be a translation invariant measure on $\{0,1\}^{Z^2}$ which satisfies the *downward FKG lattice condition. Then the following are equivalent.

(1). μ dominates ν_ρ.

(2). $\mu\{\eta \equiv 0 \text{ on } [1,n]^2\} \leq (1-\rho)^{n^2}$ for all n.

(3). For all disjoint, finite subsets A and B of "the past", we have

$$\mu\{\eta((0,0)) = 1 \mid \eta \equiv 0 \text{ on } A, \eta \equiv 1 \text{ on } B\} \geq \rho.$$

*downward FKG lattice condition means that if you condition that some of the variables are 0, then the others are "positively associated". (In the (stronger) usual FKG lattice condition, you can condition on some of the variables to be anything.)

Why the crazy definition? Not just because that is what is needed to make the proof work?

The **upper invariant measure** for the **contact process** does not satisfy the FKG lattice condition (Liggett) but does satisfy the downwards FKG lattice condition (van den Berg, Häggström and Kahn).

Theorem 8. (Liggett & S.): For large infection rates, the upper invariant measure for the contact process dominates high density product measures and hence (for $d \geq 2$) percolates. (**There is an open question here concerning the voter model.**)

In Theorem 7, we said that for measures μ on $\{0,1\}^{Z^2}$ (which satisfy certain conditions), μ dominates ν_ρ if and only if
$\mu\{\eta \equiv 0 \text{ on } [1,n]^2\} \leq (1-\rho)^{n^2}$ for all n.

Similar to before, it is not sufficient that $\mu\{\eta \equiv 1 \text{ on } [1,n]^2\} \geq \rho^{n^2}$ for all n. Interesting, for processes which satisfy conditional negative association, it is sufficient. (R. Lyons and S.)

Let P be a transition matrix for a Markov chain with state space S which is reversible w.r.t. the distribution π.

A tree-indexed Markov chain governed by the matrix P is a process $\{X_v\}_{v \in V(T)}$ taking values in S where the root has distribution π and then we use the matrix P outwards.

Remarks:

1. On each line through the tree, we see a copy of our Markov chain.
2. The plus and minus states for the Ising model are tree indexed Markov chains.

Theorem 9. (Liggett & S.):

If P is a transition matrix for a 2-state Markov chain satisfying $P(0,1) \leq P(1,1)$ (FKG), then the following are equivalent.

(1). μ_P dominates ν_ρ.
(2). $\mu_P\{\eta \equiv 0 \text{ on } T_n\} \leq (1-\rho)^{|T_n|}$ for all n.
(3). $P(0,1) \geq \rho$.

Theorem 10. (Liggett & S.):

If P and Q are transition matrices for two 2-state Markov chains, then the following are equivalent.

(1). μ_P dominates μ_Q.
(2). $P(0,1) \geq Q(0,1)$ and $P(1,1) \geq Q(1,1)$

The Ising model results can all be obtained from these by calculation.

Some words about proofs.

Theorem 2. (Liggett & S.): For all J,

$$\rho_{\text{max}}(\mu^-) = \rho_{\text{max}}(\mu^+).$$

follows fairly easily from

Theorem 7. (Liggett & S.): Let μ be a translation invariant measure on $\{0,1\}^{Z^2}$ which is satisfies *downward FKG lattice condition. Then the following are equivalent.
(1). μ dominates ν_ρ.
(2). $\mu\{\eta \equiv 0 \text{ on } [1,n]^2\} \leq (1-\rho)^{n^2}$ for all n.
(3). For all disjoint, finite subsets A and B of "the past", we have

$$\mu\{\eta((0,0)) = 1 \mid \eta \equiv 0 \text{ on } A, \eta \equiv 1 \text{ on } B\} \geq \rho.$$

basically because the probability of getting all -1's in a big box has the same exponential decay rate under the plus and minus states.

For Theorem 7, (1) implies (2) and (3) implies (1) do not need any assumptions; the first is trivial and the second is almost trivial.

For (2) implies (3), we consider Z (where the past is clear).

By downwards FKG,

$$\mu\{\eta(0) = 1 \mid \eta \equiv 0 \text{ on } A, \eta \equiv 1 \text{ on } B\} \geq$$

$$\mu\{\eta(0) = 1 \mid \eta \equiv 0 \text{ on } A \cup B\}$$

and the latter is decreasing in the set $A \cup B$. So need to show that

$$\lim_n \mu\{\eta(0) = 0 \mid \eta \equiv 0 \text{ on } [-n, -1]\} \leq (1 - \rho).$$

But

$$(\mu\{\eta(0) = 0 \mid \eta \equiv 0 \text{ on } [-\infty, -1]\})^n \sim$$

$$\mu\{\eta \equiv 0 \text{ on } [1, n]\} \leq (1 - \rho)^n.$$

QED

Thank you for your attention!

Outline of proof of

Theorem 1. (Liggett & S.): Let $X := \{X_i\}_{i \in I}$ be i.i.d. "with density ρ" and $Y := \{Y_i\}_{i \in I}$ be permutation invariant and satisfy the FKG lattice condition. Then (**) implies that $X \preceq Y$.

Letting

$u_i = P(Y_1 = \cdots = Y_i = 1, Y_{i+1} = \cdots = Y_n = 0), \quad 0$

FKG implies

$$u_i^2 \leq u_{i-1}u_{i+1}, \quad 0 < i < n.$$

This gives (after a little work) that the sequence $v_i := \frac{u_i}{\rho^i(1-\rho)^{n-i}}$ is convex.

We need to prove that for all k,

$$\sum_{i=k}^{n} \binom{n}{i} u_i \geq \sum_{i=k}^{n} \binom{n}{i} \rho^i (1-\rho)^{n-i}.$$

If it failed for some k, we would have

$$\sum_{i=k}^{n} \binom{n}{i} \rho^i (1-\rho)^{n-i} v_i < \sum_{i=k}^{n} \binom{n}{i} \rho^i (1-\rho)^{n-i}.$$

Then $v_i \leq 1$ for some $i \geq k$. $v_0 \leq 1$ and convexity gives $v_0, \ldots, v_k \leq 1$. Hence

$$\sum_{i=0}^{n} \binom{n}{i} u_i = \sum_{i=0}^{n} \binom{n}{i} \rho^i (1-\rho)^{n-i} v_i < \sum_{i=0}^{n} \binom{n}{i} \rho^i (1-\rho)^{n-i}$$

QED

Theorem 6. (Liggett & S.): Consider the Ising model on T and let J_c be the critical value for J.

(i). If $J_c < J_1 < J_2$, then $\mu^{J_2,+}$ dominates $\mu^{J_1,+}$.

(ii). For all $J_2 \geq J_c$, there exists $\alpha(J_2)$ such that

$$\{J \in [0, J_c] : \mu^{J_2,+} \text{ dominates } \mu^{J,+}\} = [\alpha(J_2), J_c].$$

((i) implies that α is a decreasing function of J_2). Moreover, the smallest $J_2 > J_c$ for which $\alpha(J_2) = 0$ (which corresponds to the smallest $J_2 > 0$ for which the plus state dominates all plus states at lower values of J) is $\log(r)$ where r is the unique real root of the cubic polynomial

$$x^3 - x^2 - x - 1.$$

(iii). For every $\rho < 1$, there exists J such that $\mu^{+,J}$ dominates ν_ρ.

Corollary. (Liggett & S.): If $Y_1, Y_2, ...$ is an infinite exchangeable Bernoulli sequence with mixing random variable W, then, for each n, $(Y_1, ..., Y_n)$ dominates the product measure with density ρ if and only if $\rho \leq 1 - ||1 - W||_n$. where $|| \cdot ||_n$ denotes the L_n norm.