Consensus Formation on Simple and Complex Networks

T. Antal, V. Sood
Sid Redner (physics.bu.edu/~redner)

100th!!! Statistical Mechanics Meeting, Rutgers, December 2008

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The classic voter model & its cousins

Voting on complex networks

- new conservation law
- two time-scale route to consensus
- short consensus time

Extensions

- zealotry, vacillation, strategic voting (>2 states)

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

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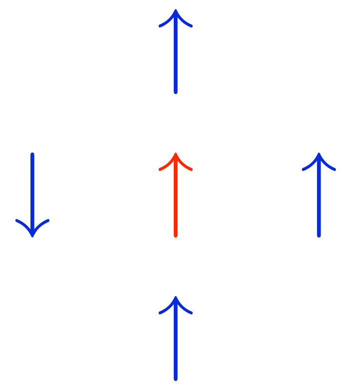

0. Binary voter variable at each site i, $\sigma_i = \pm 1$

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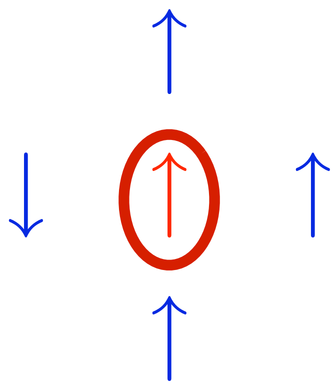

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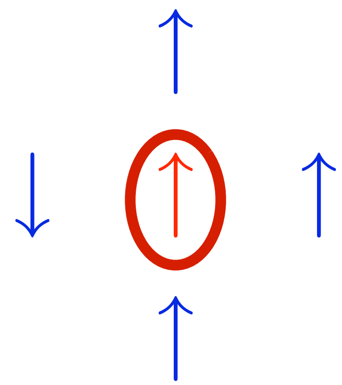

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2. Assume state of randomly-selected neighbor

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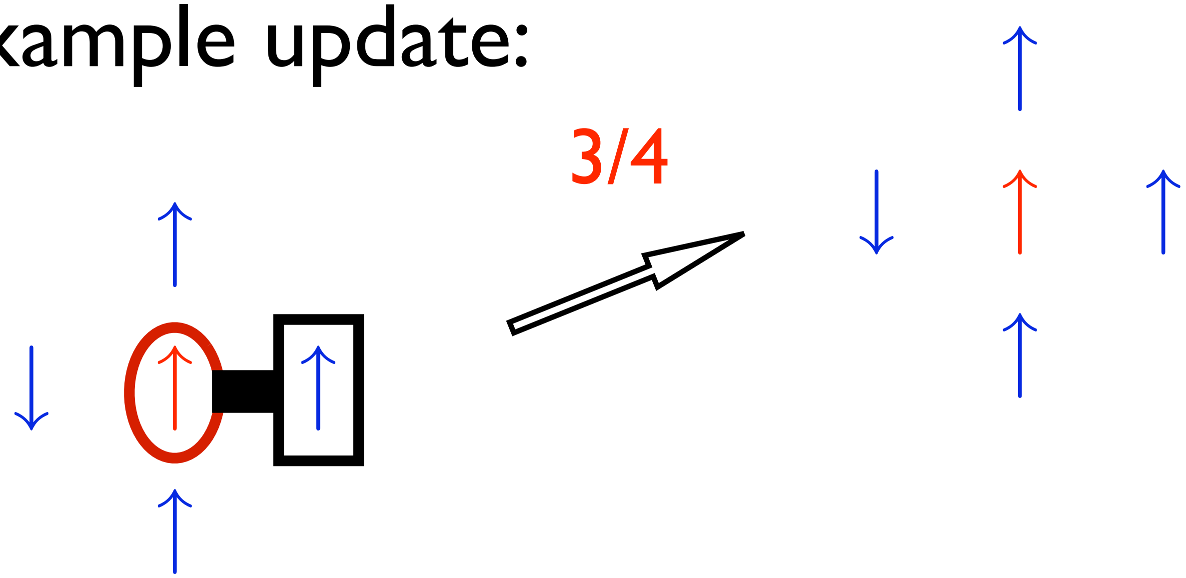
Example update:



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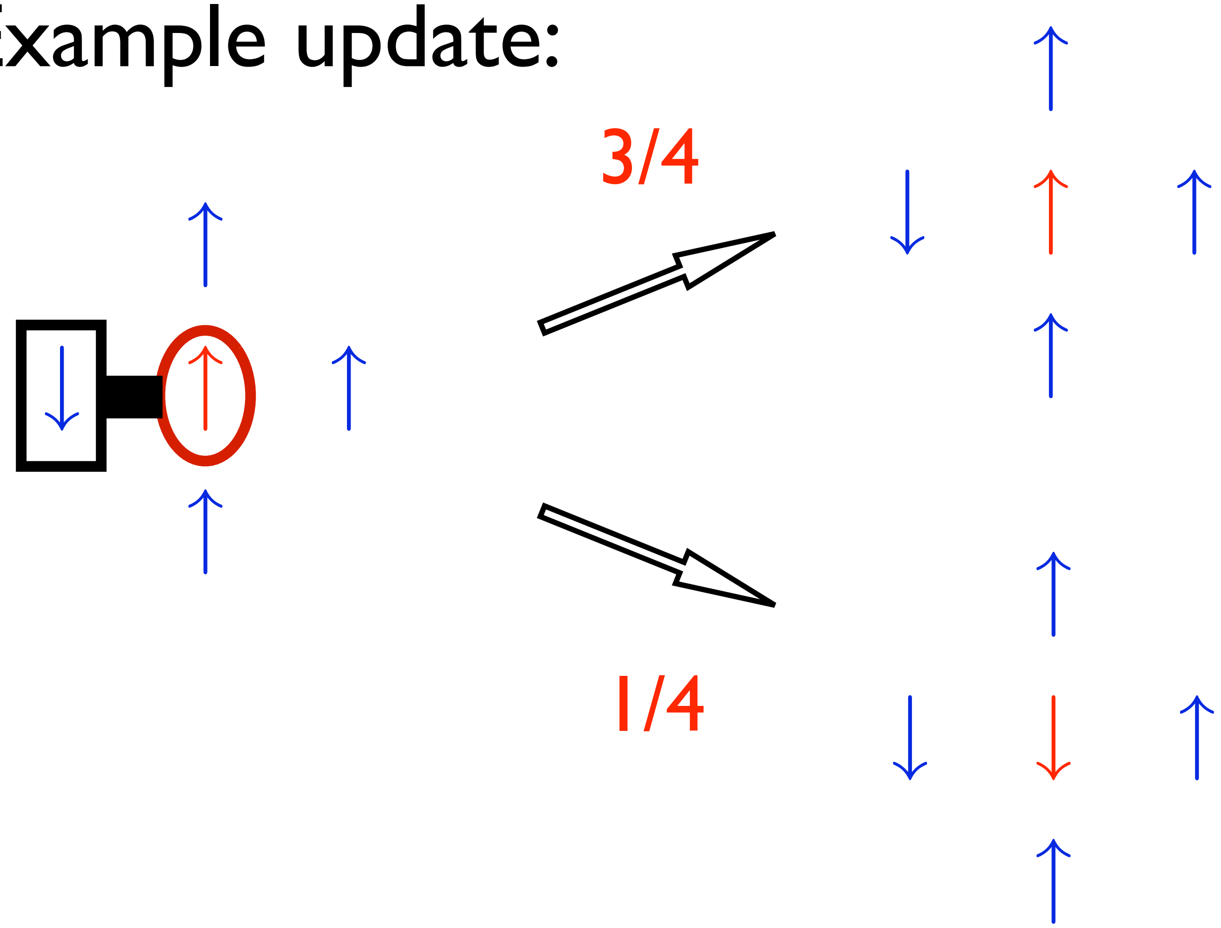

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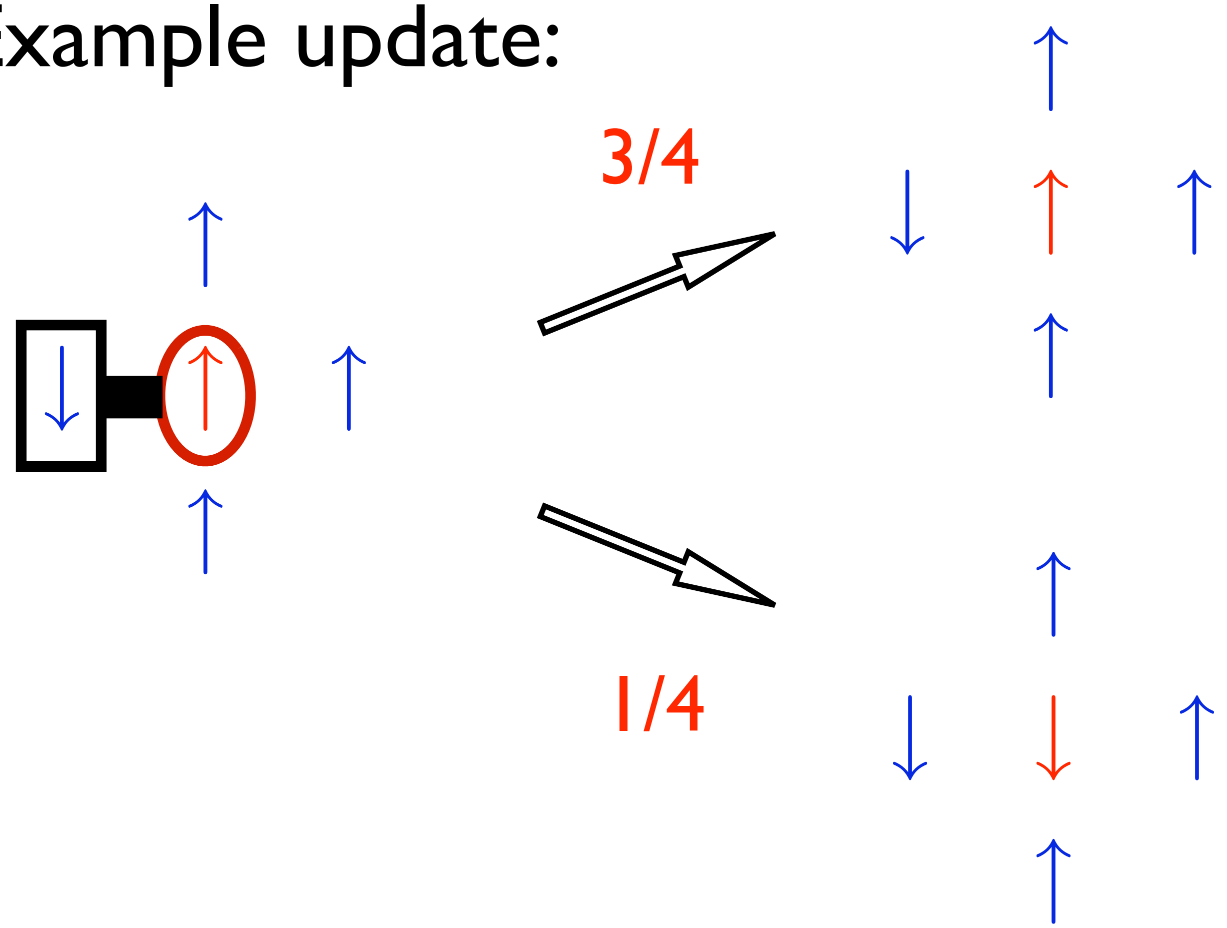

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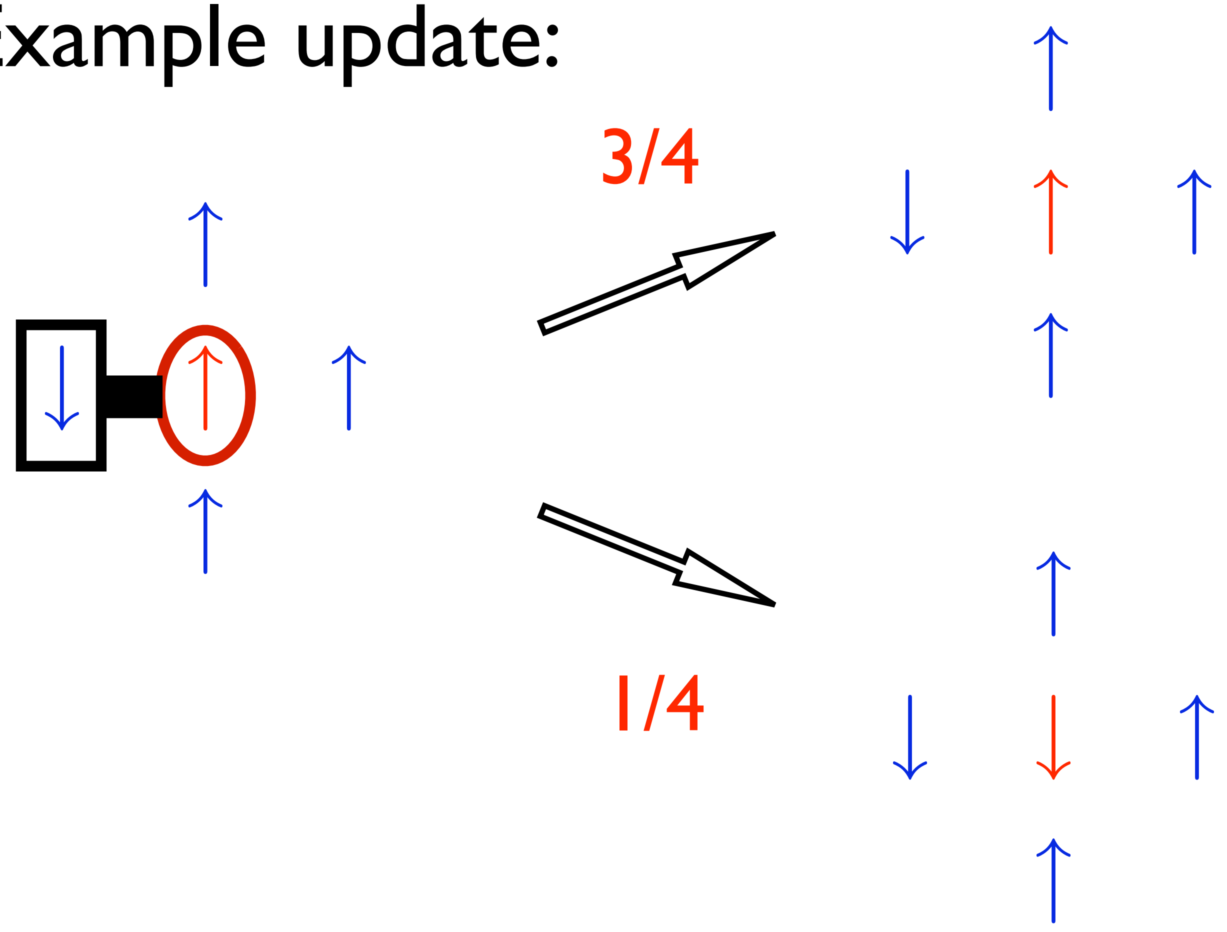

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0. Binary voter variable at each site i, $\sigma_i = \pm 1$
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system

Voter Model & Cousins

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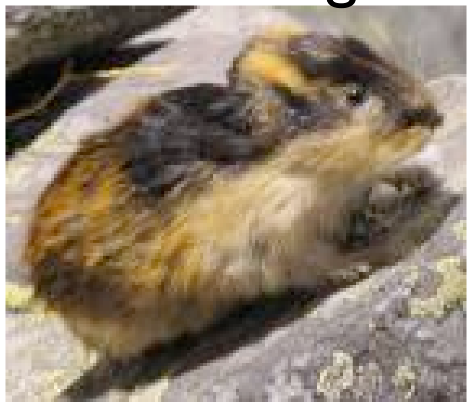
lemming



Voter Model: Tell me how to vote

Voter Model & Cousins

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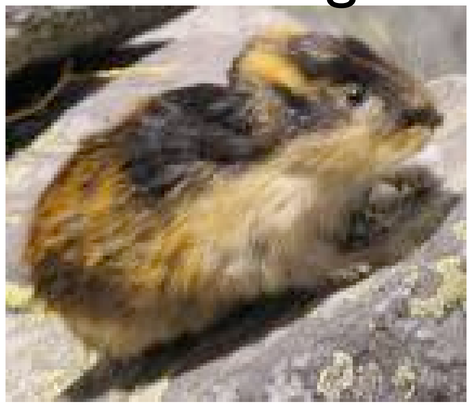

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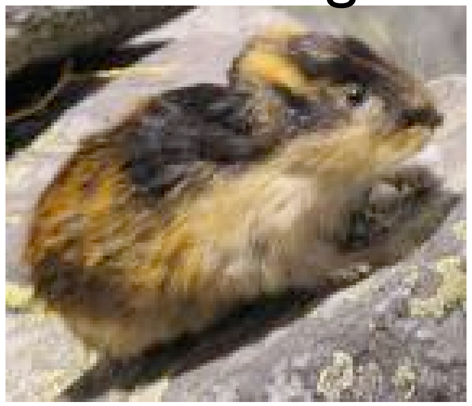

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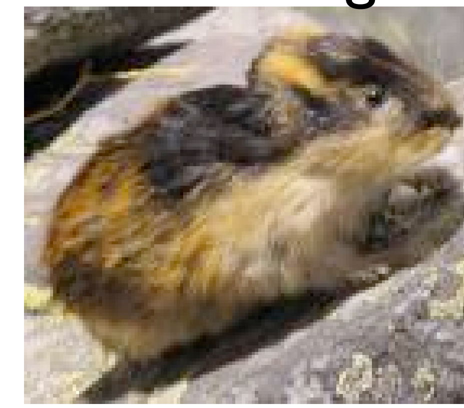

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Invasion Process: I tell you how to vote

Link Dynamics: Pick two disagreeing agents and change one at random

identical on regular lattices, *distinct* on random graphs

Suchecki, Eguiluz & San Miguel (2005), Castellano (2005), Sood & SR (2005)

Voter Model on Lattices: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0)$

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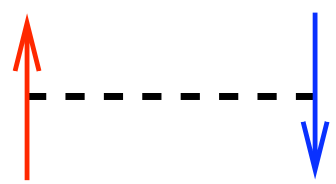

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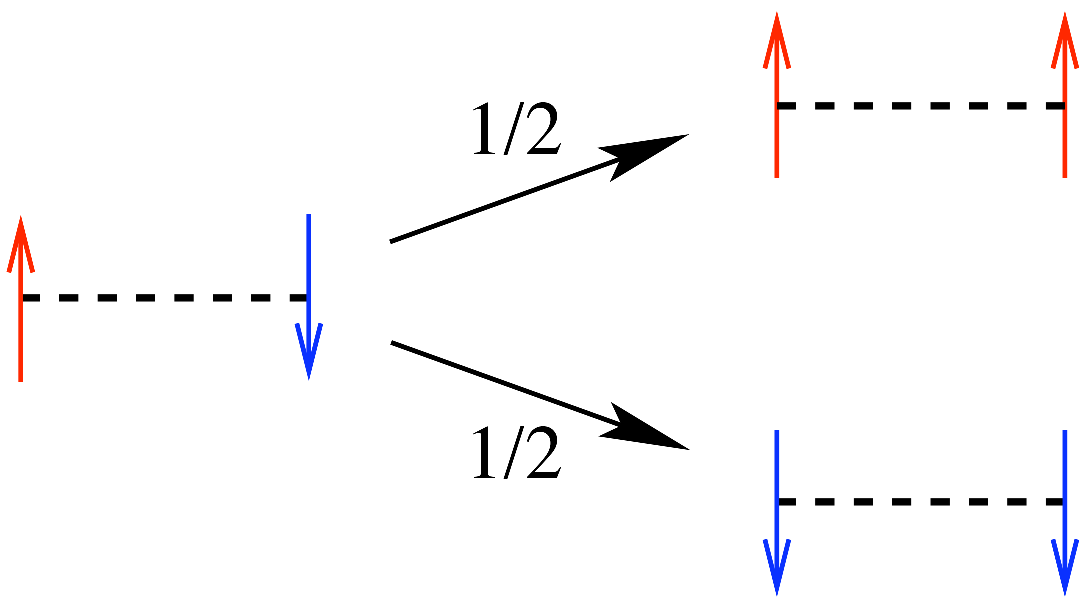

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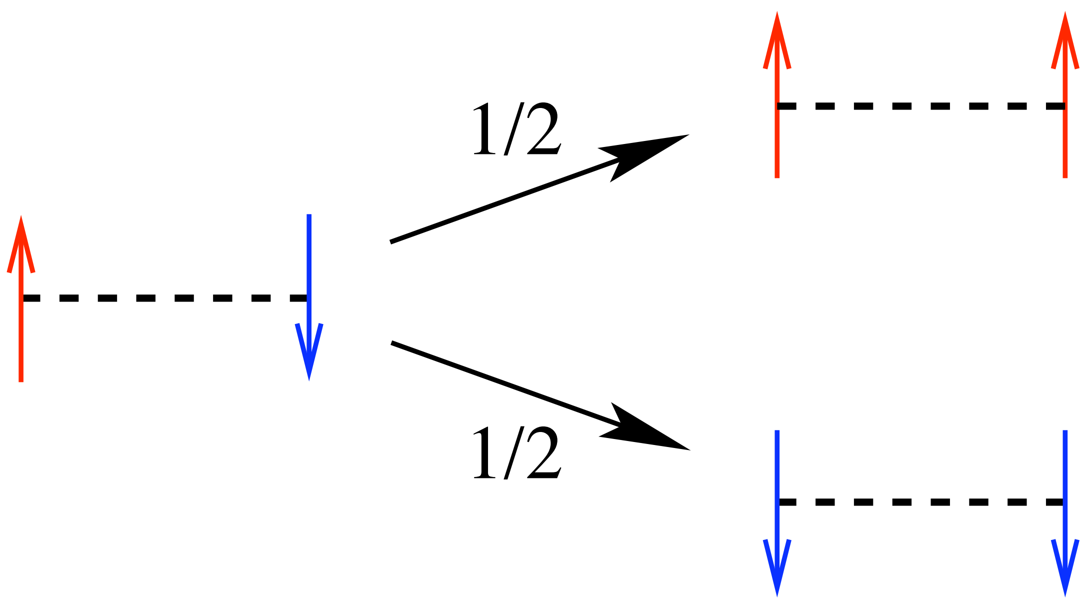

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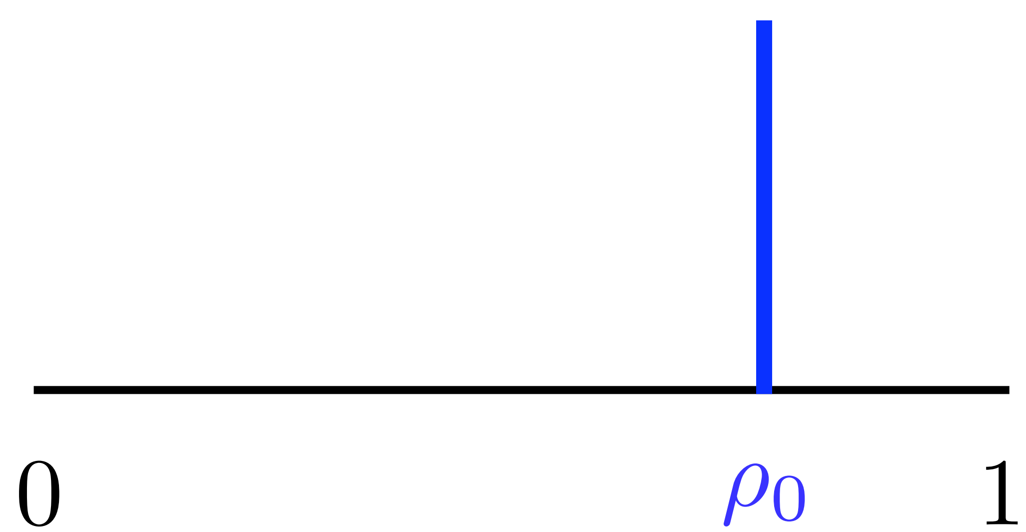

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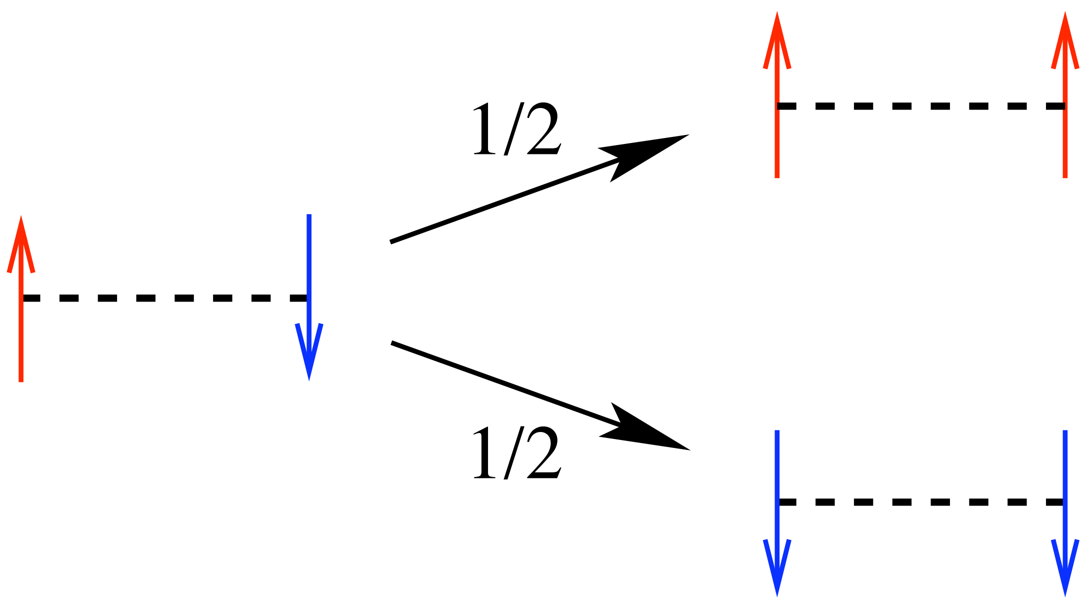

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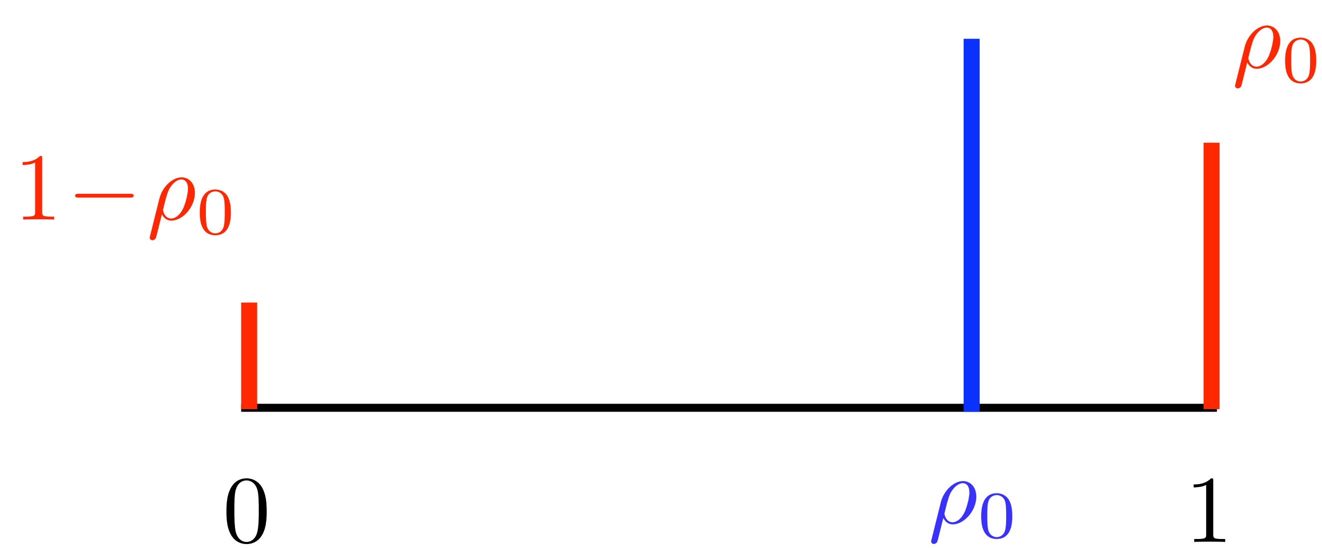

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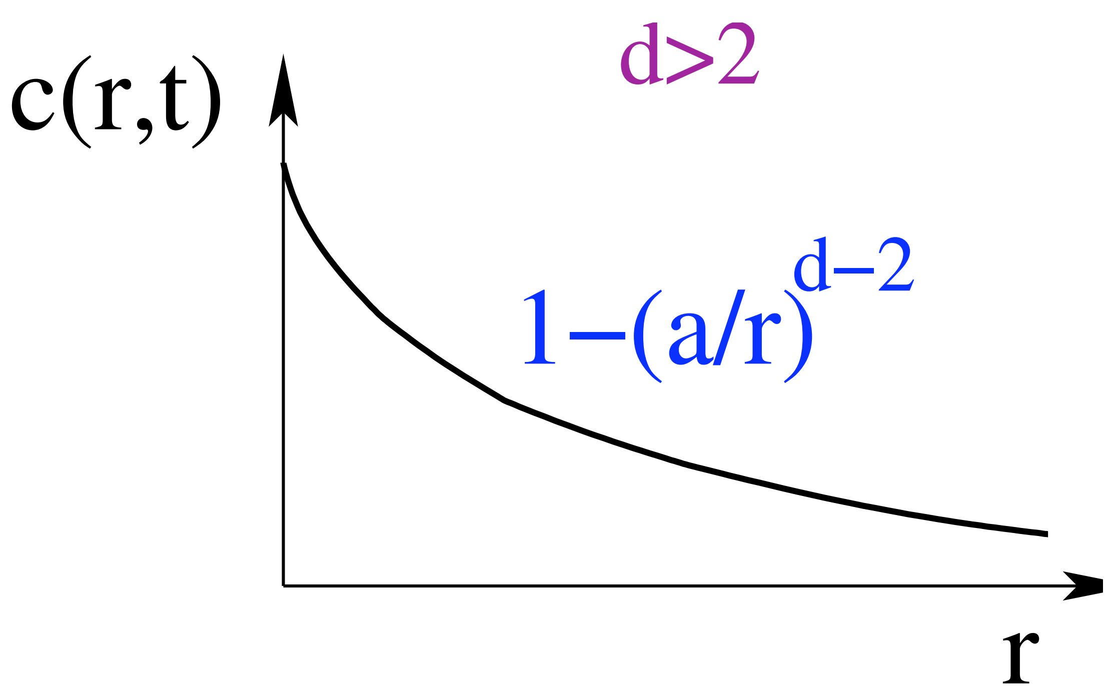

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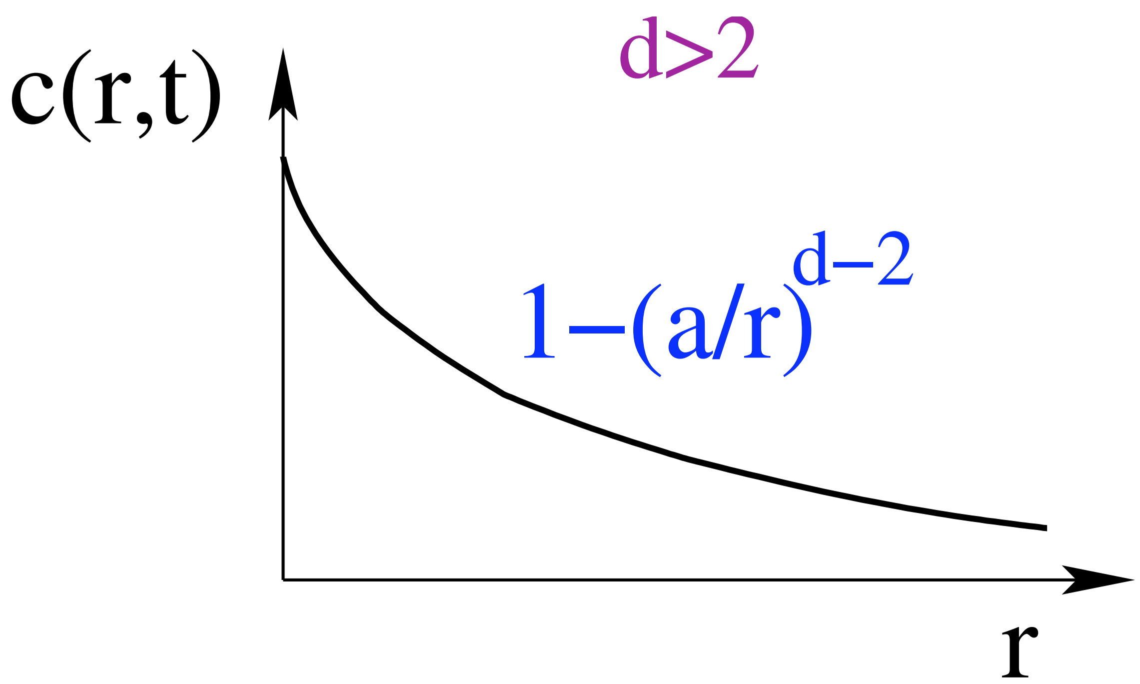

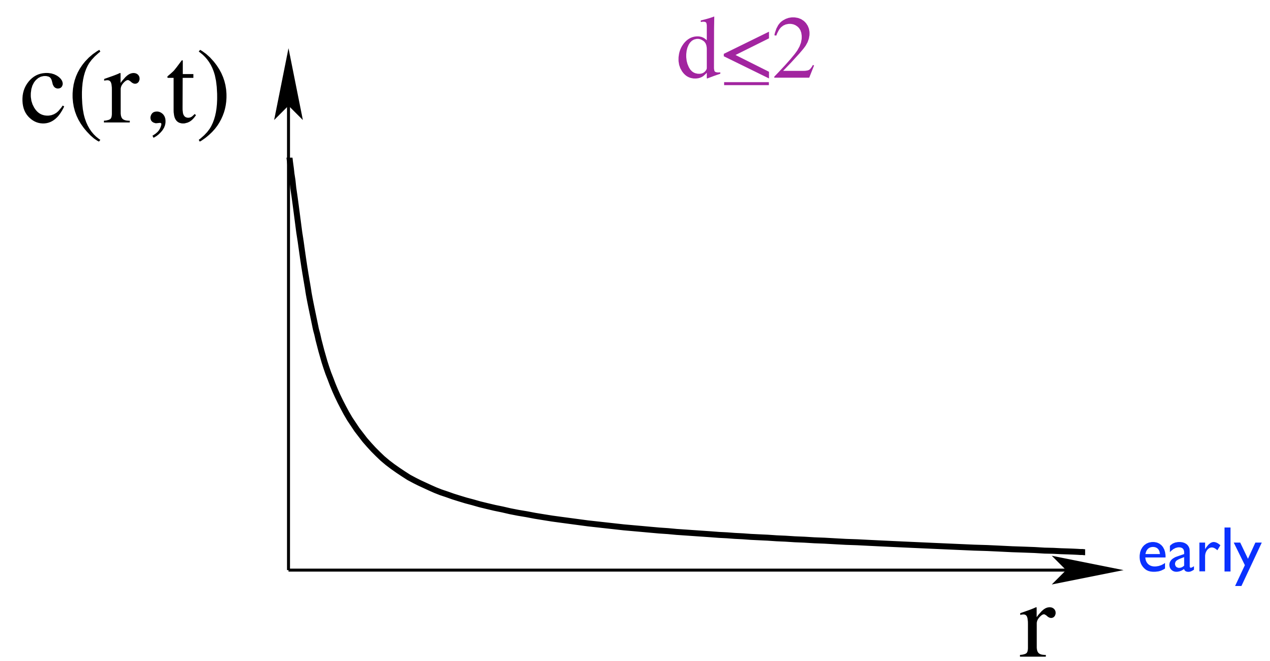

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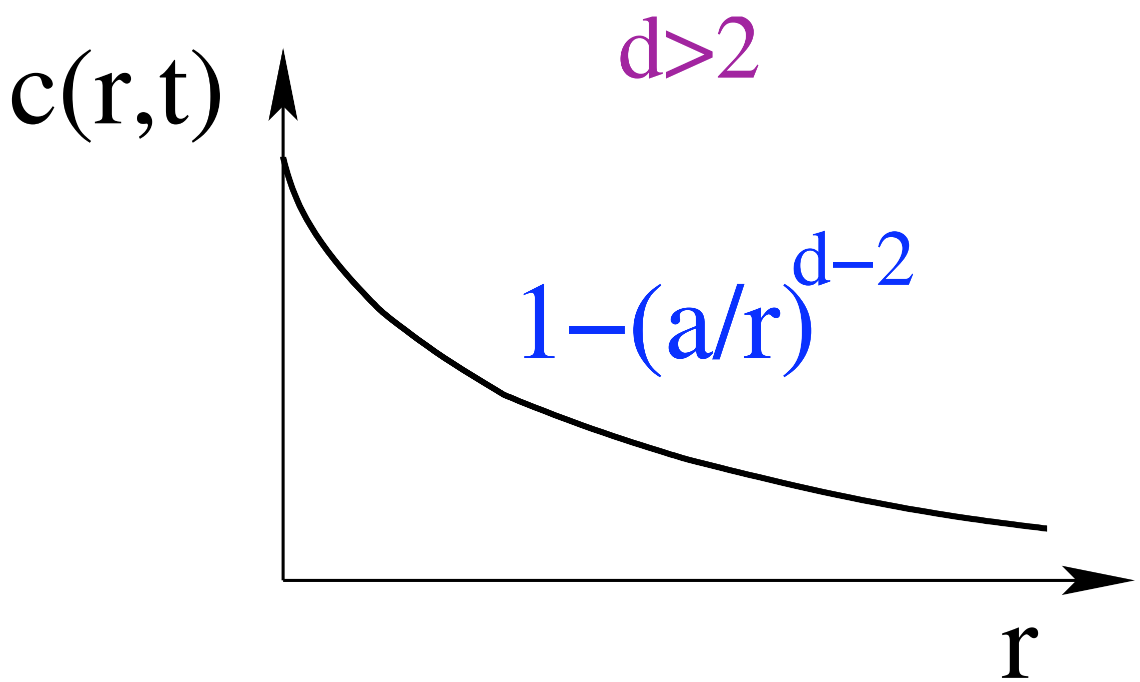

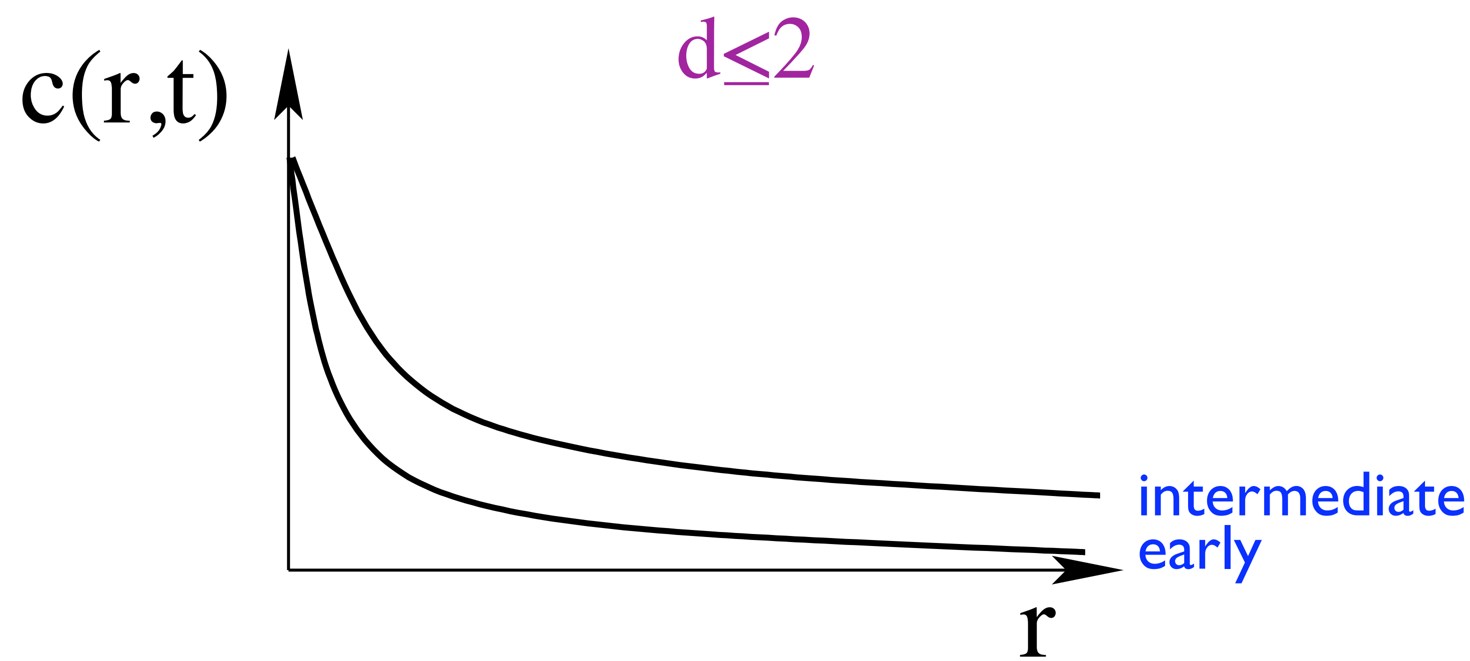

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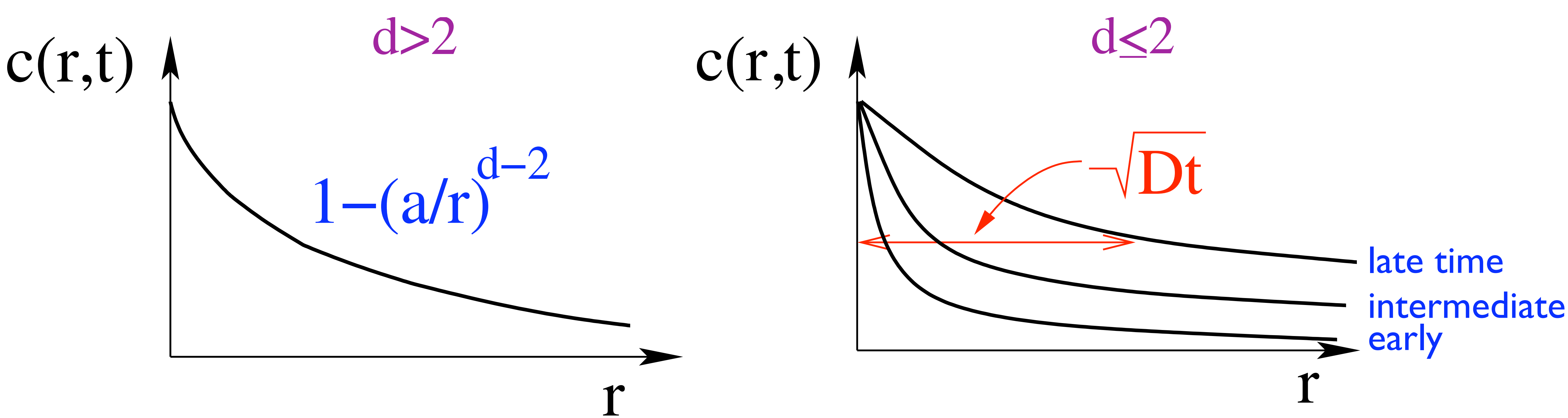

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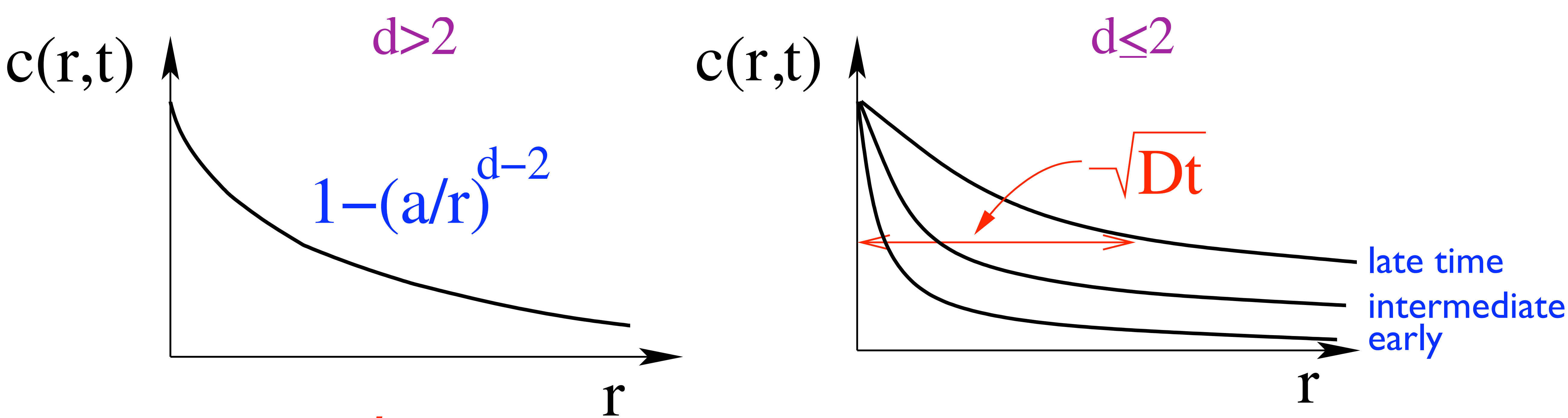

steady state

coarsening

Voter Model on Lattices: 3 Basic Properties

3. System Size Dependence of Consensus Time

Liggett (1985), Krapivsky (1992)

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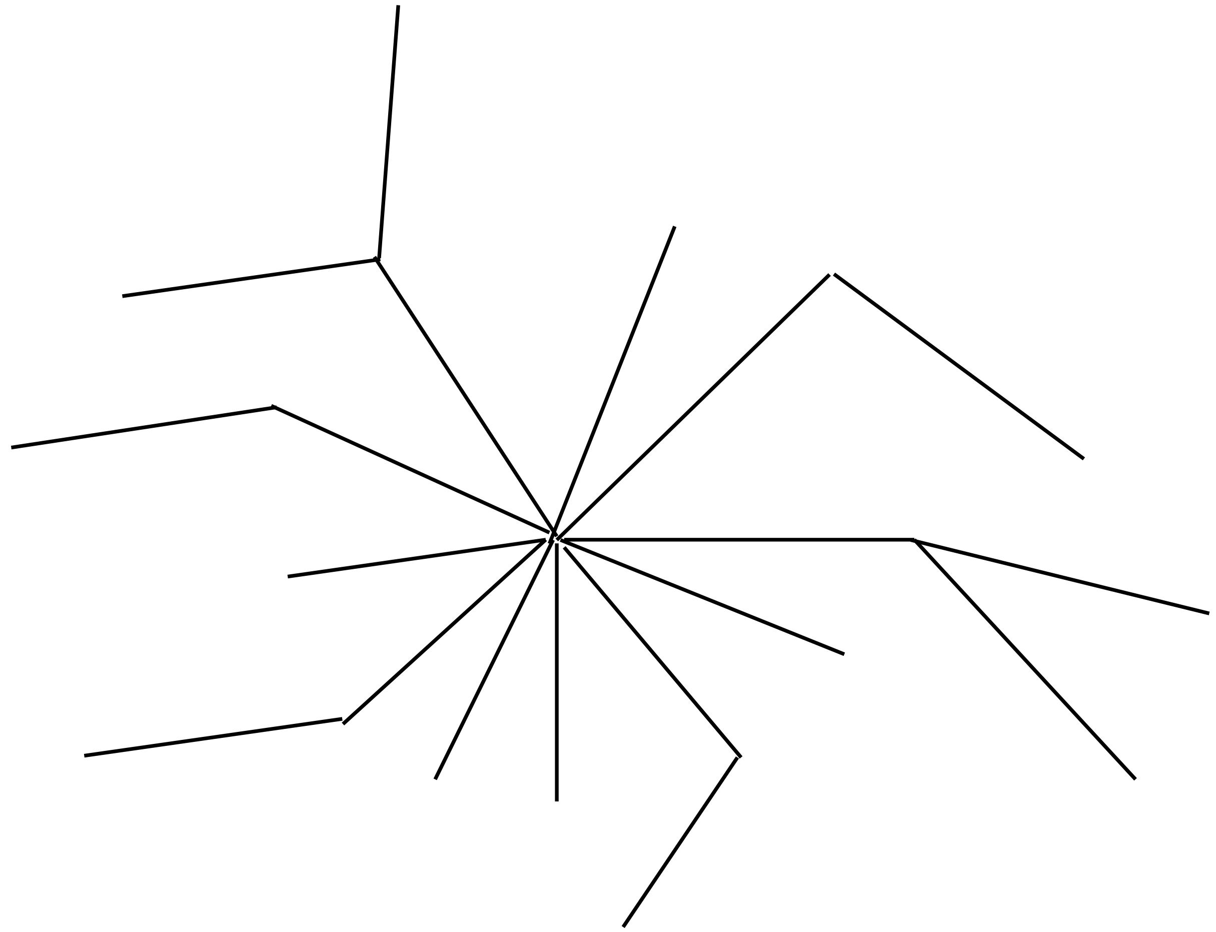
Liggett (1985), Krapivsky (1992)

$$\int^{\sqrt{Dt}} c(r,t)\, r^{d-1}\, dr = N$$

dimension	consensus time
1	N^2
2	N ln N
>2	N

Voter Model on Complex Networks

Suchecki, Eguiluz & San Miguel (2005)
Antal, Sood, SR (2005, 06, 08)



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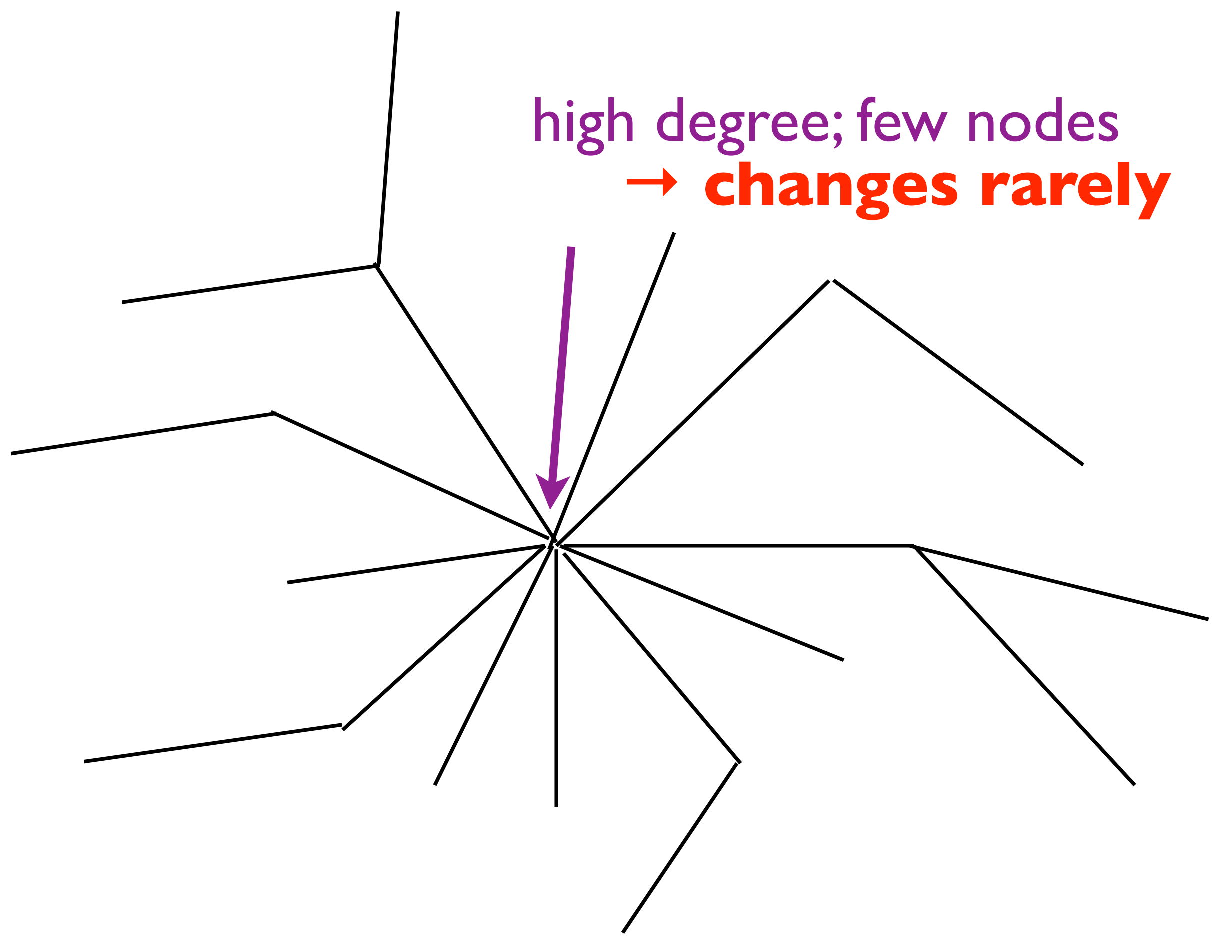

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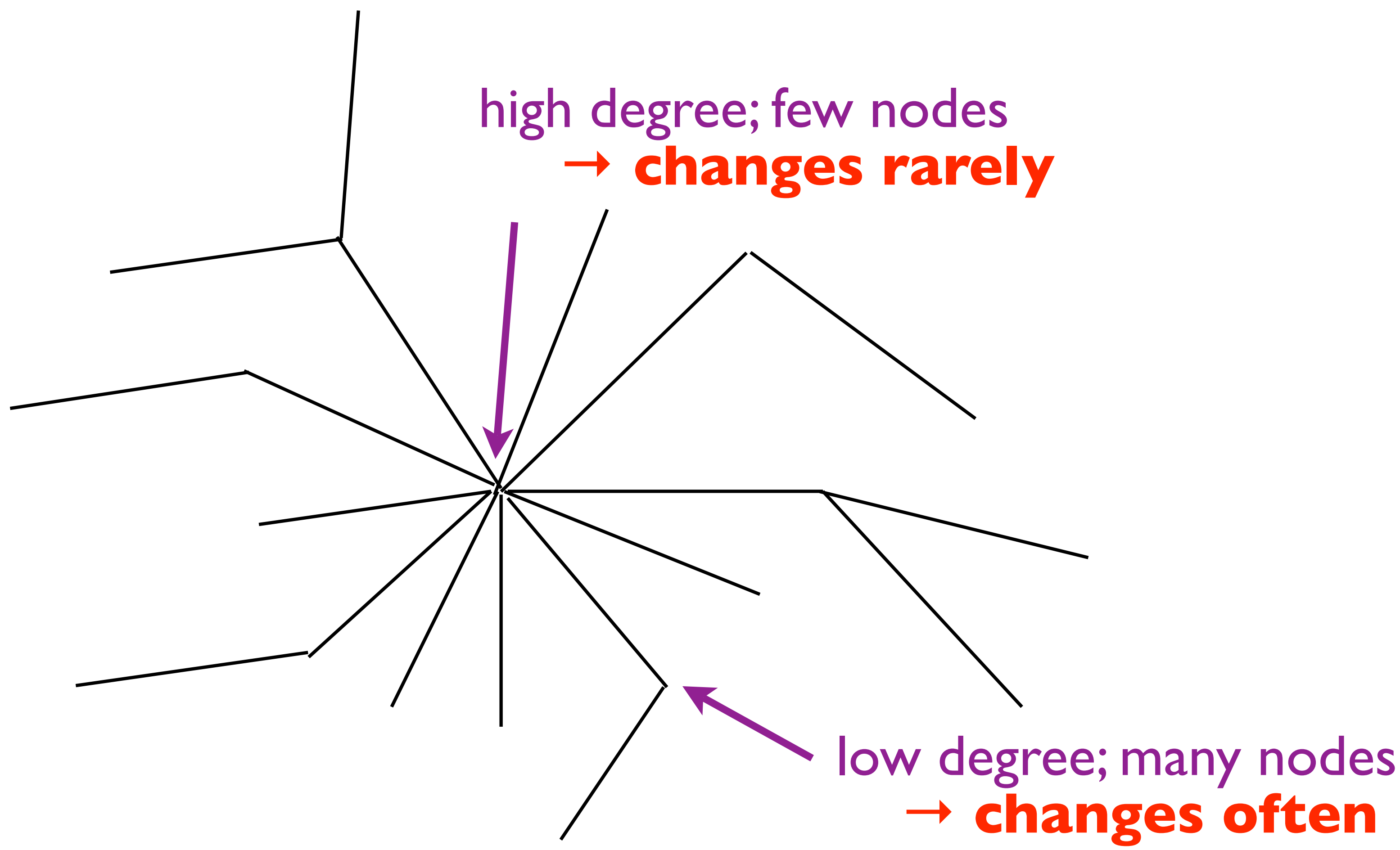

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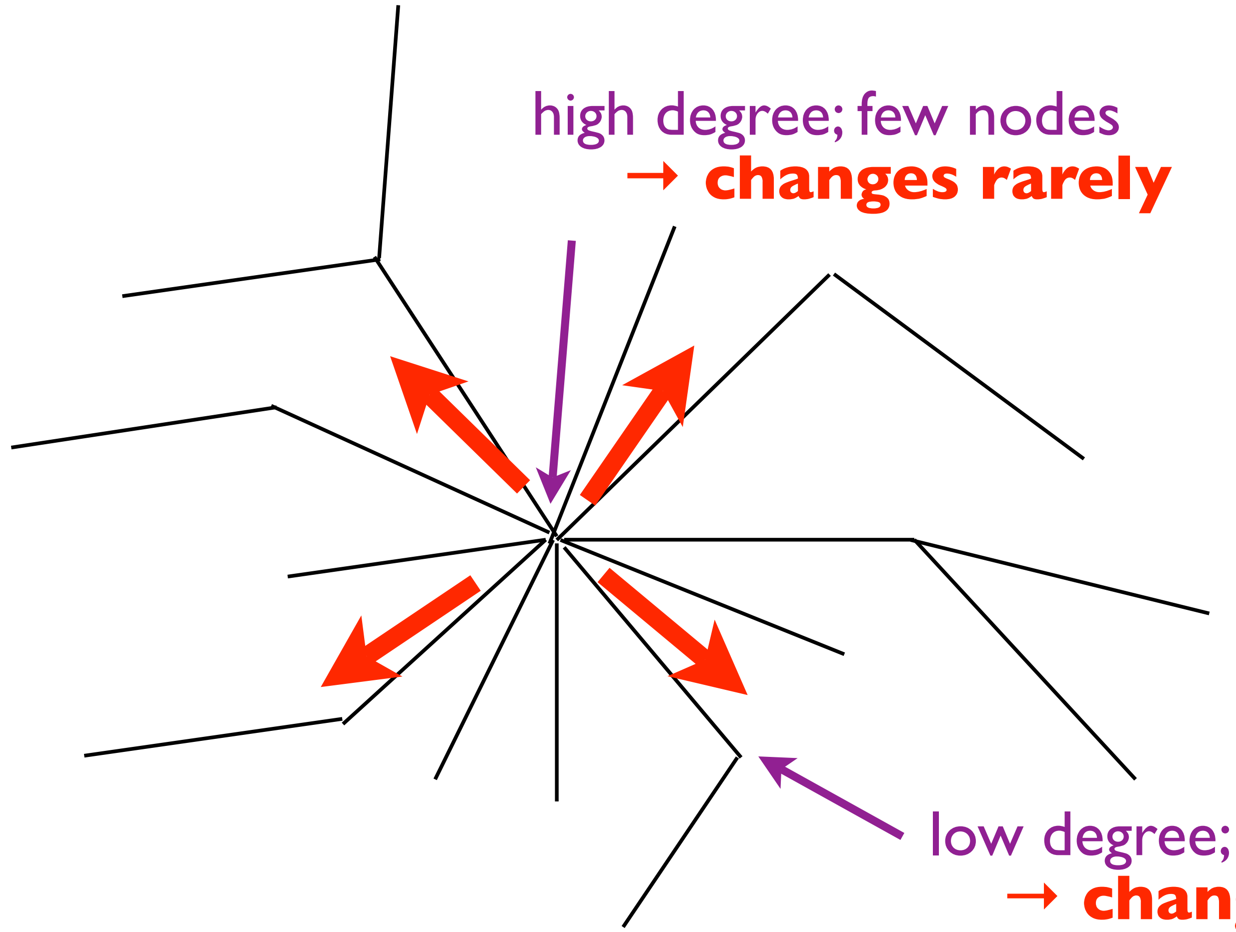

*"flow" from **high** degree to **low** degree*

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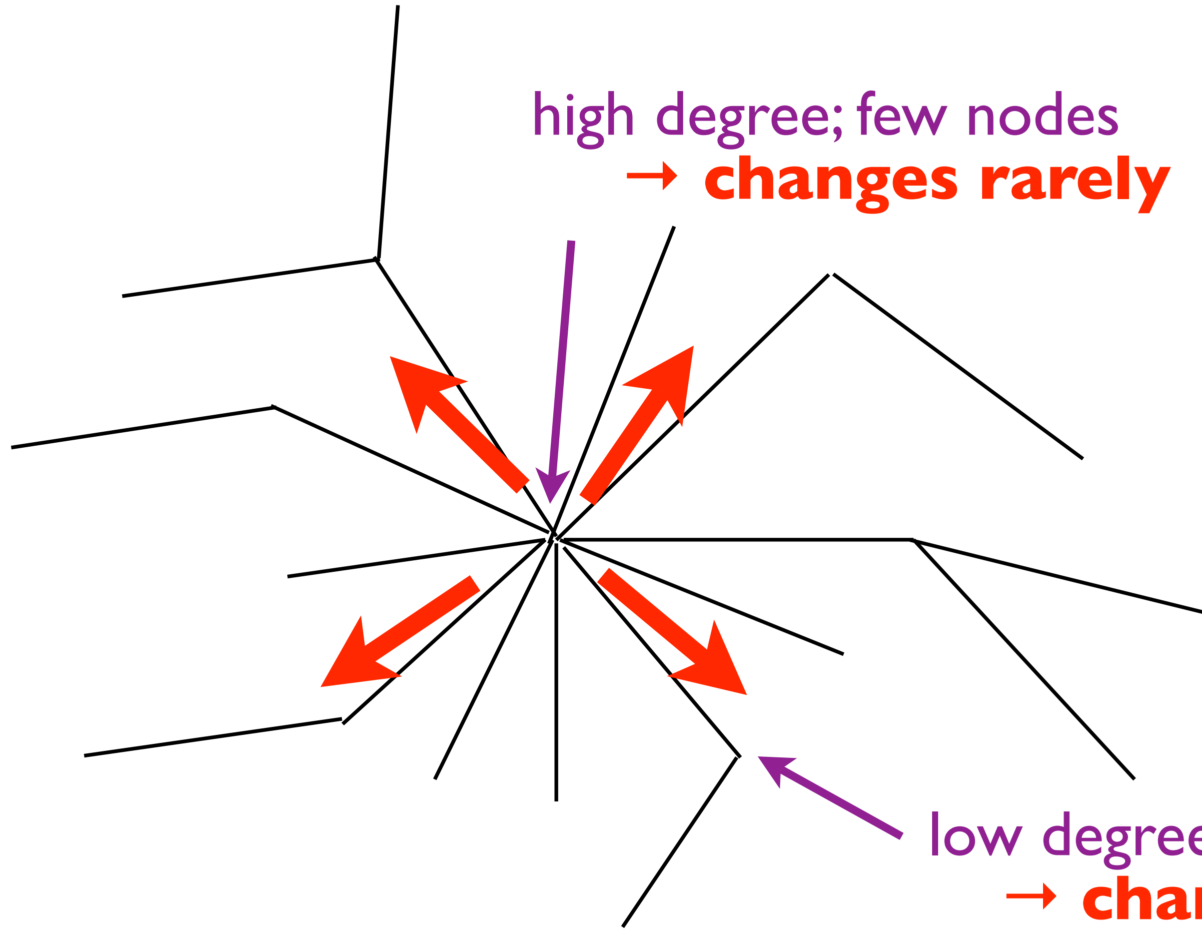


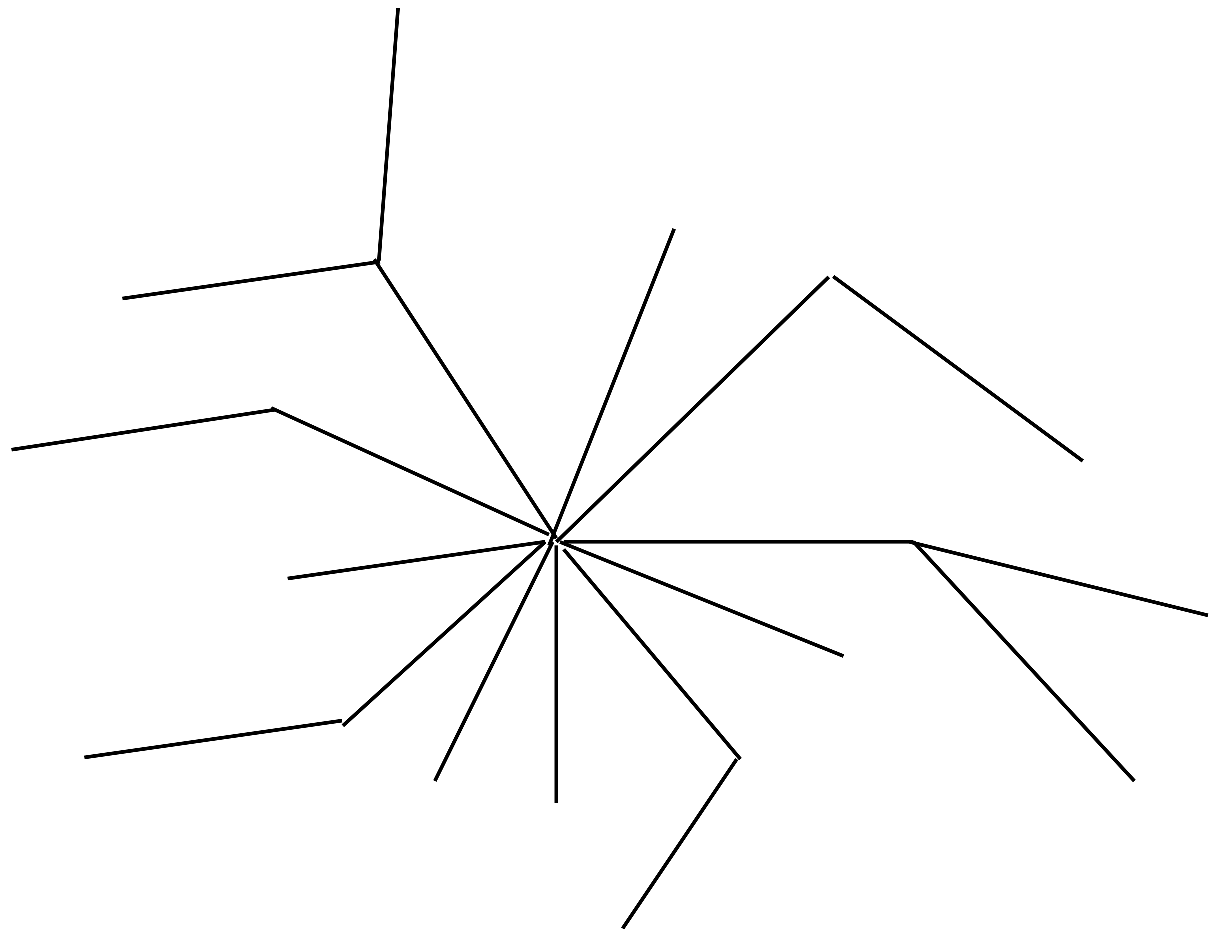
*"flow" from **high** degree to **low** degree*

degree-weighted 1st moment: $$\omega_1 = \frac{1}{N\mu_1}\sum_x k_x\,\eta(x)$$ *conserved!*

Invasion Process on Heterogeneous Networks

Castellano (2005)

Antal, Sood, SR (2005, 06, 08)



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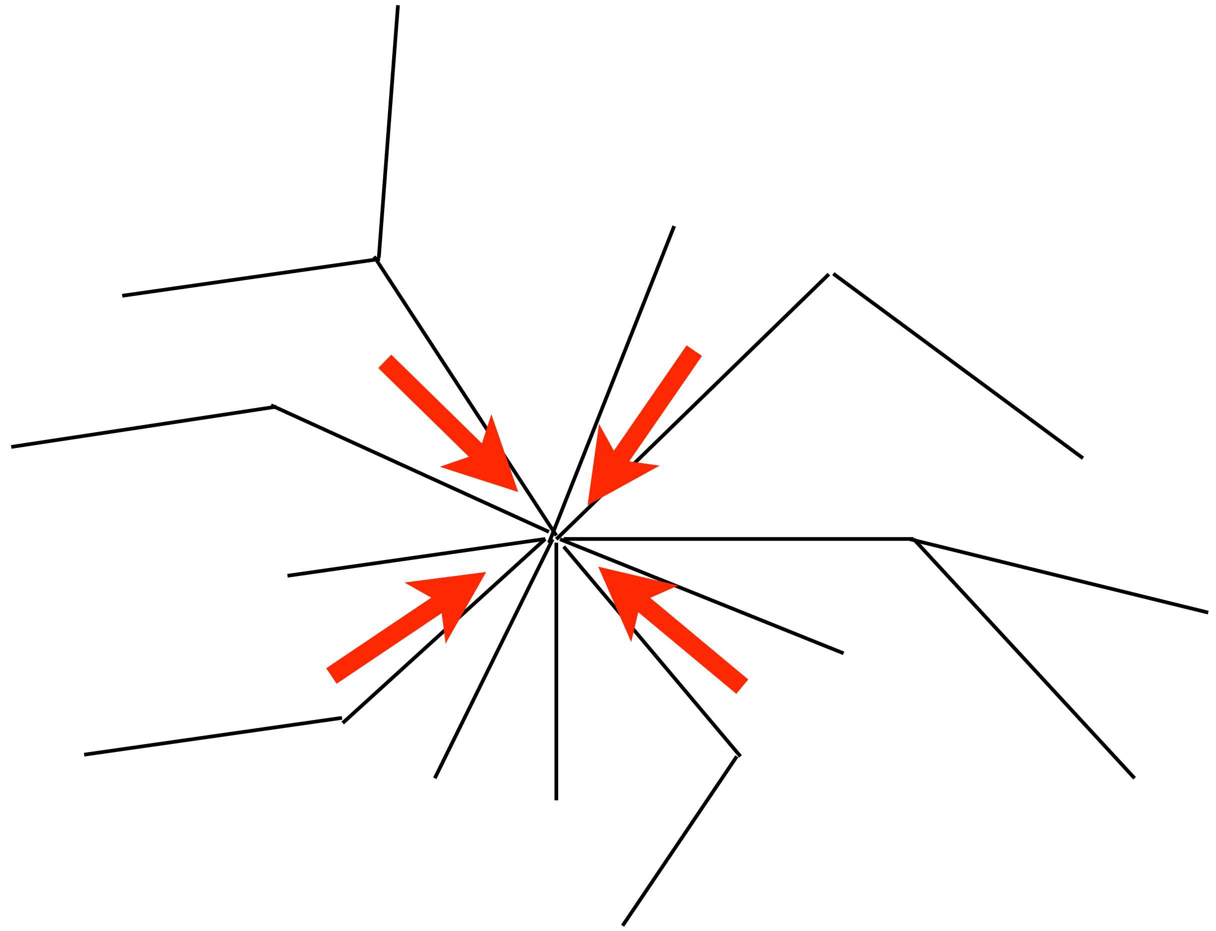

*"flow" from **low** degree to **high** degree*

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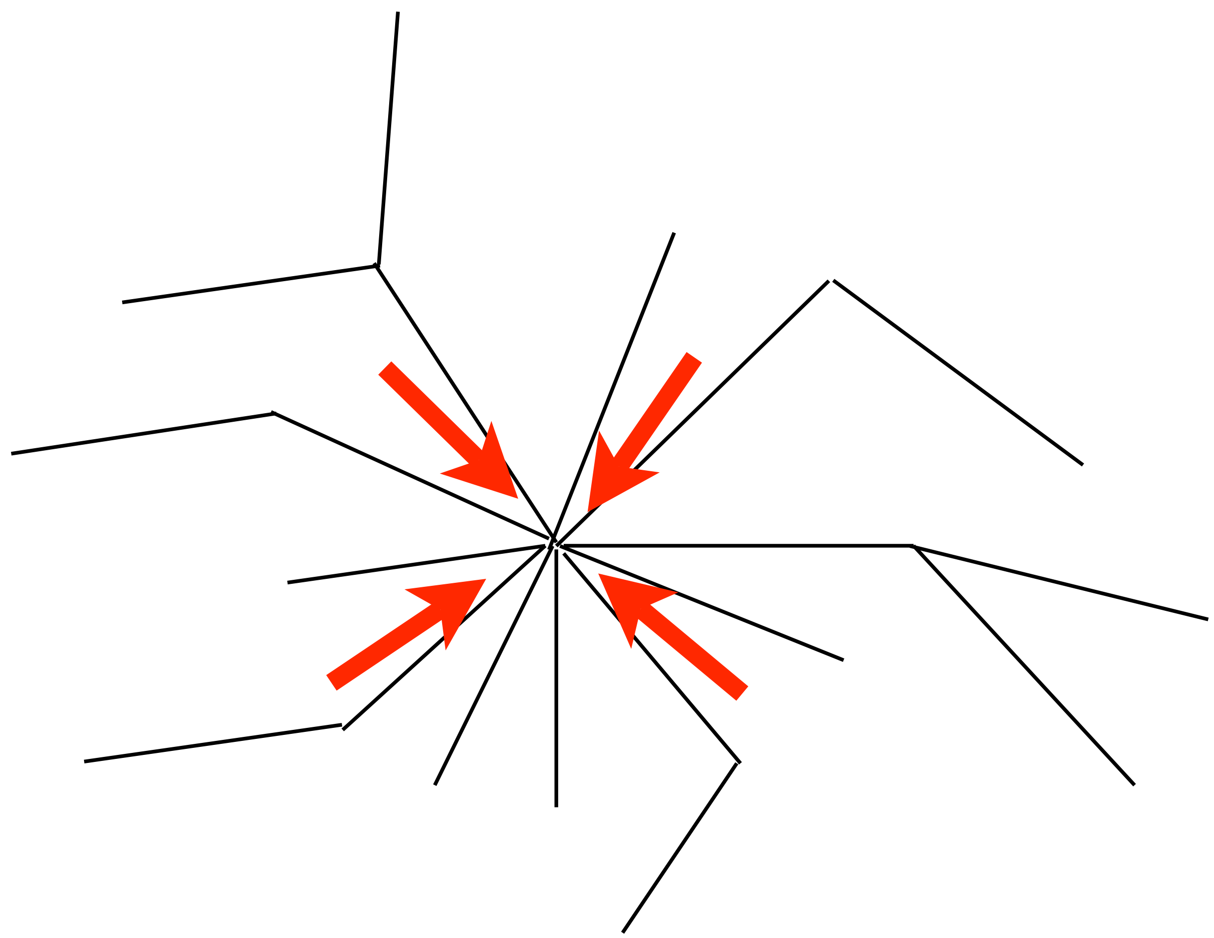

*"flow" from **low** degree to **high** degree*

degree-weighted inverse moment $$\omega_{-1} = \frac{1}{N\mu_{-1}} \sum_x k_x^{-1} \eta(x)$$ *conserved!*

Voter Model Exit Probability on Complex Graphs

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$$\mathcal{E}(\omega) = \omega$$

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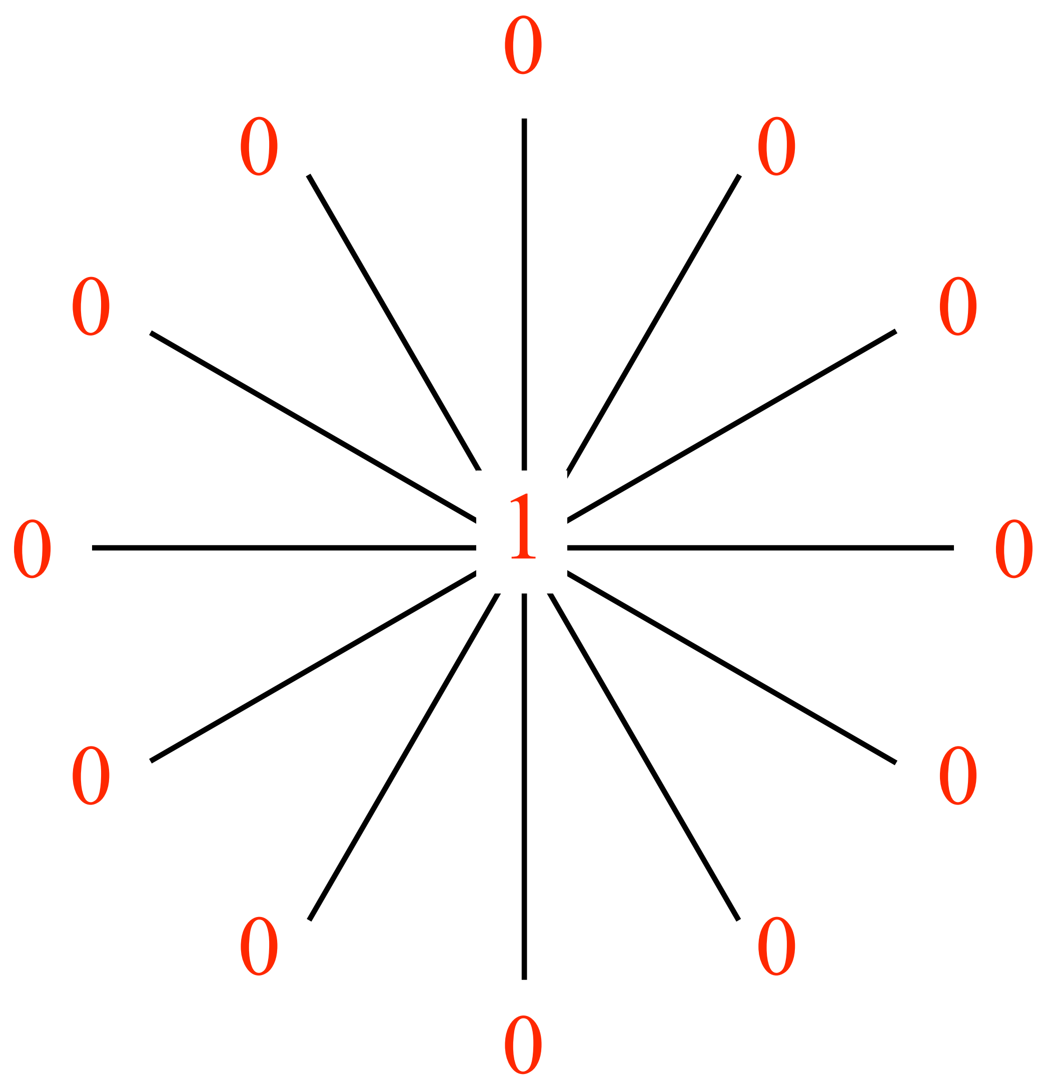
Extreme case: star graph

N nodes: degree 1
1 node: degree N



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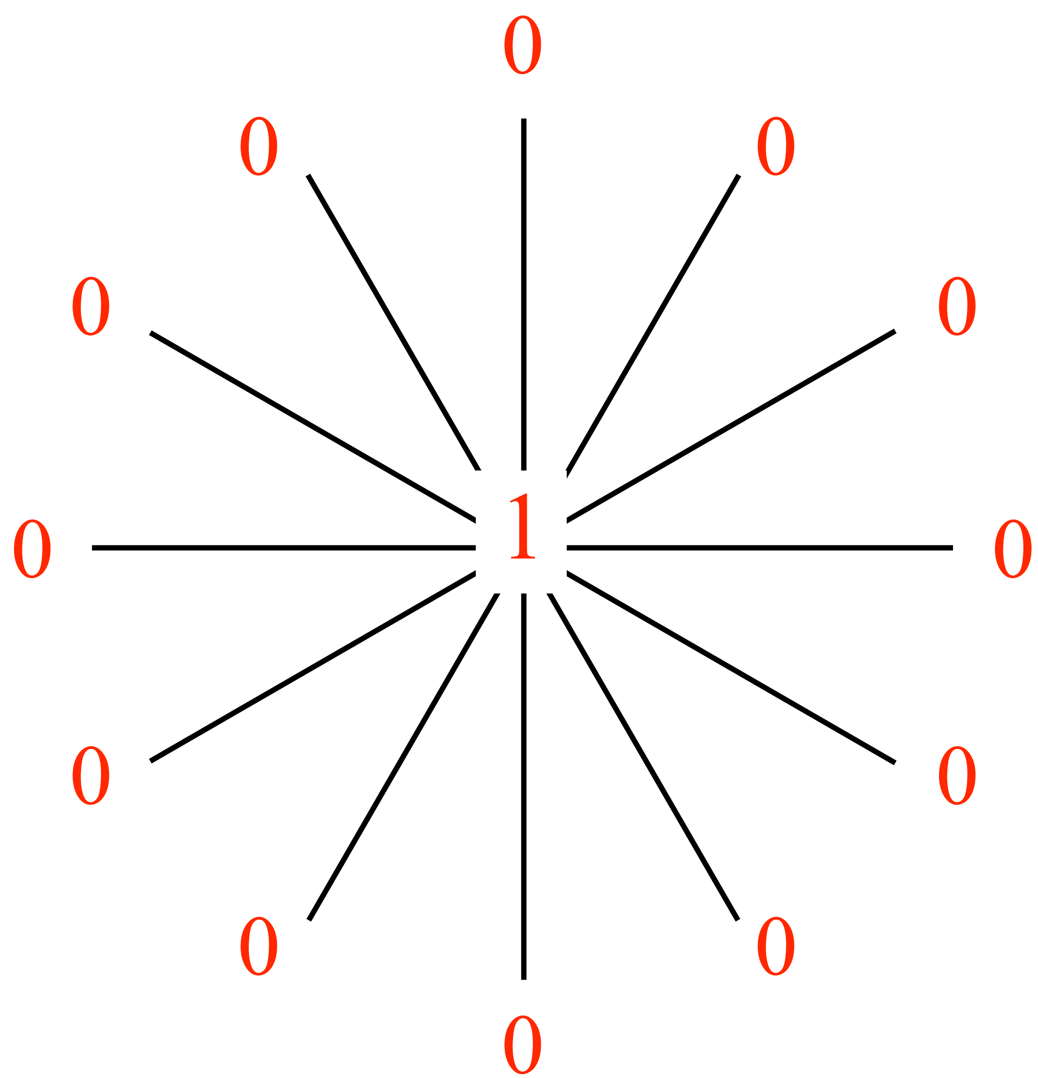
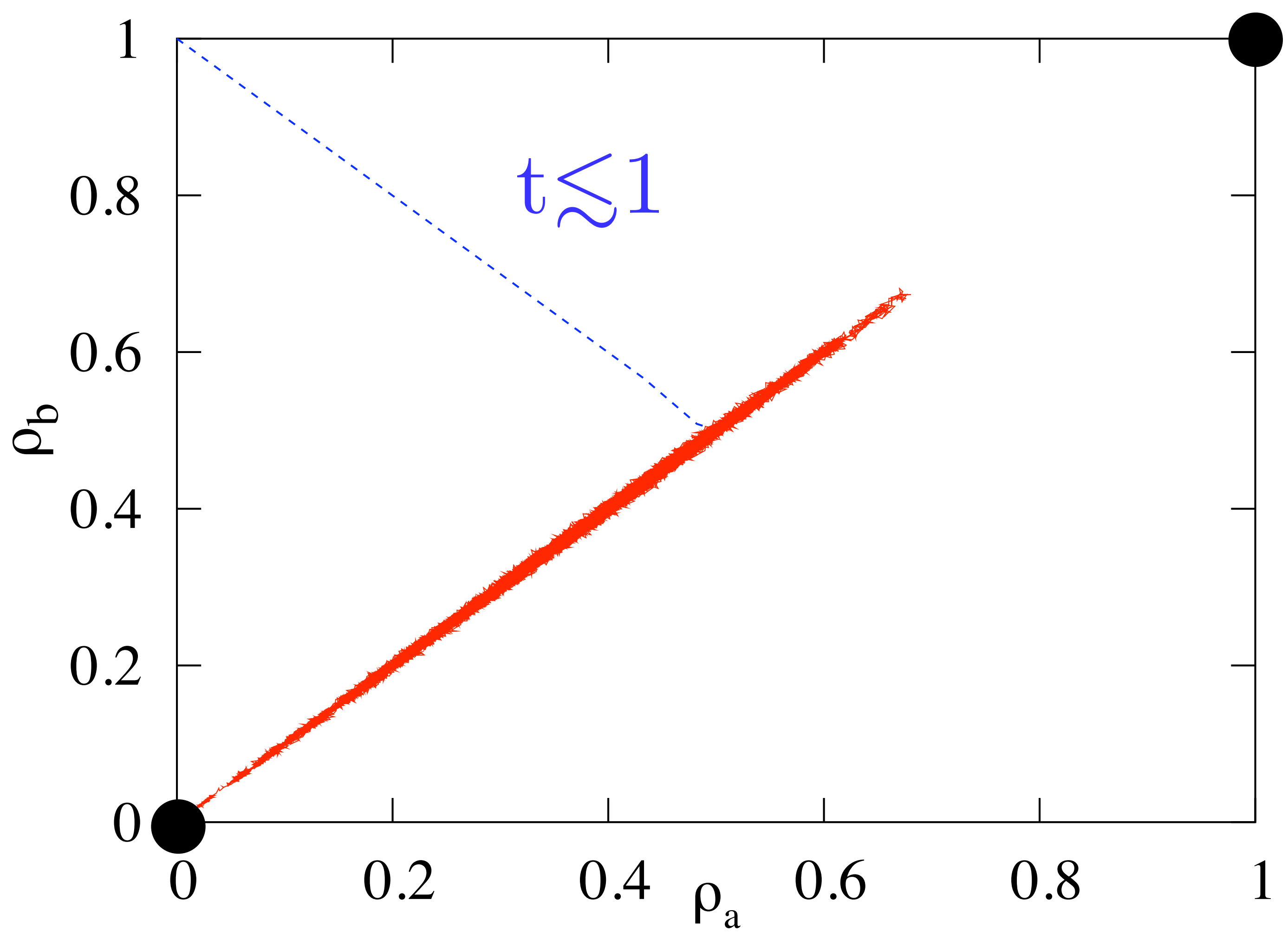


$$\omega = \frac{1}{N\mu_1}\sum_x k_x\,\eta(x) \;=\; \frac{1}{2}$$

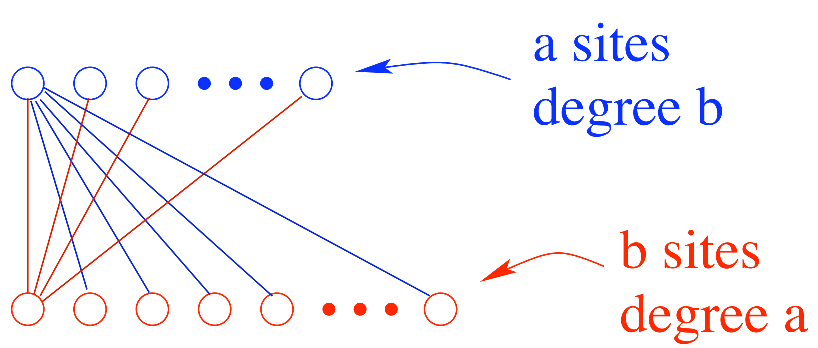
Final state: all 1 with prob. 1/2!

Route to Consensus on Complex Networks

two-time-scale trajectory



complete bipartite graph



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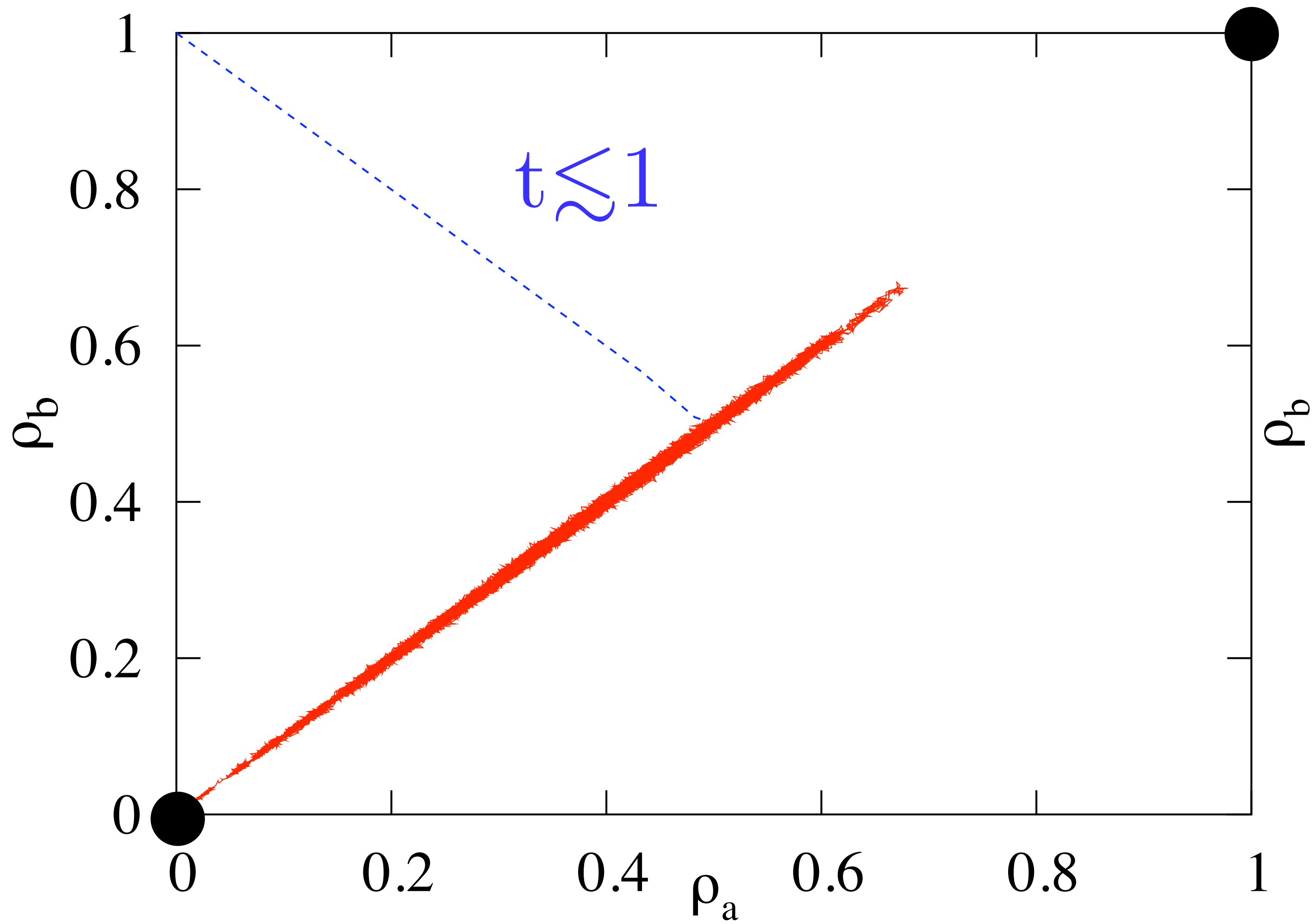

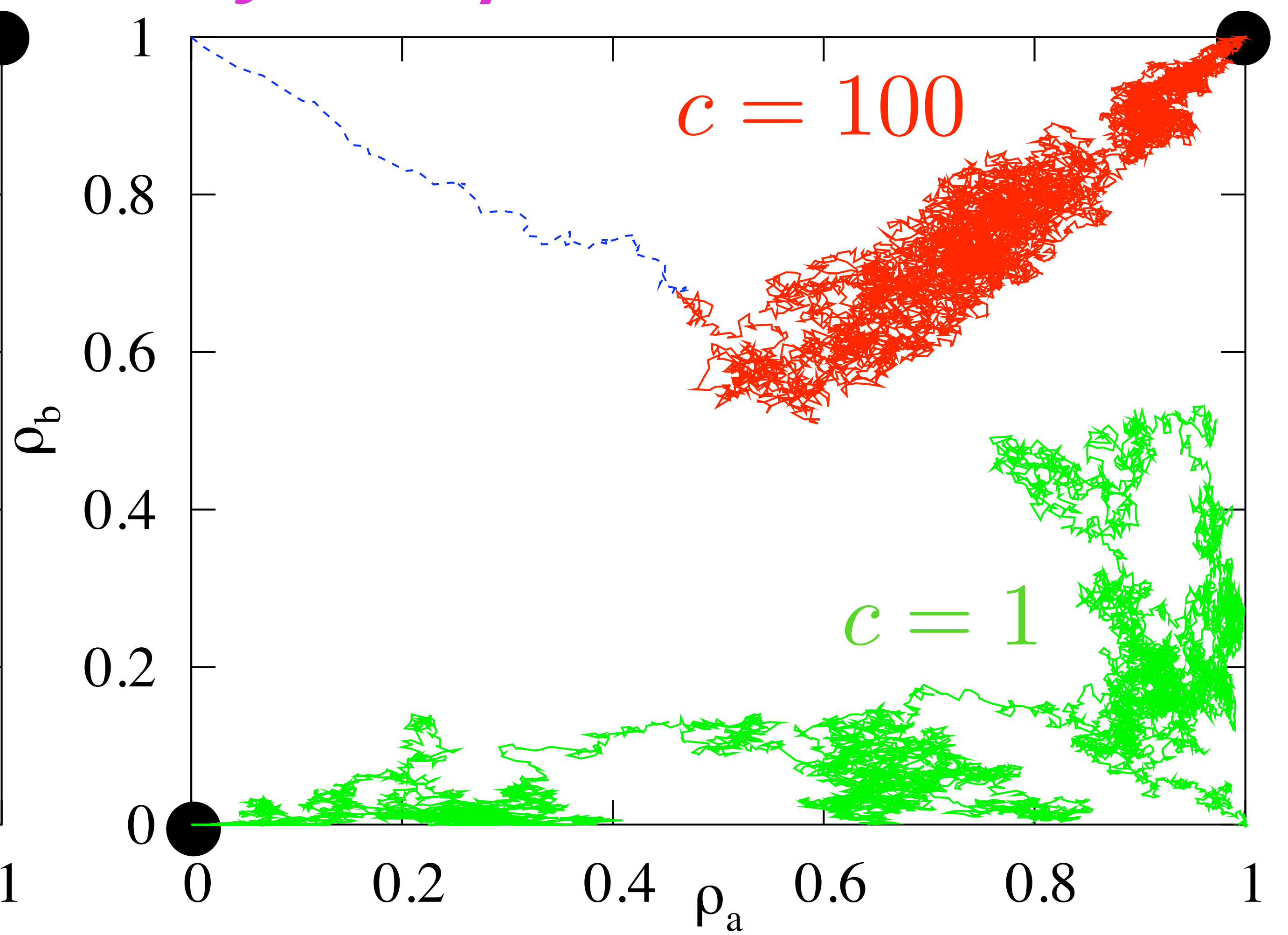

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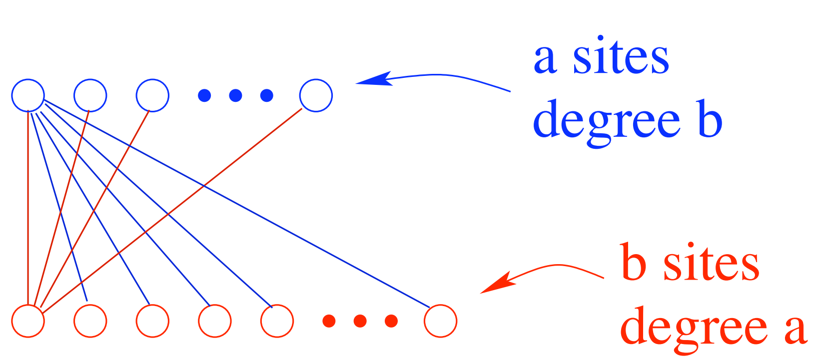
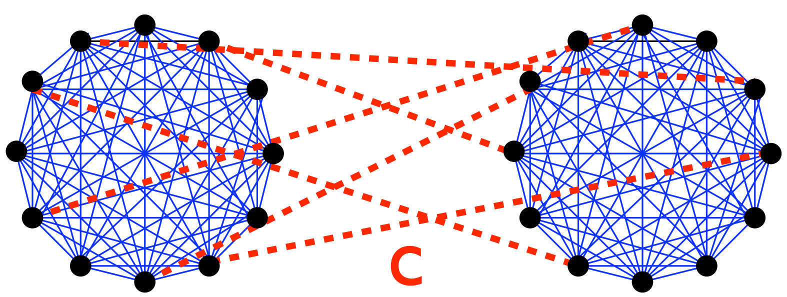


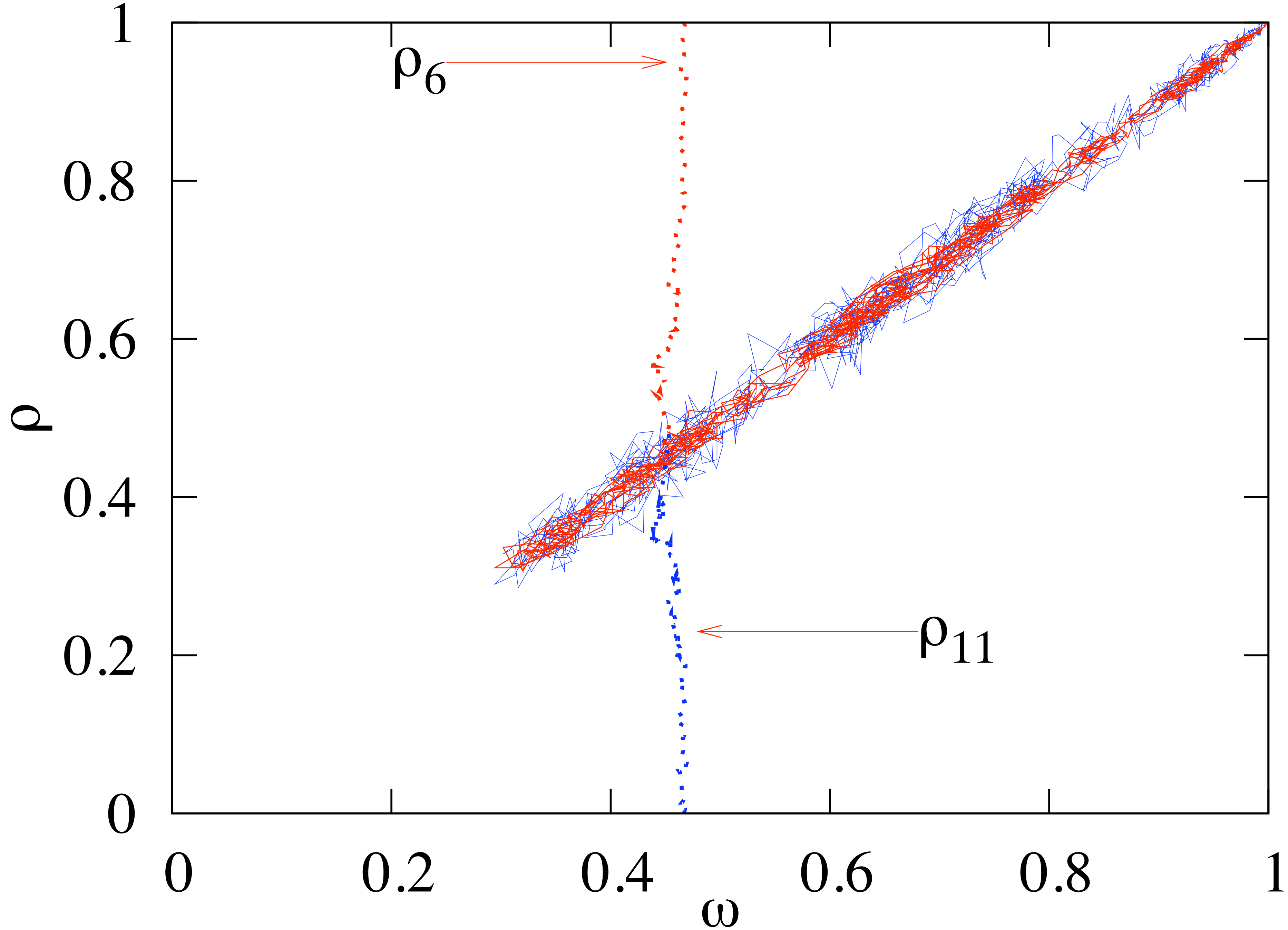
two-clique graph



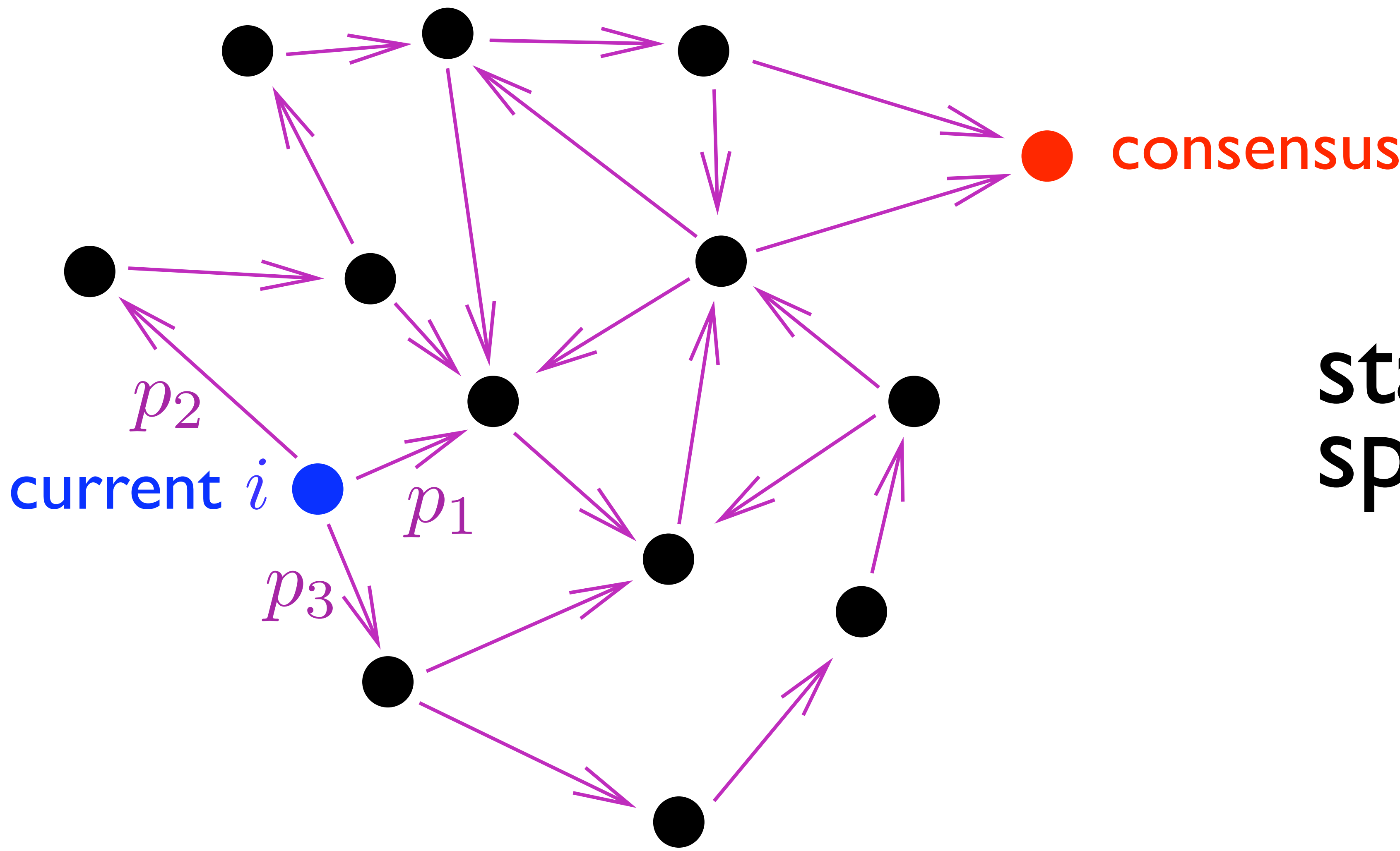
N=10000, C links/node

Molloy-Reed Scale-Free Network

$n_k \sim k^{-2.5}, \quad \mu_1 = 8$



Consensus Time Evolution Equation



state space

backward Kolmogorov equation:

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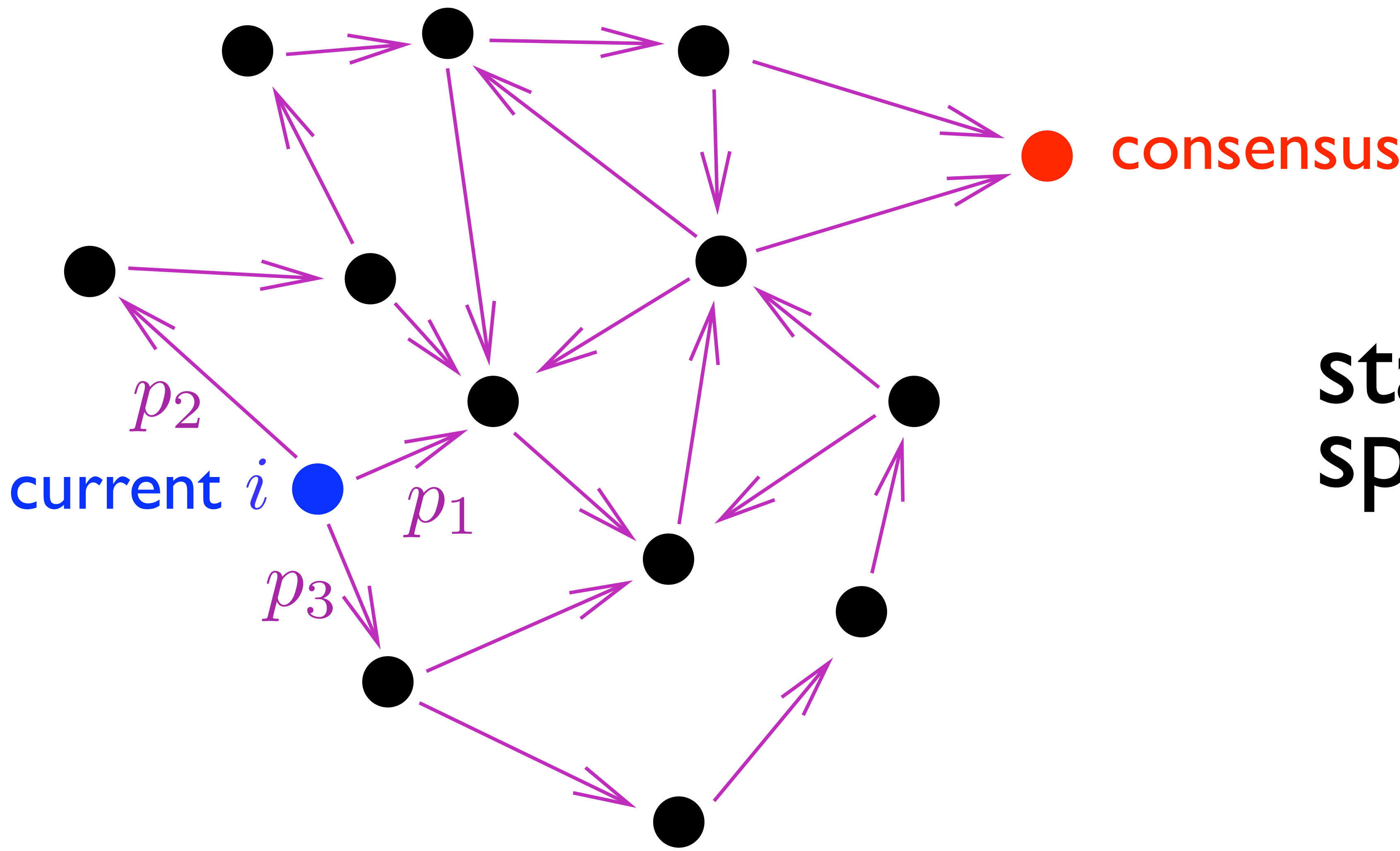

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$$T_i = p_1(T_{i_1} + 1) + p_2(T_{i_2} + 1) + p_3(T_{i_3} + 1)$$

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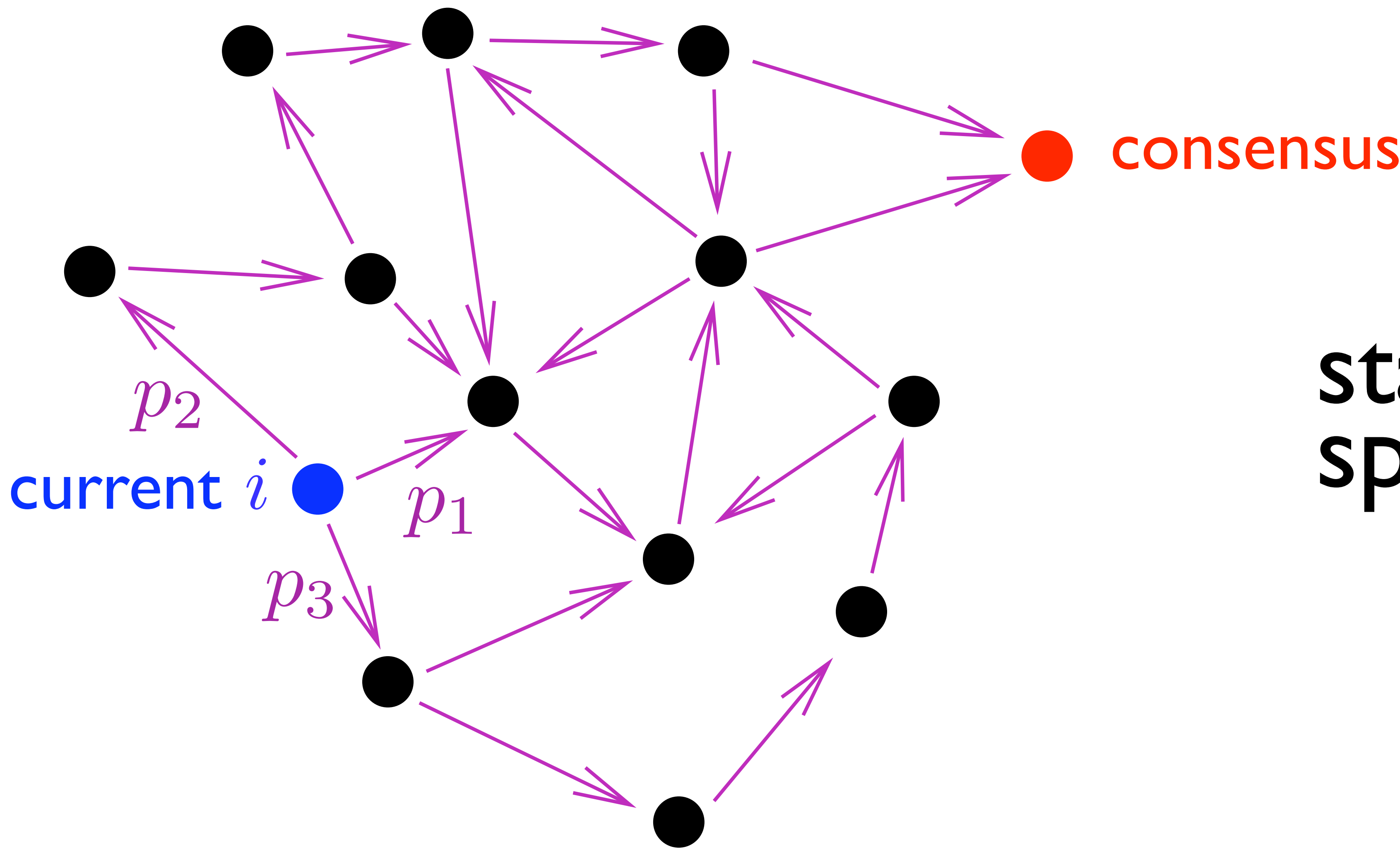

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$$\longrightarrow \nabla^2 T = -N_{\text{eff}} F(\text{initial location})$$

Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

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Voter model:

$$T_N \sim \begin{cases} N & \nu > 3, \\ N/\ln N & \nu = 3, \\ N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\ (\ln N)^2 & \nu = 2, \\ \mathcal{O}(1) & \nu < 2. \end{cases}$$

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Invasion process:

$$T_N \sim \begin{cases} N & \nu > 2, \\ N \ln N & \nu = 2, \\ N^{2-\nu} & \nu < 2. \end{cases}$$

Summary & Outlook

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Still to be done:

empirical connections & predictions

see e.g., "Scaling & University in Proportional Elections" Fortunato & Castellano, PRL (2007)